Constructions of Optimal Cyclic Codes With h-Level Hierarchical Locality

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Abstract

In order to correct different numbers of erasures in distributed storage systems, the design of locally repairable codes with hierarchical locality (H-LRCs) is crucial. In this paper, we study h-level H-LRCs where h is not limited to 2. We present four constructions of cyclic h-level H-LRCs with length ln_1 such that $\gcd(l,q) = 1$ and $n_1|(q-1)$. These four classes of cyclic h-level H-LRCs are optimal with respect to the generalized Singleton-like bound. The minimum Hamming distances of them are $d = \delta_1, \delta_1 + 1, \delta_1 + 2, \delta_1 + \delta_h$ respectively. Furthermore, these four classes of cyclic h-level H-LRCs have new and flexible parameters which are not covered in the literature. By setting the parameter l appropriately for the third construction, one can construct optimal cyclic h-level H-LRCs with length $n|(q-1)$ and distance $d = \delta_1 + \sigma$ where σ is not limited to $0, 1, 2, \delta_h$.

Index Terms

locally repairable codes, cyclic codes, hierarchical locality, h-level hierarchy, distributed storage systems

I. Introduction

In large-scale distributed storage systems, locally repairable codes (LRCs) play an important role due to their local erasure-correcting. Some companies, such as Microsoft, Facebook and so on, have employed LRCs in practical systems. For LRCs with r locality, the ith symbol can be recovered by accessing the other r symbols of its codeword. In 2012, Gopalan et al. [15] first introduced the concept of LRCs with r locality and a large number of studies on them emerged

This work was supported in part by the National Natural Science Foundation of China (Grant No. 61901288) and in part by the Fundamental Research Funds for the Central Universities (Grant No. 2019SCU12004).

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later (see [4], [9], [12], [16], [17], [18], [19], [20], [21], [24], [26], [31], [38], [39], [40]). In the same year, Prakash et al. [32] introduced the concept of (r, δ) LRCs which degenerate into LRCs with r locality when $\delta = 2$. The definition of (r, δ) LRCs can be found in Definition 1 of this paper. Prakash et al. [32] also gave an upper bound on the minimum Hamming distance of (r, δ) LRCs which can be found in (1) of this paper. By this bound, many researchers constructed optimal (r, δ) LRCs in the literature (see [2], [5], [6], [8], [10], [13], [14], [22], [23], [29], [30], [33], [34], [35], [37], [41]).

In 2015, Sasidharan et al. [36] generalized (r, δ) LRCs to LRCs with hierarchical locality (H-LRCs), which have h-level hierarchical locality. They also derived a bound on H-LRCs with h-level hierarchical locality. For $h = 2$, the H-LRC with 2-level hierarchical locality is called a 2-level H-LRC. If h is not limited to 2, then the corresponding H-LRC is called an h-level H-LRC. The definition of h-level H-LRCs can be found in Definition 3 of this paper. In [36], Sasidharan et al. utilized RS-like codes to construct optimal 2-level H-LRCs with length $n \leq q - 1$. In 2019, Ballentine et al. [1] presented a general construction of 2-level H-LRCs from maps between algebraic curves. In 2020, Luo and Cao [28] proposed optimal cyclic 2-level H-LRCs with length $n|q-1$ or $n|q+1$ based on cyclic codes. In the same year, Zhang and Liu [42] constructed 2-level H-LRCs with flexible parameters based on some optimal (r, δ) LRCs. In 2021, Chen and Barg [7] studied h-level H-LRCs and convolutional codes with locality. They first obtained optimal cyclic h-level H-LRCs with length $n|q-1$ and then constructed optimal cyclic h-level H-LRCs with unbounded length $n = q^m - 1$. In 2022, Blaum [3] gave a recursive construction of extended integrated interleaved (EII) codes into multiple layers, which suited as LRCs due to their hierarchical locality. In 2023, Chen et al. [11] constructed three classes of optimal 2-level H-LRCs with different lengths. The last one is a class of optimal cyclic 2-level H-LRCs whose length is $n|q-1$. Recently, Liu and Zeng [27] constructed three new classes of optimal cyclic 2-level H-LRCs with unbounded or large lengths.

The main objective of this paper is to study cyclic h-level H-LRCs. We first give the definition of h-level H-LRCs by considering the length and minimum distance. Then we present four classes of cyclic h-level H-LRCs which are optimal with respect to the generalized Singleton-like bound on h-level H-LRCs. All the h-level H-LRCs proposed in this paper have new and flexible parameters.

The structure of this paper is given as follows. In Section II, the definitions and notations are given. Then four constructions of optimal cyclic h-level H-LRCs with distances $d = \delta_1, \delta_1 + 1, \delta_1 + 2, \delta_1 + \delta_h$ are presented in Sections III-VI respectively. Finally, some remarks are given in Section VII.

II. Definitions and Notations

Throughout this paper, we define the following symbols:

- q —— prime power;
- GF(q) —— finite field consisting of q elements;
- GF$(q)[x]$ —— polynomial over GF(q);
- $[n, k, d]_q$ —— k-dimension linear code over a finite alphabet of size q with minimum Hamming distance d and length n;
- α —— primitive nth root of unity in some extension field of GF(q);
- $\langle x \rangle_y$ —— the smallest nonnegative residue of x module y;
- Γ —— a cyclic shift operator that $\Gamma^i(a) = (a_i, a_{\langle i+1 \rangle_n}, \cdots, a_{\langle i+n-1 \rangle_n})$ for $a = (a_0, a_1, \cdots, a_{n-1})$, $0 \leq i \leq n-1$;
- $\mathcal{C}|_S$ —— a punctured code of $\mathcal{C}$ such that $\mathcal{C}|_S = \{(c_{s_0}, c_{s_1}, \cdots, c_{s_{m-1}}) : (c_0, c_1, \cdots, c_{n-1}) \in \mathcal{C}\}$ where $S = \{s_0, s_1, \cdots, s_{m-1}\} \subseteq \{0, 1, \cdots, n-1\}$.

For an $[n,k,d]_q$ cyclic code $\mathcal{C}$, if $c \in \mathcal{C}$, then $\Gamma^{\tau}(c) \in \mathcal{C}$ for any $1 \leq \tau \leq n-1$. In the perspective of polynomial, the $[n,k,d]_q$ cyclic code $\mathcal{C}$ is a principal ideal of the quotient ring GF$(q)[x]/(x^n-1)$. $\mathcal{C}$ can be generated by a polynomial $g(x)$ called generator polynomial where $g(x)|(x^n-1)$. Then $h(x) = \frac{x^n-1}{g(x)}$ is called the parity-check polynomial of $\mathcal{C}$. The set $\{\alpha^i | g(\alpha^i) = 0, 0 \leq i \leq n-1\}$ is called the defining set of $\mathcal{C}$. For an integer i, the q-cyclotomic coset of i mod n is given by

$$C_i = \{i, iq, iq^2, \cdots, iq^{e_i - 1}\},$$

where e_i is the smallest positive integer such that $iq^{e_i} \equiv i$ mod n. Let D be the defining set of $\mathcal{C}$. It is known that the set $\{j | \alpha^j \in D, 0 \leq j \leq n-1\}$ is a union of some q-cyclotomic cosets mod n.

The following lemma is the famous BCH Bound.

Lemma 1: (BCH bound [25]) Let $\mathcal{C}$ be an $[n,k,d]_q$ cyclic code with defining set D. If $\{\alpha^{i'+i} | 0 \leq i \leq d'-2\} \subseteq D$ for some integers i', then the minimum Hamming distance of $\mathcal{C}$ is greater than or equal to d'.

The following lemma is the Hartmann-Tzeng Bound.

Lemma 2: (Hartmann-Tzeng bound [25]) Let $\mathcal{C}$ be an $[n,k,d]_q$ cyclic code with defining set D. If $\{\alpha^{i'+i_1 a + i_2 b} | 0 \leq i_1 \leq d'-2, 0 \leq i_2 \leq d''\} \subseteq D$ for some integers i', a, b such that $\gcd(n,a) = 1$ and $\gcd(n,b) = 1$, then the minimum Hamming distance of $\mathcal{C}$ is greater than or equal to $d' + d''$.

The following lemma is about the defining set of punctured code of a cyclic code [28].

Lemma 3: Let $\mathcal{C}$ be a cyclic code with length n over GF(q). Let n' be a divisor of n and $l = \frac{n}{n'}$. Suppose that α is a primitive nth root of unity and $\beta = \alpha^l$ is a primitive n'th root of unity. Let $S = \{a, a+l, a+2l, \cdots, a+(n'-1)l\}$ where $0 \leq a \leq l-1$. Clearly, $\mathcal{C}|_S$ is a cyclic code with length n' over GF(q). If the complete defining set of $\mathcal{C}$ contains $\alpha^b, \alpha^{b+n'}, \alpha^{b+2n'}, \cdots, \alpha^{b+(l-1)n'}$ where $0 \le b \le n'-1$, then the complete defining set of $\mathcal{C}|_S$ contains β^b.

The definition of (r, δ) LRCs is given as follows.

Definition 1: Let $\mathcal{C}$ be an $[n, k, d]_q$ code. For each i, $0 \le i \le n-1$, if there exists a punctured subcode of $\mathcal{C}$ with length at most $r + \delta - 1$ and minimum distance δ, whose support contains i, then ith symbol of $\mathcal{C}$ is said to have (r, δ) locality and $\mathcal{C}$ is an (r, δ) LRC.

If $\delta = 2$, then the (r, δ) LRC degenerates into an LRC with r locality. The Singleton-like bound on an (r, δ) LRC with $[n, k, d]_q$ is given as follows [32]:

$$d \le n - k + 1 - (\left\lceil \frac{k}{r} \right\rceil - 1)(\delta - 1). \tag{1}$$

Definition 2: If the parameters of an (r, δ) LRC meet the bound (1) with equality, then it is said to be optimal.

There are many optimal (r, δ) LRCs obtained in the literature (see Section I).

Similar to the definition of 2-level H-LRCs given by Luo and Cao in [28], we give the definition of h-level H-LRCs by considering the length and minimum distance.

