Langevin Delay of Singular Control Systems-Reachability and Controllability Interpretation

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Abstract

The main prospect of this proposed work is to confer an algebraic theory for analyzing fractional singular systems. A modern class of linear fractional singular delay systems with two orders are proposed. The key notion used in the enlargement is the decomposition form for matrix regular pencils. As crucial issue, a procedure for computation of reachable set, and control input of addressed system is acquainted. The considerations are illustrated by suitable examples.

Introduction

Many social, physical, organic and engineering issues are defined with the aid of fractional differential equations. Literally, fractional differential equations are taken into consideration in the act of an important tool to model nonlinear differential equations. Many researchers studied and developed more important results on fractional differential equations and its real applications.

Singular systems are endow in many fields like social systems, economic systems, network analysis and engineering systems (such as electrical circuit network, power system, aerospace engineering and chemical processing).Singular systems are classified into descriptor systems, generalized steady-state system, differential algebraic systems,etc. Moreover, In the studies of singular systems deals with significant avenues such as geometric and algebraic. In 1989, L.Dai introduced techniques to solve the singular problem via algebraic analysis and also main contribution of this work indicates the mathematical approach of singular system from system point of view for more details see [4].

Controllability play on important role in engineering, physics and control theory. In last few decades, many researchers works on this area [7, 16, 19, 20]. Since the series of works [4, 18]. In particular L.Dai[4] works gives a valuable notes to study the singular dynamical system through mathematical approach. Since the series of the paper [6, 9, 10, 17, 18] authors provides a techniques to solve the fractional differential equations with singular co-efficient. Recently, Jinde Cao [18], considered a singular fractional differential equation with delay in control and also techniques are derived for controllability.

Let us consider the singular neutral fractional linear system with control delay is of the form,

Where $\alpha+\beta>1$, ${}^{C}D^{\alpha}$ is α order caputo derivative , ${}^{C}D^{\beta}$ is β order caputo derivative; $x\in\mathbb{R}^n$ is the state variable; E is a singular real matrix defined on $\mathbb{R}^{n\times n}$, A is a non-singular real matrix defined on $\mathbb{R}^{n\times n}$ and B, H is a real matrix with $n>m$ defined on $\mathbb{R}^{n\times m}$; $\varphi(t)$ is an initial function defined on $[-\tau,0]$ and $\varphi\in\mathbf{C}([-\tau,0],\mathbb{R}^n)$, where $\mathbf{C}([-\tau,0],\mathbb{R}^n)$ denotes set of continuous functions from $[-\tau,0]$ into $\mathbb{R}^n$, $\tau>0$; $u(t)$ is a control function; $J=[0,T]$; p_0, $x_0\in\mathbb{R}^n$ and $g:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n$ be a continuous function.

The key intent of this performance is to express reachability and controllability of described system (1) via algebraic way. According to the results in Yip and Sincovec work [17] and L. Dai [4], we splitting system (1) within two subsystem by the first equivalent form (FE1) so-called slow and fast subsystems. Which is more comfortable to discuss the reachable and controllable for proposed criteria also, this work extended the facts in [4, 18] to the next level.

This manuscript is scheduled as pursues: the following section discussed about few well-known basic results for further discussion. Section 3, describes our proposed model in detail and represented structure of the admissible initial data of observed system. Next section analyzes the structure of the solution of corresponding subsystems and reachability conditions are derived. In section 5, necessary and sufficient conditions for controllability of systems given through theorems and lemma. In section 6, numerical illustrations are provided to get a close glimpse of how the state vector behaves with time varied. Finally, we spent this paper with a conclusion.

Preliminaries:

This section recalls few well-known basic definitions, properties, theorems and lemma for proving required results of following sections. For more information one can see [4],[8],[11] and [13].

Definition 2.1 [13]: The Euler gamma function $\Gamma(z)$ on the half-plane Re(z)> 0 is defined by Euler integral of second kind

$$\Gamma(z)=\int_0^{\infty}t^{z-1}e^{-t}\,dt,\quad Re(z)>0$$

For all $z\in\mathbb{C}$ $(Re(z)>0)$, the above integral is convergent.

Definition 2.2 [13]: The Caputo fractional derivative of order $\alpha\in\mathbb{C}$ with $n-1<\alpha\leq n$, $n\in\mathbb{N}$ for a function f have continuous derivative upto order n such that f^n is absolutely continuous is defined as

$$({}^{C}D^{\alpha}_{0+}f)(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}f^n(s)\,ds$$

Where the function $f^n(s)$ is $\frac{d^nf}{ds^n}$. In particular, if $0<\alpha\leq1$, then the above equation becomes

For our convenience the Caputo fractional derivative ${}^{C}D^{\alpha}_{0+}$ is written as ${}^{C}D^{\alpha}$.