Definition 3: Let $h \ge 1$, $0 < r_h < r_{h-1} < \cdots < r_1 < k$, and $1 < \delta_h < \delta_{h-1} < \cdots < \delta_1 \le d$ be integers. A code $\mathcal{C}$ of length n is called an h-level H-LRC with parameters $((r_1, \delta_1), (r_2, \delta_2), \cdots, (r_h, \delta_h))$ if for every coordinate $i \in \{0, 1, \cdots, n-1\}$ there exists a subset $S_i \subset \{0, 1, \cdots, n-1\}$, such that

(1) $i \in S_i$ and the size of S_i is at most $r_1 + \delta_1 - 1 + \sum_{j=2}^{h} \left(\left\lceil \frac{r_1}{r_j} \right\rceil - 1 \right) (\delta_j - \delta_{j+1})$ where $\delta_{h+1} = 1$,

(2) the minimum distance of $\mathcal{C}|_{S_i}$ is at least δ_1,

(3) $\mathcal{C}|_{S_i}$ is an $(h-1)$-level H-LRC with parameters $((r_2, \delta_2), (r_3, \delta_3), \cdots, (r_h, \delta_h))$.

The generalized Singleton-like bound on an $[n, k, d]_q$ h-level H-LRC with parameters $((r_1, \delta_1),$

$(r_2, \delta_2), \cdots, (r_h, \delta_h))$ is given as follows [36]:

$$d \leq n - k + 1 - \sum_{j=1}^{h} \left(\left\lceil \frac{k}{r_j} \right\rceil - 1 \right) (\delta_j - \delta_{j+1}) \tag{2}$$

where $\delta_{h+1} = 1$.

Definition 4: If the parameters of an h-level H-LRC meet the bound (2) with equality, then it is said to be optimal.

Note that (r, δ) LRCs can be seen as a special case of h-level H-LRCs for $h = 1$. There are some optimal 2-level H-LRCs ($h = 2$) in the literature (see Section I). However, to the best of our knowledge, there are very few results on optimal h-level H-LRCs (h is not limited to 2) in the literature (see [7]).

III. Construction of Optimal Cyclic h-Level H-LRCs With $d = \delta_1$

In this section, we give a construction of cyclic h-level H-LRCs with minimum Hamming distance $d = \delta_1$ which are optimal with respect to bound (2).

Let $h \geq 1$ be an integer and q a prime power. Let $n_h < n_{h-1} < \cdots < n_1$ be integers such that $n_1 | (q-1)$, and $n_{x+1} | n_x$ for $x = 1, 2, \cdots, h-1$. For an integer l with $\gcd(l, q) = 1$, the q-cyclotomic coset of i mod n is C_i where $n = ln_1$ and $0 \leq i \leq n-1$. First, we define

$$D^h = \bigcup_{i=0,1,\cdots,\frac{ln_1}{n_h}-1} C_{in_h} \cup C_{in_h+1} \cup \cdots \cup C_{in_h+\delta-1}, \tag{3}$$

where $\delta \leq n_h - 1$. For some integers $0 < \Delta_{h-1} < \Delta_{h-2} < \cdots < \Delta_1 < n_{h-1} - \frac{n_{h-1}}{n_h}\delta$, define

$$D_a^x = \bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1} \; \bigcup_{j=0,1,\cdots,\left\lfloor \frac{\Delta_x}{n_h-\delta} \right\rfloor-1} C_{in_x+jn_h+\delta} \cup C_{in_x+jn_h+\delta+1} \cup \cdots \cup C_{in_x+jn_h+n_h-1}, \tag{4}$$

$$D_b^x = \bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1} C_{in_x+\left\lfloor \frac{\Delta_x}{n_h-\delta} \right\rfloor n_h+\delta} \cup C_{in_x+\left\lfloor \frac{\Delta_x}{n_h-\delta} \right\rfloor n_h+\delta+1} \cup \cdots \cup C_{in_x+\left\lfloor \frac{\Delta_x}{n_h-\delta} \right\rfloor n_h+\delta+\langle \Delta_x \rangle_{n_h-\delta}-1}, \tag{5}$$

where $x = 1, 2, \cdots, h-1$.

Before discussing the structure of D^h, D_a^x, and D_b^x, $x = 1, 2, \cdots, h-1$, we first give the following theorem.

Theorem 1: For any n_x defined above, $x = 1, 2, \cdots, h$, we have

$$\bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1} C_{in_x+c} = \bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1} \{in_x + c\} \tag{6}$$

where $0 \leq c \leq n_x - 1$.

Proof: It is obvious that

$$\bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1} \{in_x + c\} \subseteq \bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1} C_{in_x+c}. \tag{7}$$

We only need to prove the rest that

$$\bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1} C_{in_x+c} \subseteq \bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1} \{in_x + c\}. \tag{8}$$

For any integers $0 \leq i \leq \frac{ln_1}{n_x} - 1$ and $s \geq 0$, we have $(in_x + c)q^s \pmod{n} \in C_{in_x+c}$. Since $n_x|n_1$ and $n_1|(q-1)$, we have $q = an_x + 1$ for some integer a. Then

$$(in_x + c)q^s = (in_x + c)(an_x + 1)^s = In_x + c, \tag{9}$$

where

$$I = i(an_x + 1)^s + ac \sum_{s'=1}^{s} \binom{s}{s'} (an_x)^{s'-1}.$$

It follows that

$$(in_x + c)q^s \pmod n = In_x + c \pmod{ln_1}$$
$$= \langle I \rangle_{\frac{ln_1}{n_x}} n_x + c.$$

Note that $0 \le \langle I \rangle_{\frac{ln_1}{n_x}} \le \frac{ln_1}{n_x} - 1$, which implies that

$$(in_x + c)q^s \pmod n \in \bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1} \{in_x + c\}.$$

Therefore, the (8) holds. Together with (7), the desired result is got.

□

Now we give the structure of D^h, D_a^x, and D_b^x, $x = 1, 2, \cdots, h-1$.

Theorem 2: For D^h, D_a^x, and D_b^x, $x = 1, 2, \cdots, h-1$, defined by (3), (4), and (5) respectively, we have

$$D^h = \bigcup_{i=0,1,\cdots,\frac{ln_1}{n_h}-1} \{in_h, in_h + 1, \cdots, in_h + \delta - 1\}, \tag{10}$$

$$D_a^x = \bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1} \bigcup_{j=0,1,\cdots,\left\lfloor \frac{\Delta_x}{n_h-\delta} \right\rfloor -1} \{in_x+jn_h+\delta, in_x+jn_h+\delta+1, \cdots, in_x+jn_h+n_h-1\}, \tag{11}$$

$$D_b^x = \bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1} \{in_x+\left\lfloor \frac{\Delta_x}{n_h-\delta} \right\rfloor n_h+\delta, in_x+\left\lfloor \frac{\Delta_x}{n_h-\delta} \right\rfloor n_h+\delta+1, \cdots, in_x+\left\lfloor \frac{\Delta_x}{n_h-\delta} \right\rfloor n_h+\delta+\langle \Delta_x \rangle_{n_h-\delta}-1\}. \tag{12}$$

Proof: Applying Theorem 1 to (3), (4), and (5) respectively, one can obtain those three equalities.

□

Let $\mathcal{C}$ be an $[n, k, d]_q$ cyclic code with defining set $\{\alpha^i | i \in D\}$ where

$$D = D^h \cup \left(\bigcup_{x=1,2,\cdots,h-1} D_a^x \cup D_b^x \right). \tag{13}$$

For the code $\mathcal{C}$, we have the following result.

Theorem 3: Define $\Delta_0 = \Delta_1$, $\Delta_h = 0$, $\Delta_{h+1} = -\delta$, and $n_0 = ln_1$. $\mathcal{C}$ is an optimal cyclic $[ln_1, ln_1 - \sum_{i=1}^{h} \frac{ln_1}{n_i}(\Delta_i - \Delta_{i+1}), \left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor \delta + \delta + \Delta_1 + 1]_q$ h-level H-LRC with parameters $((r_1, \delta_1 = \left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor \delta + \delta + \Delta_1 + 1), (r_2, \delta_2 = \left\lfloor \frac{\Delta_2}{n_h-\delta} \right\rfloor \delta + \delta + \Delta_2 + 1), \cdots, (r_h, \delta_h = \left\lfloor \frac{\Delta_h}{n_h-\delta} \right\rfloor \delta + \delta + \Delta_h + 1))$, where $r_j = n_j - \sum_{i=j}^{h} \frac{n_j}{n_i}(\Delta_i - \Delta_{i+1})$ for $0 \leq j \leq h$, $\frac{n_j}{n_i} = \left\lceil \frac{r_j}{r_i} \right\rceil$ for all $0 \leq j < i \leq h-1$, and

$$\frac{n_j}{n_h} = \left\lceil \frac{r_j}{r_h} \right\rceil + \sum_{i=j}^{h-1} \left\lceil \frac{r_j}{r_i} \right\rceil \left(\left\lfloor \frac{\Delta_i}{n_h - \delta} \right\rfloor - \left\lfloor \frac{\Delta_{i+1}}{n_h - \delta} \right\rfloor \right) \tag{14}$$

for all $0 \leq j \leq h-1$.

Proof: It can be checked that $\{0, 1, \cdots, \left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor n_h + \delta + \langle \Delta_1 \rangle_{n_h-\delta} - 1\} \subseteq D$. By Lemma 1, the minimum Hamming distance of $\mathcal{C}$ is at least $\left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor n_h + \delta + \langle \Delta_1 \rangle_{n_h-\delta} + 1 = \left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor \delta + \delta + \Delta_1 + 1$. The dimension of $\mathcal{C}$ is $k = n - |D| = ln_1 - \sum_{i=1}^{h} \frac{ln_1}{n_i}(\Delta_i - \Delta_{i+1})$.

Now we prove that $\mathcal{C}$ is an h-level H-LRC. For any coordinate $i \in \{0, 1, \cdots, n-1\}$, suppose that $S_i = \{\langle i \rangle_l, \langle i \rangle_l + l, \cdots, \langle i \rangle_l + (n_1 - 1)l\}$. It is clear that $i \in S_i$. Let $\beta = \alpha^l$ be a primitive n_1th root of unity. Note that $\bigcup_{j=0,1,\cdots,l-1} \{jn_1, jn_1 + 1, \cdots, jn_1 + \left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor n_h + \delta + \langle \Delta_1 \rangle_{n_h-\delta} - 1\} \subseteq D$. By Lemma 3, $\mathcal{C}|_{S_i}$ is a cyclic code with length n_1 and its complete defining set contains $\{\beta^{j'} | j' \in D'\}$ where $D' = \{0, 1, \cdots, \left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor n_h + \delta + \langle \Delta_1 \rangle_{n_h-\delta} - 1\}$. By Lemma 1, the minimum Hamming distance of $\mathcal{C}|_{S_i}$ is at least $\left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor \delta + \delta + \Delta_1 + 1$. Further, we can calculate that

$$
\begin{aligned}
& r_1 + \delta_1 - 1 + \sum_{j'=2}^{h} \left(\left\lceil \frac{r_1}{r_{j'}} \right\rceil - 1 \right) (\delta_{j'} - \delta_{j'+1}) \\
&= n_1 - \sum_{j'=1}^{h} \frac{n_1}{n_{j'}} (\Delta_{j'} - \Delta_{j'+1}) + \left(\left\lfloor \frac{\Delta_1}{n_h - \delta} \right\rfloor \delta + \delta + \Delta_1 + 1 \right) - 1 \\
&\quad + \sum_{j'=2}^{h-1} \left(\left\lceil \frac{r_1}{r_{j'}} \right\rceil - 1 \right) \left[\left(\left\lfloor \frac{\Delta_{j'}}{n_h - \delta} \right\rfloor \delta + \delta + \Delta_{j'} + 1 \right) - \left(\left\lfloor \frac{\Delta_{j'+1}}{n_h - \delta} \right\rfloor \delta + \delta + \Delta_{j'+1} + 1 \right) \right] \\
&\quad + \left(\left\lceil \frac{r_1}{r_h} \right\rceil - 1 \right) \left(\left\lfloor \frac{\Delta_h}{n_h - \delta} \right\rfloor \delta + \delta + \Delta_h + 1 - 1 \right) \\
&= n_1 - \sum_{j'=1}^{h} \frac{n_1}{n_{j'}} (\Delta_{j'} - \Delta_{j'+1}) + \left\lfloor \frac{\Delta_1}{n_h - \delta} \right\rfloor \delta + \delta + \Delta_1 \\
&\quad + \sum_{j'=2}^{h-1} \left\lceil \frac{r_1}{r_{j'}} \right\rceil \left(\left\lfloor \frac{\Delta_{j'}}{n_h - \delta} \right\rfloor \delta + \Delta_{j'} - \left\lfloor \frac{\Delta_{j'+1}}{n_h - \delta} \right\rfloor \delta - \Delta_{j'+1} \right) \\
&\quad - \left(\left\lfloor \frac{\Delta_2}{n_h - \delta} \right\rfloor \delta + \Delta_2 - \left\lfloor \frac{\Delta_h}{n_h - \delta} \right\rfloor \delta - \Delta_h \right) + \left\lceil \frac{r_1}{r_h} \right\rceil \delta - \delta \\
&= n_1 - \sum_{j'=2}^{h} \frac{n_1}{n_{j'}} (\Delta_{j'} - \Delta_{j'+1}) + \left\lfloor \frac{\Delta_1}{n_h - \delta} \right\rfloor \delta + \sum_{j'=2}^{h-1} \left\lceil \frac{r_1}{r_{j'}} \right\rceil \left(\left\lfloor \frac{\Delta_{j'}}{n_h - \delta} \right\rfloor \delta - \left\lfloor \frac{\Delta_{j'+1}}{n_h - \delta} \right\rfloor \delta \right) \\
&\quad + \sum_{j'=2}^{h-1} \left\lceil \frac{r_1}{r_{j'}} \right\rceil (\Delta_{j'} - \Delta_{j'+1}) - \left\lfloor \frac{\Delta_2}{n_h - \delta} \right\rfloor \delta + \left\lceil \frac{r_1}{r_h} \right\rceil (\Delta_h - \Delta_{h+1}) \\
&= n_1 - \sum_{j'=2}^{h-1} \left(\frac{n_1}{n_{j'}} - \left\lceil \frac{r_1}{r_{j'}} \right\rceil \right) (\Delta_{j'} - \Delta_{j'+1}) - \frac{n_1}{n_h} \delta + \sum_{j'=1}^{h-1} \left\lceil \frac{r_1}{r_{j'}} \right\rceil \left(\left\lfloor \frac{\Delta_{j'}}{n_h - \delta} \right\rfloor \delta - \left\lfloor \frac{\Delta_{j'+1}}{n_h - \delta} \right\rfloor \delta \right) \\
&\quad + \left\lceil \frac{r_1}{r_h} \right\rceil \delta \\
&= n_1
\end{aligned} \tag{15}
$$

where the last step employs the fact $\frac{n_j}{n_{j'}} = \left\lceil \frac{r_j}{r_{j'}} \right\rceil$ and equality (14) for $1 = j < j' \leq h-1$. For any coordinate $i' \in \{\langle i \rangle_l, \langle i \rangle_l + l, \cdots, \langle i \rangle_l + (n_1-1)l\}$, suppose that $\bar{S}_{i'} = \{\langle i \rangle_l + \langle i^* \rangle_{\frac{n_1}{n_2}} l, \langle i \rangle_l + (\langle i^* \rangle_{\frac{n_1}{n_2}} + \frac{n_1}{n_2})l, \cdots, \langle i \rangle_l + [\langle i^* \rangle_{\frac{n_1}{n_2}} + (n_2-1)\frac{n_1}{n_2}]l\}$ where $i' = \langle i \rangle_l + i^* l$. It is clear that $i' \in \bar{S}_{i'}$. Let $\gamma = \alpha^{\frac{l n_1}{n_2}}$ be a primitive n_2th root of unity. Note that $\bigcup_{j=0,1,\cdots,\frac{l n_1}{n_2}-1} \{jn_2, jn_2+1, \cdots, jn_2 + \left\lfloor \frac{\Delta_2}{n_h-\delta} \right\rfloor n_h +$

$\delta + \langle \Delta_2 \rangle_{n_h-\delta} - 1\} \subseteq D$. By Lemma 3, $\mathcal{C}|_{\bar{S}_{i'}}$ is a cyclic code with length n_2 and its complete defining set contains $\{\gamma^{j'} | j' \in D''\}$ where $D'' = \{0, 1, \cdots, \left\lfloor \frac{\Delta_2}{n_h-\delta} \right\rfloor n_h + \delta + \langle \Delta_2 \rangle_{n_h-\delta} - 1\}$. By Lemma 1, the minimum Hamming distance of $\mathcal{C}|_{\bar{S}_{i'}}$ is at least $\left\lfloor \frac{\Delta_2}{n_h-\delta} \right\rfloor \delta + \delta + \Delta_2 + 1$. Similar to the derivation of (15), we can obtain

$$r_2 + \delta_2 - 1 + \sum_{j'=3}^{h} \left(\left\lceil \frac{r_2}{r_{j'}} \right\rceil - 1 \right) (\delta_{j'} - \delta_{j'+1}) = n_2. \tag{16}$$

By discussing it recursively in this way, one can get similar results on the parameters (r_x, δ_x) for $x = 3, 4, \cdots, h$.

Next we prove its optimality. For the $[ln_1, ln_1 - \sum_{i=1}^{h} \frac{ln_1}{n_i}(\Delta_i - \Delta_{i+1}), d]_q$ h-level H-LRC with parameters $((r_1, \delta_1), (r_2, \delta_2), \cdots, (r_h, \delta_h))$, by bound (2) we have

$$\begin{aligned}
d \;\le\; & ln_1 - \left[ln_1 - \sum_{i=1}^{h} \frac{ln_1}{n_i}(\Delta_i - \Delta_{i+1}) \right] + 1 - \left\lceil \frac{r_0}{r_h} \right\rceil \delta + \delta \\
& - \sum_{i=1}^{h-1} \left(\left\lceil \frac{r_0}{r_i} \right\rceil - 1 \right) \left[\left(\left\lfloor \frac{\Delta_i}{n_h - \delta} \right\rfloor \delta + \delta + \Delta_i + 1 \right) - \left(\left\lfloor \frac{\Delta_{i+1}}{n_h - \delta} \right\rfloor \delta + \delta + \Delta_{i+1} + 1 \right) \right] \\
= \;& \sum_{i=1}^{h} \frac{ln_1}{n_i}(\Delta_i - \Delta_{i+1}) + 1 - \left\lceil \frac{r_0}{r_h} \right\rceil \delta + \delta + \left\lfloor \frac{\Delta_1}{n_h - \delta} \right\rfloor \delta + \Delta_1 \\
& - \sum_{i=1}^{h-1} \left\lceil \frac{r_0}{r_i} \right\rceil \left(\left\lfloor \frac{\Delta_i}{n_h - \delta} \right\rfloor \delta - \left\lfloor \frac{\Delta_{i+1}}{n_h - \delta} \right\rfloor \delta + \Delta_i - \Delta_{i+1} \right) \\
= \;& \sum_{i=1}^{h-1} \left(\frac{ln_1}{n_i} - \left\lceil \frac{r_0}{r_i} \right\rceil \right) (\Delta_i - \Delta_{i+1}) + \frac{ln_1}{n_h}\delta + 1 - \left\lceil \frac{r_0}{r_h} \right\rceil \delta + \delta + \left\lfloor \frac{\Delta_1}{n_h - \delta} \right\rfloor \delta + \Delta_1 \\
& - \sum_{i=0}^{h-1} \left\lceil \frac{r_0}{r_i} \right\rceil \left(\left\lfloor \frac{\Delta_i}{n_h - \delta} \right\rfloor \delta - \left\lfloor \frac{\Delta_{i+1}}{n_h - \delta} \right\rfloor \delta \right) \\
= \;& \left\lfloor \frac{\Delta_1}{n_h - \delta} \right\rfloor \delta + \delta + \Delta_1 + 1
\end{aligned} \tag{17}$$

where the last step employs the fact $\frac{n_j}{n_i} = \left\lceil \frac{r_j}{r_i} \right\rceil$ and equality (14) for $0 = j < i \le h - 1$. Then the minimum Hamming distance of $\mathcal{C}$ is exactly $\left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor \delta + \delta + \Delta_1 + 1$ and $\mathcal{C}$ is an optimal cyclic h-level H-LRC.