Definition 2.3 [11]: The basic definition of the Laplace transformation of a function $f(t)$ of a real variable $t \in \mathbb{R}^{+}$ given by

Some important properties of Laplace transform:

- $L[I^{\alpha}f(t)] = s^{-\alpha}F(s),\ s \in \mathbb{C}$
- The convolution formula of functions $f(t)$ and $g(t)$ given on $\mathbb{R}^{+}$ is defined for $x \in \mathbb{R}^{+}$ as

$$\int_0^t f(t-s)g(s)\ ds = (f * g)(t)$$

  The Laplace transform of a convolution is given by

- The inverse Laplace transform

$$L^{-1}[F(s)G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

- Laplace transformation of Caputo derivative is given below

$$L[{}^{C}D^{\alpha}f(t)](s) = s^{\alpha}F(s) - \sum_{k=0}^{n-1} f^{k}(0^{+})s^{\alpha-1-k}$$

- Laplace transform of Mittag-Leffler function as,

$$L[E_{\alpha}(\pm\lambda z^{\alpha})] = \frac{s^{\alpha-1}}{s^{\alpha} \pm \lambda}$$

- Inverse Laplace transform for $\Re(s) > \| A \|^{\frac{1}{2}}$

$$L^{-1}\left[\frac{s^{\alpha-\beta}}{s^{\alpha}I - A}\right] = t^{\beta-1}E_{\alpha,\beta}(At^{\alpha}),\quad \Re(s) > \| A \|^{\frac{1}{2}}$$

$$L^{-1}\left[\frac{s^{-\beta}}{s^{\alpha}I-A}\right]=t^{\alpha+\beta-1}E_{\alpha,\beta}(At^{\alpha}),\quad \Re(s)>\| A \|^{\frac{1}{2}}$$

Definition 2.4 [13]: A The Mittag-Leffler function is a complex function which depends on two complex parameters α and β. It can be defined by,

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad \alpha,\beta>0$$

and $z\in\mathbb{C}$, here $\mathbb{C}$-complex plane. The generalized Mittag-Leffler function satisfies

$$\int_0^{\infty} e^{-t}t^{\beta-1}E_{\alpha,\beta}(zt^{\alpha})\ dt=\frac{1}{1-z}\qquad \text{for}\quad |z|<1$$

Some important properties of Mittag-Leffler function:

- For $\beta=1$

$$E_{\alpha,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}$$

- For $\lambda,z\in\mathbb{C}$

$$E_{\alpha}(\lambda z^{\alpha})=\sum_{k=0}^{\infty}\frac{z^{\alpha k}\lambda^k}{\Gamma(\alpha k+1)}$$

- Caputo derivative of Mittag-Leffler functions as,

Definition 2.5 [4]: For any given two matrices $E,A\in\mathbb{R}^{n\times n}$ the pencil (E,A) is called regular if there exist a constant scalar $\alpha\in\mathbb{C}$ such that $|\lambda E+A|\neq 0$ or the polynomial $|sE+A|\not\equiv 0$.

Lemma 2.6 [4] : (E, A) is regular if and only if two non-singular matrices Q, P may be choosen such that

$$QEP = diag(I_{n_1}, M);\ QAP = diag(A_1, I_{n_2})$$

where $n_1 + n_2 = n$, $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $M \in \mathbb{R}^{n_2 \times n_2}$ is nilpotent matrix.

Linear Fractional Singular System:

This section describes our proposed model in particular and also establishes the corresponding subsystems using Weierstrass decomposition of matrix pencil principle based on the proficiencies in [4].

Assume that (E,A) is a regular pencil all over of this work. According to the result in [17], there exists a two non-singular matrix $P, Q \in \mathbb{R}^{n \times n}$, $t \in J$ such that

$$PEQQ^{-1}\ {}^{C}D^{\beta}({}^{C}D^{\alpha}x(t) - g(t, x(t))) \quad = \quad PAQQ^{-1}x(t) + PBu(t) + PHu(t - \tau)$$

satisfying

$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$; PH=

$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$; $Q^{-1}x(t) =$

$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$; $n_1 + n_2 = n$; M is a nilpotent matrix with index $\mu > 0$ (that is a least positive integer μ such that $M^{\mu} = 0$, $M^{r} \neq 0$, $r \in \{1, 2, ..., \mu - 1\}$); $A_1 \in \mathbb{R}^{n_1 \times n_1}$; $B_1, H_1 \in \mathbb{R}^{n_1 \times m_1}$ and $B_2, H_2 \in \mathbb{R}^{n_2 \times m_2}$. Then the equivalent canonical form of system (**??**) can be written as

where $x_{01}, x_{02} \in \mathbb{R}$. Let $h = [(\alpha + \beta)\mu] + 1$, here $[(\alpha + \beta)\mu]$ is integral part of $(\alpha + \beta)\mu$. Where $\mathbb{C}_p^h[[0, \infty), \mathbb{R}^m]$ be the collection of all h times differential piecewise continuous function defined on $[0, \infty)$, $\mathbb{C}^h[[-\tau, 0], \mathbb{R}^m]$ denotes h times continuously differentiable function defined on $[-\tau, 0]$.

The above equation (**??**) is known as first equivalent form(EF1) of the system (**??**), also we call the first and second equation in (**??**) as slow and fast subsystem respectively. Now we obtain the admissible initial state set $I(\varphi)$ for observed system (**??**) with $\varphi \in \mathbb{C}^h([-\tau, 0], \mathbb{R}^m)$ as follows,

$$I(\varphi) \quad := \quad \Big\{ x \in \mathbb{R}^n \mid x =$$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathrm{x}_1 \in \mathbb{R}^{n_1}$,

$\mathrm{x}_2 = \sum_{n=0}^{\mu-1}[B_2(\varphi(0) + M^n\ {}^C D^{n(\alpha+\beta)}\varphi(0)) + H_2(\varphi(-\tau) + M^n\ {}^C D^{n(\alpha+\beta)}\varphi(-\tau))]\Big\}$
(1)

Thus, the admissible initial set is denoted as

Remark 3.1 : To construct the admissible initial set by using the results in [17], In the case of slow subsystem every vector in vector space is an initial condition but its not true for descriptor system. Then the initial conditions for described system is $x_1(0) = x_{01}$ and $x_2(0) = \sum_{n=0}^{\mu-1}[B_2(\varphi(0) + M^n\ {}^C D^{n(\alpha+\beta)}\varphi(0)) + H_2(\varphi(-\tau) + M^n\ {}^C D^{n(\alpha+\beta)}\varphi(-\tau))]$. Hence by [17] the result in (1) is true.

State vector expression and state Rechability conditions:

In this section, we discussed about the state vector of the systems in (**??**) and the Rechability conditions are given by the following theorems.

Theorem 4.1: For any initial condition $(x_0, \varphi) \in \mathfrak{B}$ and the control function $u(t) \in \mathbb{C}_p^h([0,\infty), \mathbb{R}^m)$, the exact form of state vector for system (**??**) can be written us,

$$\begin{aligned} x_1(t) &= E_{\alpha+\beta,1}(A_1 t^{\alpha+\beta})x_{01} - t^{\alpha}E_{\alpha+\beta,1+\alpha}(A_1 t^{\alpha+\beta})g(0,x(0)) + \int_0^t (t-s)^{\alpha-1}E_{\alpha+\beta,\alpha}(A_1(t-s)^{\alpha+\beta}) \\ &\quad \times g_1(s,x(s))\ ds + \int_0^t E_{\alpha,\alpha}(A_1(t-s)^{\alpha+\beta})(B_1u(t) + H_1u(t-\tau)) \\ &\quad + \int_{-h}^0 E_{\alpha,\alpha}(A_1(t-h-s)^{\alpha+\beta})H_1\varphi(s)\ ds \qquad (2) \\ x_2(t) &= \sum_{k=1}^{n_2-1} M^k\{P_k(t)x_{02} + Q_k(t) + R_k(t)g(0,x(0)) + S_k(t)[B_2\varphi(0) + H_2\varphi(-\tau)\} \\ &\quad + \sum_{k=1}^{n_2-1} M^k[B_2\ {}^C D^{k(\alpha+\beta)}u(t) + H_2\ {}^C D^{k(\alpha+\beta)}u(t-\tau)] \end{aligned}$$

(3)

where $E_{\alpha,\beta}(.)$ is a Mittag-Leffler function. ${}^C D^{k(\alpha+\beta)}u(.)$ is the sequential fractional derivative.