$\square$

Remark 1: In [7], the length of cyclic h-level H-LRCs should be $n|(q-1)$ or $n = q^m - 1$ for $m \geq 1$. By our construction, the length of cyclic h-level H-LRCs is $n = ln_1$ for $\gcd(l, q) = 1$ and $n_1|(q-1)$. Thus, the parameters of cyclic h-level H-LRCs by our construction are new. The other three classes of cyclic h-level H-LRCs presented in this paper (see Sections IV, V, VI respectively) also have new parameters for the same reason.

Example 1: Let $q = 97$, $h = 4$, $n_1 = 96$, $n_2 = 48$, $n_3 = 24$, $n_4 = 6$, $l = 2$, and $\delta = 3$. For $\Delta_1 = 6$, $\Delta_2 = 4$, and $\Delta_3 = 3$, one can get

$$D^4 \cup \left(\bigcup_{x=1,2,3} D_a^x \cup D_b^x \right) = \left(\bigcup_{i=0,1,\cdots,31} \{6i, 6i+1, 6i+2\} \right) \cup \left(\bigcup_{i=0,1,\cdots,7} \{24i+3, 24i+4, 24i+5\} \right) \cup \left(\bigcup_{i=0,1,2,3} \{48i+9\} \right) \cup \left(\bigcup_{i=0,1} \{96i+10, 96i+11\} \right).$$

For a cyclic code $\mathcal{C}$ of length $ln_1 = 192$ with defining set $\{\alpha^i | i \in D^4 \cup (\bigcup_{x=1,2,3} D_a^x \cup D_b^x)\}$, one can check that $0, 1, \cdots, 14$ belong to $D^4 \cup (\bigcup_{x=1,2,3} D_a^x \cup D_b^x)$. By the BCH bound, the minimum Hamming distance of $\mathcal{C}$ is at least 16. The dimension of $\mathcal{C}$ is $k = ln_1 - |D^4 \cup (\bigcup_{x=1,2,3} D_a^x \cup D_b^x)| = 192 - 128 = 64$. By Lemmas 1,3 and Definition 3, it can be verified that $\mathcal{C}$ is an h-level H-LRC with parameters $((32, 16), (17, 11), (9, 10), (3, 4))$. By bound (2) we have

$$\begin{aligned} d \leq\ & 192 - 64 + 1 - \left(\left\lceil \frac{64}{32} \right\rceil - 1 \right)(16 - 11) - \left(\left\lceil \frac{64}{17} \right\rceil - 1 \right)(11 - 10) \\ & - \left(\left\lceil \frac{64}{9} \right\rceil - 1 \right)(10 - 4) - \left(\left\lceil \frac{64}{3} \right\rceil - 1 \right)(4 - 1) \\ =\ & 16. \end{aligned}$$

Thus, $\mathcal{C}$ is an optimal cyclic $[192, 64, 16]_{97}$ h-level H-LRC with parameters $((32, 16), (17, 11), (9, 10), (3, 4))$.

IV. Construction of Optimal Cyclic h-Level H-LRCs With $d = \delta_1 + 1$

Now we construct a class of cyclic h-level H-LRCs with $d = \delta_1 + 1$ which are optimal with respect to bound (2).

For an integer $h \geq 1$ and a prime power q, let $n_h < n_{h-1} < \cdots < n_1$ be integers such that $n_1|(q-1)$, and $n_{x+1}|n_x$, $x = 1, 2, \cdots, h-1$. For an integer l with $\gcd(l, q) = 1$, define q-cyclotomic coset of i mod n by C_i where $n = ln_1$ and $0 \leq i \leq n-1$. Let

$$E^h = \bigcup_{i=0,1,\cdots,\frac{ln_1}{n_h}-1} C_{in_h+1} \cup C_{in_h+2} \cup \cdots \cup C_{in_h+\delta}, \tag{18}$$

where $\delta \leq n_h - 2$. For some integers $0 < \Delta_{h-1} < \Delta_{h-2} < \cdots < \Delta_1 < n_{h-1} - \frac{n_{h-1}}{n_h}\delta - 1$ such that $(n_h - \delta) \nmid (\Delta_1 + 1)$, let

$$E_a^x = \bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1} \bigcup_{j=0,1,\cdots,\left\lfloor \frac{\Delta_x}{n_h-\delta} \right\rfloor -1} C_{in_x+jn_h+\delta+1} \cup C_{in_x+jn_h+\delta+2} \cup \cdots \cup C_{in_x+jn_h+n_h}, \tag{19}$$

$$E_b^x = \bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1} C_{in_x+\left\lfloor \frac{\Delta_x}{n_h-\delta} \right\rfloor n_h+\delta+1} \cup C_{in_x+\left\lfloor \frac{\Delta_x}{n_h-\delta} \right\rfloor n_h+\delta+2} \cup \cdots \cup C_{in_x+\left\lfloor \frac{\Delta_x}{n_h-\delta} \right\rfloor n_h+\delta+\langle \Delta_x \rangle_{n_h-\delta}}, \tag{20}$$

where $x = 1, 2, \cdots, h-1$.

Let $\mathcal{C}$ be an $[n, k, d]_q$ cyclic code with defining set $\{\alpha^i | i \in D\}$ where

$$D = C_0 \cup E^h \cup \left(\bigcup_{x=1,2,\cdots,h-1} E_a^x \cup E_b^x \right). \tag{21}$$

Then we can get the following result.

Theorem 4: Define $\Delta_0 = \Delta_1 + 1$, $\Delta_h = 0$, $\Delta_{h+1} = -\delta$, and $n_0 = ln_1$. $\mathcal{C}$ is an optimal cyclic $[ln_1, ln_1 - \sum_{i=1}^{h} \frac{ln_1}{n_i}(\Delta_i - \Delta_{i+1}) - 1, \left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor \delta + \delta + \Delta_1 + 2]_q$ h-level H-LRC with parameters $((r_1, \delta_1 = \left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor \delta + \delta + \Delta_1 + 1), (r_2, \delta_2 = \left\lfloor \frac{\Delta_2}{n_h-\delta} \right\rfloor \delta + \delta + \Delta_2 + 1), \cdots, (r_h, \delta_h = \left\lfloor \frac{\Delta_h}{n_h-\delta} \right\rfloor \delta + \delta + \Delta_h + 1))$, where $r_j = n_j - \sum_{i=j}^{h} \frac{n_j}{n_i}(\Delta_i - \Delta_{i+1})$ for $0 \leq j \leq h$, $\frac{n_j}{n_i} = \left\lceil \frac{r_j}{r_i} \right\rceil$ for all $0 \leq j < i \leq h-1$, and

$$\frac{n_j}{n_h}=\left\lceil\frac{r_j}{r_h}\right\rceil+\sum_{i=j}^{h-1}\left\lceil\frac{r_j}{r_i}\right\rceil\left(\left\lfloor\frac{\Delta_i}{n_h-\delta}\right\rfloor-\left\lfloor\frac{\Delta_{i+1}}{n_h-\delta}\right\rfloor\right)\tag{22}$$

for all $0\leq j\leq h-1$.

Proof: Note that $C_0=\{0\}$. Similar to Theorem 2, we can obtain

$$E^h=\bigcup_{i=0,1,\cdots,\frac{ln_1}{n_h}-1}\{in_h+1,in_h+2,\cdots,in_h+\delta\},\tag{23}$$

$$E_a^x=\bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1}\ \bigcup_{j=0,1,\cdots,\left\lfloor\frac{\Delta_x}{n_h-\delta}\right\rfloor-1}\{in_x+jn_h+\delta+1,in_x+jn_h+\delta+2,\cdots,in_x+jn_h+n_h\},\tag{24}$$

$$E_b^x=\bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1}\{in_x+\left\lfloor\frac{\Delta_x}{n_h-\delta}\right\rfloor n_h+\delta+1,in_x+\left\lfloor\frac{\Delta_x}{n_h-\delta}\right\rfloor n_h+\delta+2,\cdots,in_x+\left\lfloor\frac{\Delta_x}{n_h-\delta}\right\rfloor n_h+\delta+\langle\Delta_x\rangle_{n_h-\delta}\},\tag{25}$$

where $x=1,2,\cdots,h-1$. It is easy to check that $\{0,1,\cdots,\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor n_h+\delta+\langle\Delta_1\rangle_{n_h-\delta}\}\subseteq D$. By Lemma 1, the minimum Hamming distance of $\mathcal{C}$ is at least $\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor n_h+\delta+\langle\Delta_1\rangle_{n_h-\delta}+2=\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor\delta+\delta+\Delta_1+2$. The dimension of $\mathcal{C}$ is $k=n-|D|=ln_1-\sum_{i=1}^{h}\frac{ln_1}{n_i}(\Delta_i-\Delta_{i+1})-1$.

Similar to the proof of Theorem 3, it follows that $\mathcal{C}$ is an h-level H-LRC with parameters $((r_1,\delta_1),(r_2,\delta_2),\cdots,(r_h,\delta_h))$. For the $[ln_1,ln_1-\sum_{i=1}^{h}\frac{ln_1}{n_i}(\Delta_i-\Delta_{i+1})-1,d]_q$ h-level H-LRC with parameters $((r_1,\delta_1),(r_2,\delta_2),\cdots,(r_h,\delta_h))$, by bound (2) we have

$$\begin{aligned}d\ \leq\ & ln_1-\left[ln_1-\sum_{i=1}^{h}\frac{ln_1}{n_i}(\Delta_i-\Delta_{i+1})-1\right]+1-\left\lceil\frac{r_0}{r_h}\right\rceil\delta+\delta\\&-\sum_{i=1}^{h-1}\left(\left\lceil\frac{r_0}{r_i}\right\rceil-1\right)\left[\left(\left\lfloor\frac{\Delta_i}{n_h-\delta}\right\rfloor\delta+\delta+\Delta_i+1\right)-\left(\left\lfloor\frac{\Delta_{i+1}}{n_h-\delta}\right\rfloor\delta+\delta+\Delta_{i+1}+1\right)\right]\\=\ &\sum_{i=1}^{h}\frac{ln_1}{n_i}(\Delta_i-\Delta_{i+1})+2-\left\lceil\frac{r_0}{r_h}\right\rceil\delta+\delta+\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor\delta+\Delta_1\\&-\sum_{i=1}^{h-1}\left\lceil\frac{r_0}{r_i}\right\rceil\left(\left\lfloor\frac{\Delta_i}{n_h-\delta}\right\rfloor\delta-\left\lfloor\frac{\Delta_{i+1}}{n_h-\delta}\right\rfloor\delta+\Delta_i-\Delta_{i+1}\right).\end{aligned}\tag{26}$$