Proof. Let us take first equation from (**??**) as,

Now employing Laplace transform of above equation (**??**), we have

$$\begin{aligned} X_1(s)(s^{\alpha+\beta} - A_1) &= s^{\alpha+\beta-1}x_{01} - s^{\beta}\mathcal{L}[g_1(t,x(t))] - s^{\beta-1}g(0,x(0)) + B_1U(s) + H_1\mathcal{L}[u(t-\tau)] \\ X_1(s) &= \frac{s^{\alpha+\beta-1}}{(s^{\alpha+\beta} - A_1)}x_{01} - \frac{s^{\beta}}{(s^{\alpha+\beta} - A_1)}g(0,x(0)) - \frac{s^{\beta-1}}{(s^{\alpha+\beta} - A_1)}\mathcal{L}[g_1(t,x(t))] \\ &\quad + \frac{1}{(s^{\alpha+\beta} - A_1)}\{B_1U(s) + H_1\mathcal{L}[u(t-\tau)]\} \end{aligned}$$

Taking inverse Laplace transform of both sides of above equation, we get

$$\begin{aligned} x_1(t) &= \mathcal{L}^{-1}\Big[\frac{s^{\alpha+\beta-1}}{(s^{\alpha+\beta} - A_1)}\Big]x_{01} - \mathcal{L}^{-1}\Big[\frac{s^{\beta}}{(s^{\alpha+\beta} - A_1)}\Big] * \mathcal{L}^{-1}[\mathcal{L}[g_1(t,x(t))]] - \mathcal{L}^{-1}\Big[\frac{s^{\beta-1}}{(s^{\alpha+\beta} - A_1)}\Big] \\ &\quad \times g(0,x(0)) + B_1\mathcal{L}^{-1}[(s^{\alpha+\beta} - A_1)^{-1}] * \mathcal{L}^{-1}[U(s)] \\ &\quad + H_1\mathcal{L}^{-1}[(s^{\alpha+\beta} - A_1)^{-1}] * \mathcal{L}^{-1}[\mathcal{L}[u(t-\tau)]] \end{aligned}$$

(4)

From definitions and properties in preliminaries of section 2, we define the following

$$\begin{aligned} \mathcal{L}^{-1}\Big[\frac{s^{\alpha+\beta-1}}{(s^{\alpha+\beta} - A_1)}\Big] &= E_{\alpha+\beta,1}(A_1t^{\alpha+\beta}) \\ \mathcal{L}^{-1}\Big[\frac{s^{\beta}}{(s^{\alpha+\beta} - A_1)}\Big] &= t^{\alpha-1}E_{\alpha+\beta,\alpha}(A_1(t-s)^{\alpha+\beta}) \\ \mathcal{L}^{-1}\Big[\frac{s^{\beta-1}}{(s^{\alpha+\beta} - A_1)}\Big] &= t^{\alpha}E_{\alpha+\beta,1+\alpha}(A_1t^{\alpha+\beta}) \\ L^{-1}[(s^{\alpha+\beta} - A_1)^{-1}] &= E_{\alpha,\alpha}(A_1(t-h-s)^{\alpha+\beta}) \end{aligned}$$

By applying the above results in (4) we have,

$$\begin{aligned} x_1(t) &= E_{\alpha+\beta,1}(A_1t^{\alpha+\beta})x_{01} - t^{\alpha}E_{\alpha+\beta,1+\alpha}(A_1t^{\alpha+\beta})g(0,x(0)) + \int_0^t (t-s)^{\alpha-1} \\ &\quad \times E_{\alpha+\beta,\alpha}(A_1(t-s)^{\alpha+\beta})g_1(s,x(s))\ ds + \int_0^t E_{\alpha,\alpha}(A_1(t-s)^{\alpha+\beta})(B_1u(t)) \\ &\quad + H_1u(t-\tau))\ ds + \int_{-h}^0 E_{\alpha,\alpha}(A_1(t-h-s)^{\alpha+\beta})H_1\varphi(s)\ ds \end{aligned}$$

(5)