Since $(n_h-\delta)\nmid(\Delta_1+1)$, we have

$$\left\lfloor\frac{\Delta_0}{n_h-\delta}\right\rfloor=\left\lfloor\frac{\Delta_0-1}{n_h-\delta}\right\rfloor=\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor \tag{27}$$

which leads to

$$\begin{aligned} d \leq & \sum_{i=1}^{h-1}\left(\frac{ln_1}{n_i}-\left\lceil\frac{r_0}{r_i}\right\rceil\right)(\Delta_i-\Delta_{i+1})+\frac{ln_1}{n_h}\delta+2-\left\lceil\frac{r_0}{r_h}\right\rceil\delta+\delta+\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor\delta+\Delta_1 \\ & -\sum_{i=0}^{h-1}\left\lceil\frac{r_0}{r_i}\right\rceil\left(\left\lfloor\frac{\Delta_i}{n_h-\delta}\right\rfloor\delta-\left\lfloor\frac{\Delta_{i+1}}{n_h-\delta}\right\rfloor\delta\right) \\ = & \left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor\delta+\delta+\Delta_1+2. \end{aligned} \tag{28}$$

The last step of (28) employs the fact $\frac{n_j}{n_i}=\left\lceil\frac{r_j}{r_i}\right\rceil$ and equality (22) for $0=j<i\leq h-1$. Then the minimum Hamming distance of $\mathcal{C}$ is exactly $\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor\delta+\delta+\Delta_1+2$ and $\mathcal{C}$ is an optimal cyclic h-level H-LRC.

□

Example 2: Let $q=113$, $h=4$, $n_1=112$, $n_2=56$, $n_3=28$, $n_4=7$, $l=2$, and $\delta=3$. For $\Delta_1=8$, $\Delta_2=5$, and $\Delta_3=4$, it follows that

$$\begin{aligned} D= & C_0\cup E^4\cup\left(\bigcup_{x=1,2,3}E_a^x\cup E_b^x\right) \\ = & \{0\}\cup\left(\bigcup_{i=0,1,\cdots,31}\{7i+1,7i+2,7i+3\}\right)\cup\left(\bigcup_{i=0,1,\cdots,7}\{28i+4,28i+5,28i+6,28i+7\}\right) \\ & \cup\left(\bigcup_{i=0,1,2,3}\{56i+11\}\right)\cup\left(\bigcup_{i=0,1}\{112i+12,112i+13,112i+14\}\right). \end{aligned}$$

For a cyclic code $\mathcal{C}$ of length $ln_1=224$ with defining set $\{\alpha^i|i\in D\}$, it is easily checked that $0,1,\cdots,17$ belong to D. By Lemma 1, the minimum Hamming distance of $\mathcal{C}$ is at least 19. The dimension of $\mathcal{C}$ is $k=ln_1-|D|=224-139=85$. By Lemmas 1,3 and Definition 3, one can verify that $\mathcal{C}$ is an h-level H-LRC with parameters $((43,18),(23,12),(12,11),(4,4))$. By bound (2) we have

$$\begin{aligned} d \ \leq\ & 224-85+1-\left(\left\lceil\frac{85}{43}\right\rceil-1\right)(18-12)-\left(\left\lceil\frac{85}{23}\right\rceil-1\right)(12-11) \\ & -\left(\left\lceil\frac{85}{12}\right\rceil-1\right)(11-4)-\left(\left\lceil\frac{85}{4}\right\rceil-1\right)(4-1) \\ =\ & 19. \end{aligned}$$

Thus, $\mathcal{C}$ is an optimal cyclic $[224,85,19]_{113}$ h-level H-LRC with parameters $((43,18),(23,12),(12,11),(4,4))$.

V. Construction of Optimal Cyclic h-Level H-LRCs With $d=\delta_1+2$

Next we present a construction of optimal cyclic h-level H-LRCs with $d=\delta_1+2$.

Let $h\geq 1$ be an integer and q a prime power. Let $n_h<n_{h-1}<\cdots<n_1$ and l be integers such that $\gcd(l,q)=1$, $n_1|(q-1)$, and $n_{x+1}|n_x$, $x=1,2,\cdots,h-1$. Define q-cyclotomic coset of i mod n by C_i where $n=ln_1$ and $0\leq i\leq n-1$. Let $\delta,\Delta_1,\Delta_2,\cdots,\Delta_{h-1}$ be integers such that $\delta\leq n_h-3$, $0<\Delta_{h-1}<\Delta_{h-2}<\cdots<\Delta_1<n_{h-1}-\frac{n_{h-1}}{n_h}\delta-1$, $\langle\Delta_1\rangle_{n_h-\delta}<n_h-\delta-2$, and $[\frac{l}{\gcd(l,\frac{q-1}{n_1})}]\mid\left(\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor n_h+\delta+\langle\Delta_1\rangle_{n_h-\delta}+1\right)$. Assume that E^h, E_a^x, and E_b^x, $x=1,2,\cdots,h-1$, are defined by (18), (19), and (20) respectively.

Let $\mathcal{C}$ be an $[n,k,d]_q$ cyclic code with defining set $\{\alpha^i|i\in D\}$ where

$$D=C_0\cup C_{\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor n_h+\delta+\langle\Delta_1\rangle_{n_h-\delta}+1}\cup E^h\cup\left(\bigcup_{x=1,2,\cdots,h-1}E_a^x\cup E_b^x\right). \tag{29}$$

Then the following theorem is got.

Theorem 5: Define $\Delta_0=\Delta_1+2$, $\Delta_h=0$, $\Delta_{h+1}=-\delta$, and $n_0=ln_1$. $\mathcal{C}$ is an optimal cyclic $[ln_1,ln_1-\sum_{i=1}^{h}\frac{ln_1}{n_i}(\Delta_i-\Delta_{i+1})-2,\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor\delta+\delta+\Delta_1+3]_q$ h-level H-LRC with parameters $((r_1,\delta_1=\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor\delta+\delta+\Delta_1+1),(r_2,\delta_2=\left\lfloor\frac{\Delta_2}{n_h-\delta}\right\rfloor\delta+\delta+\Delta_2+1),\cdots,(r_h,\delta_h=\left\lfloor\frac{\Delta_h}{n_h-\delta}\right\rfloor\delta+\delta+$

$\Delta_h+1)$), where $r_j = n_j - \sum_{i=j}^{h} \frac{n_j}{n_i}(\Delta_i - \Delta_{i+1})$ for $0 \le j \le h$, $\frac{n_j}{n_i} = \left\lceil \frac{r_j}{r_i} \right\rceil$ for all $0 \le j < i \le h-1$, and

$$\frac{n_j}{n_h} = \left\lceil \frac{r_j}{r_h} \right\rceil + \sum_{i=j}^{h-1} \left\lceil \frac{r_j}{r_i} \right\rceil \left(\left\lfloor \frac{\Delta_i}{n_h - \delta} \right\rfloor - \left\lfloor \frac{\Delta_{i+1}}{n_h - \delta} \right\rfloor \right) \tag{30}$$

for all $0 \le j \le h-1$.

Proof: First, we prove that $C_{\left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor n_h + \delta + \langle \Delta_1 \rangle_{n_h-\delta} + 1} = \{ \left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor n_h + \delta + \langle \Delta_1 \rangle_{n_h-\delta} + 1 \}$. Since $[\frac{l}{\gcd(l, \frac{q-1}{n_1})}] \mid \left(\left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor n_h + \delta + \langle \Delta_1 \rangle_{n_h-\delta} + 1 \right)$, we have

$$\begin{aligned}
&\left(\left\lfloor \frac{\Delta_1}{n_h - \delta} \right\rfloor n_h + \delta + \langle \Delta_1 \rangle_{n_h-\delta} + 1 \right) q \pmod{ln_1} \\
&= \left\lfloor \frac{\Delta_1}{n_h - \delta} \right\rfloor n_h + \delta + \langle \Delta_1 \rangle_{n_h-\delta} + 1 + \frac{\left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor n_h + \delta + \langle \Delta_1 \rangle_{n_h-\delta} + 1}{ln_1/(q-1)} \cdot ln_1 \pmod{ln_1} \\
&= \left\lfloor \frac{\Delta_1}{n_h - \delta} \right\rfloor n_h + \delta + \langle \Delta_1 \rangle_{n_h-\delta} + 1 \\
&\quad + \frac{\left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor n_h + \delta + \langle \Delta_1 \rangle_{n_h-\delta} + 1}{l/\gcd(l, \frac{q-1}{n_1})} \cdot \frac{\frac{q-1}{n_1}}{\gcd(l, \frac{q-1}{n_1})} \cdot ln_1 \pmod{ln_1} \\
&= \left\lfloor \frac{\Delta_1}{n_h - \delta} \right\rfloor n_h + \delta + \langle \Delta_1 \rangle_{n_h-\delta} + 1.
\end{aligned} \tag{31}$$

It is well known that $C_0 = \{0\}$. Similar to the proof of Theorem 4, we can check that $\{0, 1, \cdots, \left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor n_h + \delta + \langle \Delta_1 \rangle_{n_h-\delta} + 1\} \subseteq D$. By Lemma 1, the minimum Hamming distance of $\mathcal{C}$ is at least $\left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor n_h + \delta + \langle \Delta_1 \rangle_{n_h-\delta} + 3 = \left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor \delta + \delta + \Delta_1 + 3$. The dimension of $\mathcal{C}$ is $k = n - |D| = ln_1 - \sum_{i=1}^{h} \frac{ln_1}{n_i}(\Delta_i - \Delta_{i+1}) - 2$.

Similarly, it can be verified that $\mathcal{C}$ is an h-level H-LRC with parameters $((r_1, \delta_1), (r_2, \delta_2), \cdots, (r_h, \delta_h))$. For the $[ln_1, ln_1 - \sum_{i=1}^{h} \frac{ln_1}{n_i}(\Delta_i - \Delta_{i+1}) - 2, d]_q$ h-level H-LRC with parameters

$((r_1,\delta_1),(r_2,\delta_2),\cdots,(r_h,\delta_h))$, by bound (2) we have

$$\begin{aligned}
d \ \le\ & ln_1 - \left[ln_1 - \sum_{i=1}^{h} \frac{ln_1}{n_i}(\Delta_i - \Delta_{i+1}) - 2 \right] + 1 - \left\lceil \frac{r_0}{r_h} \right\rceil \delta + \delta \\
& - \sum_{i=1}^{h-1} \left(\left\lceil \frac{r_0}{r_i} \right\rceil - 1 \right) \left[\left(\left\lfloor \frac{\Delta_i}{n_h - \delta} \right\rfloor \delta + \delta + \Delta_i + 1 \right) - \left(\left\lfloor \frac{\Delta_{i+1}}{n_h - \delta} \right\rfloor \delta + \delta + \Delta_{i+1} + 1 \right) \right] \\
=\ & \sum_{i=1}^{h} \frac{ln_1}{n_i}(\Delta_i - \Delta_{i+1}) + 3 - \left\lceil \frac{r_0}{r_h} \right\rceil \delta + \delta + \left\lfloor \frac{\Delta_1}{n_h - \delta} \right\rfloor \delta + \Delta_1 \\
& - \sum_{i=1}^{h-1} \left\lceil \frac{r_0}{r_i} \right\rceil \left(\left\lfloor \frac{\Delta_i}{n_h - \delta} \right\rfloor \delta - \left\lfloor \frac{\Delta_{i+1}}{n_h - \delta} \right\rfloor \delta + \Delta_i - \Delta_{i+1} \right).
\end{aligned} \tag{32}$$

Note that $\langle \Delta_1 \rangle_{n_h-\delta} < n_h - \delta - 2$. Thus

$$\begin{aligned}
\left\lfloor \frac{\Delta_0}{n_h - \delta} \right\rfloor &= \left\lfloor \frac{\Delta_1 + 2}{n_h - \delta} \right\rfloor \\
&= \left\lfloor \frac{\left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor (n_h - \delta) + \langle \Delta_1 \rangle_{n_h-\delta} + 2}{n_h - \delta} \right\rfloor \\
&= \left\lfloor \frac{\Delta_1}{n_h - \delta} \right\rfloor + \left\lfloor \frac{\langle \Delta_1 \rangle_{n_h-\delta} + 2}{n_h - \delta} \right\rfloor \\
&= \left\lfloor \frac{\Delta_1}{n_h - \delta} \right\rfloor.
\end{aligned} \tag{33}$$

This implies that

$$\begin{aligned}
d \ \le\ & \sum_{i=1}^{h-1} \left(\frac{ln_1}{n_i} - \left\lceil \frac{r_0}{r_i} \right\rceil \right) (\Delta_i - \Delta_{i+1}) + \frac{ln_1}{n_h}\delta + 3 - \left\lceil \frac{r_0}{r_h} \right\rceil \delta + \delta + \left\lfloor \frac{\Delta_1}{n_h - \delta} \right\rfloor \delta + \Delta_1 \\
& - \sum_{i=0}^{h-1} \left\lceil \frac{r_0}{r_i} \right\rceil \left(\left\lfloor \frac{\Delta_i}{n_h - \delta} \right\rfloor \delta - \left\lfloor \frac{\Delta_{i+1}}{n_h - \delta} \right\rfloor \delta \right) \\
=\ & \left\lfloor \frac{\Delta_1}{n_h - \delta} \right\rfloor \delta + \delta + \Delta_1 + 3
\end{aligned} \tag{34}$$

where the last step employs the fact $\frac{n_j}{n_i} = \left\lceil \frac{r_j}{r_i} \right\rceil$ and equality (30) for $0 = j < i \le h-1$. Therefore, the minimum Hamming distance of $\mathcal{C}$ is exactly $\left\lfloor \frac{\Delta_1}{n_h-\delta} \right\rfloor \delta + \delta + \Delta_1 + 3$ and $\mathcal{C}$ is an optimal cyclic h-level H-LRC.

□

Remark 2: One can relax the restriction $[\frac{l}{\gcd(l,\frac{q-1}{n_1})}] \mid \left(\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor n_h+\delta+\langle\Delta_1\rangle_{n_h-\delta}+1\right)$ to $[\frac{l}{\gcd(l,\frac{q-1}{n_1})}] \mid \left(\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor n_h+\delta+\langle\Delta_1\rangle_{n_h-\delta}+1+in_1\right)$ such that $\gcd(l, in_1+1)=1$ for some $0\le i\le l-1$, and substitute $C_{\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor n_h+\delta+\langle\Delta_1\rangle_{n_h-\delta}+1}$ with $C_{\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor n_h+\delta+\langle\Delta_1\rangle_{n_h-\delta}+1+in_1}$. By the Hartmann-Tzeng bound in Lemma 2, an optimal cyclic h-level H-LRC with the same distance $d=\delta_1+2$ can be obtained.

Remark 3: By setting $l|\frac{q-1}{n_1}$, it is well known that $[\frac{l}{\gcd(l,\frac{q-1}{n_1})}] \mid \left(\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor n_h+\delta+\langle\Delta_1\rangle_{n_h-\delta}+\omega\right)$ always holds for $\omega=1,2,\cdots$. One can substitute $C_{\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor n_h+\delta+\langle\Delta_1\rangle_{n_h-\delta}+1}$ with $C_{\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor n_h+\delta+\langle\Delta_1\rangle_{n_h-\delta}+1}$ $\cup C_{\left\lfloor\frac{\Delta_1}{n_h-\delta}\right\rfloor n_h+\delta+\langle\Delta_1\rangle_{n_h-\delta}+2}\cup\cdots$. Then more optimal cyclic h-level H-LRCs with distance $d=\delta_1+\sigma$ $(\sigma>2)$ can be obtained although the length of them is $n|(q-1)$.

Example 3: Let $q=73$, $h=3$, $n_1=72$, $n_2=36$, $n_3=9$, $l=4$, and $\delta=2$. For $\Delta_1=7$ and $\Delta_2=1$, we have

$$
\begin{aligned}
D &= C_0\cup C_{12}\cup E^3\cup\left(\bigcup_{x=1,2} E_a^x\cup E_b^x\right)\\
&=\{0\}\cup\{12\}\cup\left(\bigcup_{i=0,1,\cdots,31}\{9i+1,9i+2\}\right)\cup\left(\bigcup_{i=0,1,\cdots,7}\{36i+3\}\right)\\
&\quad\cup\left(\bigcup_{i=0,1,2,3}\{72i+4,72i+5,\cdots,72i+9\}\right).
\end{aligned}
$$

For a cyclic code $\mathcal{C}$ of length $ln_1=288$ with defining set $\{\alpha^i|i\in D\}$, it is obvious that $0,1,\cdots,12$ belong to D. By Lemma 1, the minimum Hamming distance of $\mathcal{C}$ is at least 14. The dimension of $\mathcal{C}$ is $k=ln_1-|D|=288-98=190$. By Lemmas 1,3 and Definition 3, it can be verified that $\mathcal{C}$ is an h-level H-LRC with parameters $((48,12),(27,4),(7,3))$. By bound

(2) we have

$$
\begin{aligned}
d \leq\ & 288-190+1-\left(\left\lceil\frac{190}{48}\right\rceil-1\right)(12-4)-\left(\left\lceil\frac{190}{27}\right\rceil-1\right)(4-3) \\
& -\left(\left\lceil\frac{190}{7}\right\rceil-1\right)(3-1) \\
=\ & 14.
\end{aligned}
$$

Hence, $\mathcal{C}$ is an optimal cyclic $[288,190,14]_{73}$ h-level H-LRC with parameters $((48,12),(27,4),(7,3))$.

VI. Construction of Optimal Cyclic h-Level H-LRCs With $d=\delta_1+\delta_h$

Finally, we present a construction of optimal cyclic h-level H-LRCs which have minimum Hamming distance $d=\delta_1+\delta_h$.

For an integer $h\geq 1$ and a prime power q, let $n_h<n_{h-1}<\cdots<n_1$ be integers such that $n_1|(q-1)$, and $n_{x+1}|n_x$, $x=1,2,\cdots,h-1$. For an integer l with $\gcd(l,q)=1$, let C_i be q-cyclotomic coset of i mod n where $n=ln_1$ and $0\leq i\leq n-1$. Define

$$
F^h=\bigcup_{i=0,1,\cdots,\frac{ln_1}{n_h}-1} C_{in_h+2}\cup C_{in_h+3}\cup\cdots\cup C_{in_h+n_h-1}. \tag{35}
$$

For some integers $0<\Delta_{h-1}<\Delta_{h-2}<\cdots<\Delta_1<\frac{2n_{h-1}}{n_h}-1$ such that $2\nmid\Delta_1$, define

$$
F_a^x=\bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1}\ \bigcup_{j=0,1,\cdots,\left\lfloor\frac{\Delta_x}{2}\right\rfloor-1} C_{in_x+jn_h+n_h}\cup C_{in_x+jn_h+n_h+1}, \tag{36}
$$

$$
F_b^x=\begin{cases}
\varnothing, & \text{if } \langle\Delta_x\rangle_2=0,\\
\bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1} C_{in_x+\left\lfloor\frac{\Delta_x}{2}\right\rfloor n_h+n_h}, & \text{if } \langle\Delta_x\rangle_2=1 \text{ and } x\neq 1,\\
\bigcup_{i=0,1,\cdots,l-1} C_{in_1+1}, & \text{otherwise,}
\end{cases} \tag{37}
$$

where $x=1,2,\cdots,h-1$.

Let $\mathcal{C}$ be an $[n,k,d]_q$ cyclic code with defining set $\{\alpha^i | i \in D\}$ where

$$D = C_0 \cup F^h \cup \left(\bigcup_{x=1,2,\cdots,h-1} F_a^x \cup F_b^x \right). \tag{38}$$

Then the following theorem is obtained.