Moreover, let us consider the second equation in (??) and apply Laplace transform yields,

$$
\begin{aligned}
(Ms^{\alpha+\beta}-I)X_2(s) &= Ms^{\alpha+\beta-1}x_{02}+Ms^{\beta}\mathcal{L}[g_2(t,x(t))]-Ms^{\beta-1}g_2(0,x(0)) \\
&\quad +B_2U(s)+H_2\mathcal{L}[u(t-\tau)] \\
X_2(s) &= Ms^{\alpha+\beta-1}(Ms^{\alpha+\beta}-I)^{-1}x_{02}+Ms^{\beta}(Ms^{\alpha+\beta}-I)^{-1}\mathcal{L}[g_2(t,x(t))] \\
&\quad \times g_2(0,x(0))-Ms^{\beta-1}(Ms^{\alpha+\beta}-I)^{-1}+(Ms^{\alpha+\beta}-I)^{-1}B_2U(s) \\
&\quad +(Ms^{\alpha+\beta}-I)^{-1}H_2\mathcal{L}[u(t-\tau)]
\end{aligned}
$$

(6)

Employing inverse Laplace transform of above equation, we get

$$
\begin{aligned}
x_2(t) &= \mathcal{L}^{-1}\left[Ms^{\alpha+\beta-1}(Ms^{\alpha+\beta}-I)^{-1}\right]x_{02}+\mathcal{L}^{-1}\left[Ms^{\beta}(Ms^{\alpha+\beta}-I)^{-1}\right]*\mathcal{L}^{-1}[\mathcal{L}[g_2(t,x(t))]] \\
&\quad -\mathcal{L}^{-1}\left[Ms^{\beta-1}(Ms^{\alpha+\beta}-I)^{-1}\right]g_2(0,x(0))+\mathcal{L}^{-1}\left[(Ms^{\alpha+\beta}-I)^{-1}\right]*B_2\mathcal{L}^{-1}[U(s)] \\
&\quad +\mathcal{L}^{-1}\left[(Ms^{\alpha+\beta}-I)^{-1}\right]H_2*\mathcal{L}^{-1}[\mathcal{L}[u(t-\tau)]]
\end{aligned}
$$

(7)

For our convenience we consider the following notations

$$
\begin{aligned}
\mathcal{L}^{-1}\left[Ms^{\alpha+\beta-1}(Ms^{\alpha+\beta}-I)^{-1}\right]x_{02} &= \sum_{k=1}^{n_2-1} M^k P_k(t)x_{02} \\
\mathcal{L}^{-1}\left[Ms^{\beta}(Ms^{\alpha+\beta}-I)^{-1}\mathcal{L}[g_2(t,x(t))]\right] &= \sum_{k=1}^{n_2-1} M^k Q_k(t) \\
\mathcal{L}^{-1}\left[Ms^{\beta-1}(Ms^{\alpha+\beta}-I)^{-1}\right]g_2(0,x(0)) &= \sum_{k=1}^{n_2-1} M^k R_k(t)g_2(0,x(0))
\end{aligned}
$$

where

$$
\begin{aligned}
P_k(t) &= \mathcal{L}^{-1}\left[s^{(\alpha+\beta)n-1};\ t\right] \\
Q_k(t) &= \mathcal{L}^{-1}\left[s^{n(\alpha+\beta)-\alpha}\mathcal{L}(g_2(t,x(t)));\ t\right] \\
R_k(t) &= \mathcal{L}^{-1}\left[s^{n(\alpha+\beta)-\alpha-1};\ t\right]
\end{aligned}
$$

The following facts are used to prove the theorem:
Fact 1:

$$
\begin{aligned}
B_2(Ms^{\alpha+\beta}-I)^{-1}U(s)) &= B_2(Ms^{\alpha+\beta}-I)^{-1}\int_0^\infty e^{-st}u(t)\ dt \\
&= B_2\sum_{k=0}^{n_2-1}M^k s^{k(\alpha+\beta)}[\frac{u(0)}{s}+1/s\mathcal{L}(u'(t))] \\
&= B_2\Big\{\sum_{k=0}^{n_2-1}M^k s^{k(\alpha+\beta)-1}u(0)+\sum_{k=0}^{n_2-1}M^k s^{k(\alpha+\beta)-1}\mathcal{L}u'(t)\Big\}
\end{aligned}
$$

Apply inverse Laplace transform, we get

$$
\begin{aligned}
\mathcal{L}^{-1}\big[B_2(Ms^{\alpha+\beta}-I)^{-1}U(s))\big] &= B_2\Big\{\sum_{k=0}^{n_2-1}M^k\frac{t^{-k(\alpha+\beta)}}{\Gamma(1-k(\alpha+\beta))}u(0) \\
&\quad +\sum_{k=0}^{n_2-1}M^k\int_0^\infty\frac{t^{-k(\alpha+\beta)}}{\Gamma(1-k(\alpha+\beta))}u'(s)\ ds\Big\} \\
&= B_2\sum_{k=0}^{n_2-1}M^k\frac{t^{-k(\alpha+\beta)}}{\Gamma(1-k(\alpha+\beta))}u(0)+\sum_{k=0}^{n_2-1}M^k\ {}^CD^{k(\alpha+\beta)}u(t) \\
&= B_2\sum_{k=0}^{n_2-1}M^k S_k(t)u(0)+\sum_{k=0}^{n_2-1}M^k\ {}^CD^{k(\alpha+\beta)}u(t) \\
\text{where}\qquad S_k(t) &= \frac{t^{-k(\alpha+\beta)}}{\Gamma(1-k(\alpha+\beta))}
\end{aligned}
$$