Theorem 6: Define $\Delta_0 = \Delta_1 + 1$, $\Delta_h = 0$, $\Delta_{h+1} = 2 - n_h$, and $n_0 = ln_1$. $\mathcal{C}$ is an optimal cyclic $[ln_1, ln_1 - \sum_{i=1}^{h} \frac{ln_1}{n_i}(\Delta_i - \Delta_{i+1}) - 1, \frac{(\Delta_1+3)n_h}{2} - 1]_q$ h-level H-LRC with parameters $((r_1, \delta_1 = (\lfloor \frac{\Delta_1}{2} \rfloor + 1)(n_h - 2) + \Delta_1 + 1), (r_2, \delta_2 = (\lfloor \frac{\Delta_2}{2} \rfloor + 1)(n_h - 2) + \Delta_2 + 1), \cdots, (r_h, \delta_h = (\lfloor \frac{\Delta_h}{2} \rfloor + 1)(n_h - 2) + \Delta_h + 1))$, where $r_j = n_j - \sum_{i=j}^{h} \frac{n_j}{n_i}(\Delta_i - \Delta_{i+1})$ for $0 \le j \le h$, $\frac{n_j}{n_i} = \left\lceil \frac{r_j}{r_i} \right\rceil$ for all $0 \le j < i \le h-1$, and

$$\frac{n_j}{n_h} = \left\lceil \frac{r_j}{r_h} \right\rceil + \sum_{i=j}^{h-1} \left\lceil \frac{r_j}{r_i} \right\rceil \left(\left\lfloor \frac{\Delta_i}{2} \right\rfloor - \left\lfloor \frac{\Delta_{i+1}}{2} \right\rfloor \right) \tag{39}$$

for all $0 \le j \le h-1$.

Proof: Similar to Theorem 2, we have

$$F^h = \bigcup_{i=0,1,\cdots,\frac{ln_1}{n_h}-1} \{in_h + 2, in_h + 3, \cdots, in_h + n_h - 1\}, \tag{40}$$

$$F_a^x = \bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1} \bigcup_{j=0,1,\cdots,\lfloor \frac{\Delta_x}{2} \rfloor - 1} \{in_x + jn_h + n_h, in_x + jn_h + n_h + 1\}, \tag{41}$$

$$F_b^x = \begin{cases} \emptyset, & \text{if } \langle \Delta_x \rangle_2 = 0, \\ \bigcup_{i=0,1,\cdots,\frac{ln_1}{n_x}-1} \{in_x + \lfloor \frac{\Delta_x}{2} \rfloor n_h + n_h\}, & \text{if } \langle \Delta_x \rangle_2 = 1 \text{ and } x \neq 1, \\ \bigcup_{i=0,1,\cdots,l-1} \{in_1 + 1\}, & \text{otherwise,} \end{cases} \tag{42}$$

where $x = 1, 2, \cdots, h-1$. Since $C_0 = \{0\}$, one can check that $\{ln_1 - n_h + 2, ln_1 - n_h + 3, \cdots, ln_1 - 1, 0, 1, \cdots, \lfloor \frac{\Delta_1}{2} \rfloor n_h + n_h - 1\} \subseteq D$. By Lemma 1, the minimum Hamming distance of $\mathcal{C}$ is at least $\lfloor \frac{\Delta_1}{2} \rfloor n_h + 2n_h - 1 = \frac{(\Delta_1+3)n_h}{2} - 1$. The dimension of $\mathcal{C}$ is $k = n - |D| =$

$ln_1 - \sum_{i=1}^{h} \frac{ln_1}{n_i}(\Delta_i - \Delta_{i+1}) - 1$.

Similarly, one can prove that $\mathcal{C}$ is an h-level H-LRC with parameters $((r_1,\delta_1),(r_2,\delta_2),\cdots,(r_h,\delta_h))$. For the $[ln_1, ln_1 - \sum_{i=1}^{h} \frac{ln_1}{n_i}(\Delta_i - \Delta_{i+1}) - 1, d]_q$ h-level H-LRC with parameters $((r_1,\delta_1),(r_2,\delta_2),\cdots,$ $(r_h,\delta_h))$, by bound (2) we have

$$\begin{aligned} d \;\le\; & ln_1 - \left[ln_1 - \sum_{i=1}^{h} \frac{ln_1}{n_i}(\Delta_i - \Delta_{i+1}) - 1\right] + 1 - \left(\left\lceil \frac{r_0}{r_h} \right\rceil - 1\right)(n_h - 2) \\ & - \sum_{i=1}^{h-1} \left(\left\lceil \frac{r_0}{r_i} \right\rceil - 1\right)\left[\left(\left\lfloor \frac{\Delta_i}{2} \right\rfloor - \left\lfloor \frac{\Delta_{i+1}}{2} \right\rfloor\right)(n_h - 2) + (\Delta_i - \Delta_{i+1})\right] \\ = \; & \sum_{i=1}^{h} \frac{ln_1}{n_i}(\Delta_i - \Delta_{i+1}) + 2 - \left(\left\lceil \frac{r_0}{r_h} \right\rceil - 1\right)(n_h - 2) + \left\lfloor \frac{\Delta_1}{2} \right\rfloor (n_h - 2) + \Delta_1 \\ & - \sum_{i=1}^{h-1} \left\lceil \frac{r_0}{r_i} \right\rceil \left[\left(\left\lfloor \frac{\Delta_i}{2} \right\rfloor - \left\lfloor \frac{\Delta_{i+1}}{2} \right\rfloor\right)(n_h - 2) + (\Delta_i - \Delta_{i+1})\right]. \end{aligned} \tag{43}$$

It should be noted that $2 \mid \Delta_0$, which implies

$$\left\lfloor \frac{\Delta_0}{2} \right\rfloor - \left\lfloor \frac{\Delta_1}{2} \right\rfloor = \frac{\Delta_0}{2} - \left\lfloor \frac{\Delta_0 - 1}{2} \right\rfloor = 1. \tag{44}$$

Thus

$$\begin{aligned} d \;\le\; & \sum_{i=1}^{h-1} \left(\frac{ln_1}{n_i} - \left\lceil \frac{r_0}{r_i} \right\rceil\right)(\Delta_i - \Delta_{i+1}) + \frac{ln_1}{n_h}(n_h - 2) + 2 \\ & - \left\lceil \frac{r_0}{r_h} \right\rceil (n_h - 2) + n_h - 2 + \left\lfloor \frac{\Delta_1}{2} \right\rfloor (n_h - 2) + \Delta_1 \\ & - \sum_{i=0}^{h-1} \left\lceil \frac{r_0}{r_i} \right\rceil \left(\left\lfloor \frac{\Delta_i}{2} \right\rfloor - \left\lfloor \frac{\Delta_{i+1}}{2} \right\rfloor\right)(n_h - 2) + n_h - 2 \\ = \; & \left\lfloor \frac{\Delta_1}{2} \right\rfloor (n_h - 2) + \Delta_1 + 2n_h - 2 \\ = \; & \frac{(\Delta_1 + 3)n_h}{2} - 1, \end{aligned} \tag{45}$$

where the first equality employs the fact $\frac{n_j}{n_i} = \left\lceil \frac{r_j}{r_i} \right\rceil$ and equality (39) for $0 = j < i \le h - 1$.

Hence, the minimum Hamming distance of $\mathcal{C}$ is exactly $\frac{(\Delta_1+3)n_h}{2}-1$ and $\mathcal{C}$ is an optimal cyclic h-level H-LRC.

$\square$

Remark 4: Note that the minimum Hamming distance of h-level H-LRCs in this section is $d=\delta_1+\delta_h$ which is different from those of the other classes of h-level H-LRCs in this paper.

Example 4: Let $q=25$, $h=3$, $n_1=24$, $n_2=12$, $n_3=4$, and $l=2$. For $\Delta_1=3$ and $\Delta_2=2$, one can obtain

$$\begin{aligned} D &= C_0 \cup F^3 \cup \left(\bigcup_{x=1,2} F_a^x \cup F_b^x\right) \\ &= \{0\} \cup \left(\bigcup_{i=0,1,\cdots,11} \{4i+2, 4i+3\}\right) \cup \left(\bigcup_{i=0,1,2,3} \{12i+4, 12i+5\}\right) \\ &\quad \cup \left(\bigcup_{i=0,1} \{24i+1\}\right). \end{aligned}$$

For a cyclic code $\mathcal{C}$ of length $ln_1=48$ with defining set $\{\alpha^i | i \in D\}$, it can easily be seen that $46,47,0,1,2,3,4,5,6,7$ belong to D. By Lemma 1, the minimum Hamming distance of $\mathcal{C}$ is at least 11. The dimension of $\mathcal{C}$ is $k=ln_1-|D|=48-35=13$. By Lemmas 1,3 and Definition 3, it can be verified that $\mathcal{C}$ is an h-level H-LRC with parameters $((7,8),(4,7),(2,3))$. By bound (2) we have

$$\begin{aligned} d &\leq 48-13+1-\left(\left\lceil\frac{13}{7}\right\rceil-1\right)(8-7)-\left(\left\lceil\frac{13}{4}\right\rceil-1\right)(7-3) \\ &\quad -\left(\left\lceil\frac{13}{2}\right\rceil-1\right)(3-1) \\ &= 11. \end{aligned}$$

Therefore, $\mathcal{C}$ is an optimal cyclic $[48,13,11]_{25}$ h-level H-LRC with parameters $((7,8),(4,7),(2,3))$.

Table 1. The lengths and distances of cyclic h-level H-LRCs in [7] and this paper

Length n	Distance d	Remark
$n\|(q-1)$	$d \geq \delta_1$	[7], Remark 3
$n = q^m - 1$, $m \geq 1$	$d = \delta_1 + 1$	[7]
$n = ln_1$, $\gcd(l, q) = 1$, $n_1\|(q-1)$	$d = \delta_1$	Theorem 3
$n = ln_1$, $\gcd(l, q) = 1$, $n_1\|(q-1)$	$d = \delta_1 + 1$	Theorem 4
$n = ln_1$, $\gcd(l, q) = 1$, $n_1\|(q-1)$, $[\frac{l}{\gcd(l, \frac{q-1}{n_1})}] \| \left(\left\lfloor \frac{\Delta_1}{n_h - \delta} \right\rfloor n_h + \delta + \langle \Delta_1 \rangle_{n_h - \delta} + 1 + in_1 \right)$ and $\gcd(l, in_1 + 1) = 1$ for some $0 \leq i \leq l - 1$	$d = \delta_1 + 2$	Theorem 5
$n = ln_1$, $\gcd(l, q) = 1$, $n_1\|(q-1)$	$d = \delta_1 + \delta_h$	Theorem 6

VII. Conclusions

In this paper, by considering the length and minimum distance the definition of h-level H-LRCs was given. Then four classes of optimal cyclic h-level H-LRCs were constructed, which have minimum Hamming distances $d = \delta_1, \delta_1 + 1, \delta_1 + 2, \delta_1 + \delta_h$ respectively. Compared with cyclic h-level H-LRCs in [7], our cyclic h-level H-LRCs have new and flexible parameters. By setting $l|\frac{q-1}{n_1}$ for the third construction, more optimal cyclic h-level H-LRCs with distance $d = \delta_1 + \sigma$ (σ is not limited to $0, 1, 2, \delta_h$) can be obtained, but the length of them is $n|(q-1)$. It is interesting to construct optimal cyclic h-level H-LRCs with any Hamming distance and length in future work.