Fact 2:

$$
\begin{aligned}
H_2(Ms^{\alpha+\beta}-I)^{-1}\mathcal{L}(u(t-\tau)) &= H_2(Ms^{\alpha+\beta}-I)^{-1}\int_0^\infty e^{-st}u(t-\tau)\ dt \\
&= H_2(Ms^{\alpha+\beta}-I)^{-1}\Big[e^{-s\tau}\int_0^\infty e^{-sh}u(h)\ dh+\int_{-\tau}^0 e^{-sh}\varphi(h)\ dh\Big] \\
&= H_2(Ms^{\alpha+\beta}-I)^{-1}\Big[\frac{\varphi(-\tau)}{s}+\int_0^\infty e^{-st}u'(t-\tau)\ dt\Big]
\end{aligned}
$$

Apply inverse Laplace transform, we get

$$\begin{aligned}
\mathcal{L}^{-1}\big[H_2(Ms^{\alpha+\beta}-I)^{-1}\mathcal{L}(u(t-\tau))\big] &= H_2\Big\{\sum_{k=0}^{n_2-1} M^k \int_0^\infty \frac{t^{-k(\alpha+\beta)}}{\Gamma(1-k(\alpha+\beta))} u'(s-\tau)\ ds \\
&\quad + \sum_{k=0}^{n_2-1} M^k \frac{t^{-k(\alpha+\beta)}}{\Gamma(1-k(\alpha+\beta))}\varphi(-\tau)\Big\}
\end{aligned}$$

$$\begin{aligned}
&= H_2 \sum_{k=0}^{n_2-1} M^k \frac{t^{-k(\alpha+\beta)}}{\Gamma(1-k(\alpha+\beta))}\varphi(-\tau) \\
&\quad + \sum_{k=0}^{n_2-1} M^k\ {}^C D^{k(\alpha+\beta)} u(t-\tau) \\
&= H_2 \sum_{k=0}^{n_2-1} M^k T_k(t)\varphi(-\tau) + \sum_{k=0}^{n_2-1} M^k\ {}^C D^{k(\alpha+\beta)} u(t-\tau) \\
\textit{where} \qquad S_k(t) &= \frac{t^{-k(\alpha+\beta)}}{\Gamma(1-k(\alpha+\beta))}
\end{aligned}$$

Using the above results and facts equation (7) yields,

$$\begin{aligned}
x_2(t) &= \sum_{k=1}^{n_2-1} M^k\ {}^C D^{k(\alpha+\beta)} u(t) + \sum_{k=1}^{n_2-1} M^k\ {}^C D^{k(\alpha+\beta)} u(t-\tau) + \sum_{k=1}^{n_2-1} M^k \\
&\quad \times\{P_k(t)x_{02} + Q_k(t) + R_k(t)g(0,x(0)) + S_k(t)[B_2\varphi(0) + H_2\varphi(-\tau))\}
\end{aligned}$$

Thus the equations (2) and (3) holds. Hence the theorem.

Remark 4.2 : The existence and uniqueness of system (2) is drawn from the result in [17]. Whenever (A,E) is regular matrix then the system corresponding to (A,E) is solvable. So, by this result system (2) has a unique solution.

Given matrices $A_1 \in \mathbb{R}^{n_1\times n_1}$, B_1, $H_1 \in \mathbb{R}^{n_1\times m_1}$. For any polynomial $f(s) \neq 0$, we define $W(f,t) : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ such that

$$\begin{aligned}
W(f,t) &= \int_0^{t-\tau} f(s)\{E_{\alpha,\alpha}[A_1(t-s)^{\alpha+\beta}]B_1B_1^* E_{\alpha,\alpha}[A_1^*(t-s)^{\alpha+\beta}] + E_{\alpha,\alpha}[A_1(t-\tau-s)^{\alpha+\beta}]H_1H_1^* \\
&\quad \times E_{\alpha,\alpha}[A_1^*(t-\tau-s)^{\alpha+\beta}]\}f(s)\ ds + \int_{t-\tau}^{t} f(s)E_{\alpha,\alpha}[A_1(t-s)^{\alpha+\beta}]B_1B_1^* \\
&\quad \times E_{\alpha,\alpha}[A_1^*(t-s)^{\alpha+\beta}]f(s)\ ds
\end{aligned}$$

(8)

where * denotes the matrix transpose.
Some notations are introduced for solving the following lemma as, the range of f denote Im(f), that is

$$Im(f) = \{y \mid f(x) = y,\ \forall x \in \mathbb{R}^n\}$$

Here $\langle A|B,H\rangle$ denote as,

$$\langle A|B,H\rangle \quad = \quad Im(B) + Im(AB) + ... + Im(A^{n-1}B) + Im(H) + Im(AH) + ... + Im(A^{n-1}H)$$

Then the space $\langle A|B,H\rangle$ is spanned by the column vectors

$$[B, AB, ...A^{n-1}B, H, AH, ...A^{n-1}H]$$

Lemma 4.3 Given matrices $A_1 \in \mathbb{R}^{n_1\times n_1}$, B_1, $H_1 \in \mathbb{R}^{n_1\times m_1}$, then

Proof. The equation (**??**) holds if and only if

$$kerW(f,t) = \bigcap_{i=0}^{n_1-1} kerB_1^*(A_1^*)^i \bigcap_{i=0}^{n_1-1} kerH_1^*(A_1^*)^i$$

suppose take $x \in kerW(f,t)$ then

$$x^*W(f,t)x = 0$$

(9)

Insert (8) in (**??**), we get

$$\int_0^{t-\tau} x^*f(s)\{E_{\alpha,\alpha}(A_1(t-s)^{\alpha+\beta})B_1B_1^*E_{\alpha,\alpha}(A_1^*(t-s)^{\alpha+\beta}) + E_{\alpha,\alpha}(A_1(t-\tau-s)^{\alpha+\beta})H_1H_1^*$$
$$\times E_{\alpha,\alpha}(A_1^*(t-\tau-s)^{\alpha+\beta})\}f(s)x\ ds \int_{t-\tau}^{t} x^*f(s)E_{\alpha,\alpha}[A_1(t-s)^{\alpha+\beta}]B_1B_1^*$$
$$\times E_{\alpha,\alpha}[A_1^*(t-s)^{\alpha+\beta}]f(s)x\ ds = 0$$

(10)

Put $\|x\|_2 = (x^*x)^{1/2}$ in equation (10), then

$$\int_0^{t-\tau} \left\| \left\{ B_1^* E_{\alpha,\alpha}[A_1^*(t-s)^{\alpha+\beta}] + H_1^* E_{\alpha,\alpha}[A_1^*(t-\tau-s)^{\alpha+\beta}] \right\} f(s)x \right\|_2^2 + \int_{t-\tau}^{t} \left\| B_1^* E_{\alpha,\alpha}(A_1^*(t-s)^{\alpha+\beta}) f(s)x \right\|_2^2 = 0$$

(11)

since $\|x\|_2 = 0 \Leftrightarrow x = 0$. From the above equation (11) we have the following results

since the non-zero polynomial $f(s)$ has finite number of zeroes on $s \in [0, t-\tau]$ and $s \in [t-\tau, t]$, then for $t-\tau \leq s \leq t$ equation (**??**) becomes

For $s \in [0, t-\tau]$ equation (**??**) becomes

For $0 \leq s \leq t-\tau$, taking caputo derivative simultaneously along with (**??**) as follows,

stick $t = s$ on equation (**??**) yields

$$B_1^*(A_1^*)^i x = 0, i = 1, 2, ...n_1 - 1$$

Therefore, which implies that $x \in ker B_1^*(A_1^*)^i, i = 1, 2, ...n_1 - 1$
From the Cayley-Hamilton theorem

For $s \in [0, t-\tau]$ insert (**??**) in (**??**) we get,

For $s \in [0, t-\tau]$ put (**??**) into (**??**) we get,

Continuously taking the Caputo fractional derivative on equation (**??**) and stick $s = t - \tau$

which implies that $x \in ker H_1^*(A_1^*)^i, i = 1, 2, ...n_1 - 1$. Hence,

On the other hand, let us take $x \in \bigcap_{i=0}^{n_1-1} ker B_1^*(A_1^*)^i \bigcap_{i=0}^{n_1-1} ker H_1^*(A_1^*)^i$ with $x \neq 0$. Thus, $x \in ker B_1^*(A_1^*)^i$ and $x \in ker H_1^*(A_1^*)^i$, for $i = 1, 2, ...n_1 - 1$, that is