Table 1 summarizes the lengths and distances of cyclic h-level H-LRCs in [7] and this paper. It can be seen that the parameters of cyclic h-level H-LRCs in this paper are new and more flexible than those in [7].

References

[1] S. Ballentine, A. Barg, and S. Vlăduţ, "Codes with hierarchical locality from covering maps of curves," *IEEE Trans. Inf. Theory*, vol. 65, no. 10, pp. 6056-6071, Oct. 2019.

[2] A. Barg, I. Tamo, and S. Vlăduţ, “Locally recoverable codes on algebraic curves,” *IEEE Trans. Inf. Theory*, vol. 63, no. 8, pp. 4928-4939, Aug. 2017.

[3] M. Blaum, “Multiple-layer integrated interleaved codes: a class of hierarchical locally recoverable codes,” *IEEE Trans. Inf. Theory*, vol. 68, no. 8, pp. 5098-5111, Aug. 2022.

[4] V.R. Cadambe and A. Mazumdar, “Bounds on the size of locally recoverable codes,” *IEEE Trans. Inf. Theory*, vol. 61, no. 11, pp. 5787-5794, Nov. 2015.

[5] H. Cai, Y. Miao, M. Schwartz, and X. Tang, “On optimal locally repairable codes with super-linear length,” *IEEE Trans. Inf. Theory*, vol. 66, no. 8, pp. 4853-4868, Aug. 2020.

[6] H. Cai and M. Schwartz, “On optimal locally repairable codes and generalized sector-disk codes,” *IEEE Trans. Inf. Theory*, vol. 67, no. 2, pp. 686-704, Feb. 2021.

[7] Z. Chen and A. Barg, “Cyclic and convolutional codes with locality,” *IEEE Trans. Inf. Theory*, vol. 67, no. 2, pp. 755-769, Feb. 2021.

[8] B. Chen, W. Fang, S. Xia, and F. Fu, “Constructions of optimal (r, δ) locally repairable codes via constacyclic codes,” *IEEE Trans. Commun.*, vol. 67, no. 8, pp. 5253-5263, Aug. 2019.

[9] B. Chen, W. Fang, S. Xia, J. Hao, and F. Fu, “Improved bounds and Singleton-optimal constructions of locally repairable codes with minimum distance 5 and 6,” *IEEE Trans. Inf. Theory*, vol. 67, no. 1, pp. 217-231, Jan. 2021.

[10] B. Chen, S. Xia, J. Hao, and F. Fu, “Constructions of optimal cyclic (r, δ) locally repairable codes,” *IEEE Trans. Inf. Theory*, vol. 64, no. 4, pp. 2499-2511, Apr. 2018.

[11] B. Chen, G. Zhang, and W. Li, “New optimal linear codes with hierarchical locality,” *IEEE Trans. Inf. Theory*, vol. 69, no. 3, pp. 1544-1550, Mar. 2023.

[12] H. Chen, J. Weng, W. Luo, and L. Xu, “Long optimal and small-defect LRC codes with unbounded minimum distance,” *IEEE Trans. Inf. Theory*, vol. 67, no. 5, pp. 2786-2792, May 2021.

[13] T. Ernvall, T. Westerbäck, R. Freij-Hollanti, and C. Hollanti, “Constructions and properties of linear locally repairable codes,” *IEEE Trans. Inf. Theory*, vol. 62, no. 3, pp. 1129-1143, Mar. 2016.

[14] W. Fang and F. Fu, “Optimal cyclic (r, δ) locally repairable codes with unbounded length,” *2018 IEEE Information Theory Workshop (ITW)*, Guangzhou, 2018.

[15] P. Gopalan, C. Huang, H. Simitci, and S. Yekhanin, "On the locality of codeword symbols," *IEEE Trans. Inf. Theory*, vol. 58, no. 11, pp. 6925-6934, Nov. 2012.

[16] S. Goparaju and R. Calderbank, "Binary cyclic codes that are locally repairable," *2014 IEEE International Symposium on Information Theory*, Honolulu, 2014.

[17] M. Grezet, R. Freij-Hollanti, T. Westerback, and C. Hollanti, "Alphabet-dependent bounds for linear locally repairable codes based on residual codes," *IEEE Trans. Inf. Theory*, vol. 65, no. 10, pp. 6089-6100, Oct. 2019.

[18] V. Guruswami, C. Xing, and C. Yuan, "How long can optimal locally repairable codes be?" *IEEE Trans. Inf. Theory*, vol. 65, no. 6, pp. 3662-3670, Jun. 2019.

[19] J. Hao, S. Xia, K. Shum, B. Chen, F. Fu, and Y. Yang, "Bounds and constructions of locally repairable codes: parity-check matrix approach," *IEEE Trans. Inf. Theory*, vol. 66, no. 12, pp. 7465-7474, Dec. 2020.

[20] L. Jin, "Explicit construction of optimal locally recoverable codes of distance 5 and 6 via binary constant weight codes," *IEEE Trans. Inf. Theory*, vol. 65, no. 8, pp. 4658-4663, Aug. 2019.

[21] L. Jin, L. Ma, and C. Xing, "Construction of optimal locally repairable codes via automorphism groups of rational function fields," *IEEE Trans. Inf. Theory*, vol. 66, no. 1, pp. 210-221, Jan. 2020.

[22] X. Kong, X. Wang, and G. Ge, "New constructions of optimal locally repairable codes with super-linear length," *IEEE Trans. Inf. Theory*, vol. 67, no. 10, pp. 6491-6506, Oct. 2021.

[23] M. Krishnan, B. Puranik, P. Kumar, I. Tamo, and A. Barg, "Exploiting locality for improved decoding of binary cyclic codes," *IEEE Trans. Commun.*, vol. 66, no. 6, pp. 2346-2358, Jun. 2018.

[24] X. Li, L. Ma, and C. Xing, "Optimal locally repairable codes via elliptic curves," *IEEE Trans. Inf. Theory*, vol. 65, no. 1, pp. 108-117, Jan. 2019.

[25] S. Ling and C. Xing, *Coding Theory: A First Course*. Cambridge, U.K.: Cambridge Univ. Press, 2004.

[26] J. Liu, S. Mesnager, and L. Chen, "New constructions of optimal locally recoverable codes via good polynomials," *IEEE Trans. Inf. Theory*, vol. 64, no. 2, pp. 889-899, Feb. 2018.

[27] X. Liu and Q. Zeng, "New classes of optimal cyclic codes with hierarchical locality," submitted to *IEEE Trans. Inf. Theory*.

[28] G. Luo and X. Cao, "Optimal cyclic codes with hierarchical locality," *IEEE Trans. Commun.*, vol. 68, no. 6, pp. 3302-3310, Jun. 2020.

[29] G. Luo, M.F. Ezerman, and S. Ling, "Three new constructions of optimal locally repairable codes from matrix-product codes," *IEEE Trans. Inf. Theory*, vol. 69, no. 1, pp. 75-85, Jan. 2023.

[30] Y. Luo, C. Xing, and C. Yuan, "Optimal locally repairable codes of distance 3 and 4 via cyclic codes," *IEEE Trans. Inf. Theory*, vol. 65, no. 2, pp. 1048-1053, Feb. 2019.

[31] G. Micheli, "Constructions of locally recoverable codes which are optimal," *IEEE Trans. Inf. Theory*, vol. 66, no. 1, pp. 167-175, Jan. 2020.

[32] N. Prakash, G. Kamath, V. Lalitha, and P. Kumar, "Optimal linear codes with a local-error-correction property," *Proc. IEEE Int. Symp. Inf. Theory*, Cambridge, MA, USA. pp. 2776-2780, 2012.

[33] J. Qian and L. Zhang, "New optimal cyclic locally recoverable codes of length $n = 2(q+1)$," *IEEE Trans. Inf. Theory*, vol. 66, no. 1, pp. 233-239, Jan. 2020.

[34] J. Qiu, W. Fang, and F. Fu, "New lower bounds for the minimum distance of cyclic codes and applications to locally repairable codes," *IEEE Trans. Inf. Theory*, 2024, early access (DOI: 10.1109/TIT.2024.3349560).

[35] J. Qiu, D. Zheng, and F. Fu, "New constructions of optimal cyclic (r, δ) locally repairable codes from their zeros," *IEEE Trans. Inf. Theory*, vol. 67, no. 3, pp. 1596-1608, Mar. 2021.

[36] B. Sasidharan, G.K. Agarwal, and P.V. Kumar, "Codes with hierarchical locality," *2015 IEEE International Symposium on Information Theory (ISIT)*, Hong Kong, 2015.

[37] N. Silberstein, A. S. Rawat, O. O. Koyluoglu, and S. Vishwanath, "Optimal locally repairable codes via rank-metric codes," *2013 IEEE International Symposium on Information Theory (ISIT)*, Istanbul, 2013.

[38] W. Song, S. H. Dau, C. Yuen, and T. J. Li, "Optimal locally repairable linear codes," *IEEE J. Sel. Areas Commun.*, vol. 32, no. 5, pp. 1019-1036, May 2014.

[39] I. Tamo and A. Barg, "A family of optimal locally recoverable codes," *IEEE Trans. Inf. Theory*, vol. 60, no. 8, pp. 4661-4676, Aug. 2014.

[40] I. Tamo, D. Papailiopoulos, and A. Dimakis, "Optimal locally repairable codes and connections to matroid theory," *IEEE Trans. Inf. Theory*, vol. 62, no. 12, pp. 6661-6671, Dec. 2016.

[41] C. Xing and C. Yuan, "Construction of optimal (r, δ)-locally recoverable codes and connection with graph theory," *IEEE Trans. Inf. Theory*, vol. 68, no. 7, pp. 4320-4328, Jul. 2022.

[42] G. Zhang and H. Liu, "Constructions of optimal codes with hierarchical locality," *IEEE Trans. Inf. Theory*,

vol. 66, no. 12, pp. 7333-7340, Dec. 2020.