For $s \in [t-\tau, t]$ from equation (**??**), we get

$$\sum_{l=0}^{n_1-1} B_1^* \eta_l(t-s)(A_1^*)^l x = 0, \qquad i = 1, 2, ...n_1 - 1$$

For $s \in [0, t-\tau]$, apply the same argument in equation (**??**) yields

Hence $x \in KerW(f, t)$.
Therefore, we get

From the results in the equations (??) and (??) we conclude that

Hence the theorem.

Definition 4.4 : Any vector $v \in \mathbb{R}^n$ in n-dimensional vector space is said to be reachable, there exists an admissible initial data $(x_0, \varphi, g(0, x(0))) \in \mathscr{B}$, admissible control input $u(t) \in C_p^h([0, \infty), \mathbb{R}^m)$ and $t_w > 0$ such that the solution of system (??) or (??) satisfies $x(t_w, x_0, \varphi, g(0, x(0))) = v$

Let $\mathfrak{R}(x_0, \varphi, g(0, x(0)))$ be the reachable set from any initial data $(x_0, \varphi,$ $g(0, x(0))) \in \mathscr{B}$, then we have

Fixing the reachable set from the initial state $x_0 = 0$, $g(0, x(0)) = 0$ and $\varphi \equiv 0, t \in [-\tau, 0]$, we derive the following theorem for reachable set.

Theorem 4.5 : For the system (??) the reachable set $\mathfrak{R}(0, 0, 0)$ with zero initial state ($x_0 = 0$, $g(0, x(0)) = 0$ and $\varphi \equiv 0$) can be represented as

$$\mathfrak{R}(0, 0, 0) = \langle A_1 | B_1, H_1 \rangle \oplus \langle M | B_2, H_2 \rangle$$

where $\oplus$- direct sum of vector space.

Proof. Let $v =$ $\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \in \mathfrak{R}(0, 0, 0)$. From equations (5), (8) and (??) there exists $t_w > 0$ and $u(t) \in C_p^h([0, \infty), \mathbb{R}^m)$ such that

By using the Cayley-Hamilton theorem for above equation yields which implies $v_1 \in \langle A_1 \mid B_1, H_1\rangle$, similarly from (8) and (**??**), we have

$$v_2 \quad = \quad \sum_{k=0}^{n_2-1} M^k[B_2\ {}^C D^{k(\alpha+\beta)}u(t_w) + H_2{}^C D^{k(\alpha+\beta)}u(t_w-\tau)]$$

which implies $v_2 \in \langle M \mid B_2, H_2\rangle$. Hence,

$$\mathfrak{R}(0,0,0) \subseteq \langle A_1|B_1,H_1\rangle \oplus \langle M|B_2,H_2\rangle$$

Conversly, suppose that $v =$

$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \in \langle A_1|B_1,H_1\rangle \oplus \langle M|B_2,H_2\rangle$. Here $v_1 \in \langle A_1 \mid B_1, H_1\rangle$, $v_2 \in \langle M \mid B_2, H_2\rangle$ with $v_1 \neq 0, v_2 \neq 0$. For initial condition $x_1(0) = 0$; $\varphi(t) \equiv 0$ for all $t \in [-\tau, 0]$; $g_1(t, x(t)) \equiv 0$ for all $t \in [0,t]$ and take $u(s) = u_1(s) + u_2(s)$, then from equations (5) and (8) as,

$$\begin{aligned} x_1(t) &= \int_0^{t-\tau} E_{\alpha,\alpha}(A_1(t-s)^{\alpha+\beta})B_1[u_1(s)+u_2(s)]\ ds + \int_0^{t-\tau} E_{\alpha,\alpha}(A_1(t-\tau-s)^{\alpha+\beta}) \\ &\quad \times H_1[u_1(s)+u_2(s)]\ ds + \int_{t-\tau}^{t} E_{\alpha,\alpha}(A_1(t-s)^{\alpha+\beta})B_1[u_1(s)+u_2(s)]\ ds \qquad (12) \\ x_2(t) &= \sum_{k=0}^{n_2-1} M^k\{B_2{}^C D^{k(\alpha+\beta)}[u_1(s)+u_2(s)] + H_2{}^C D^{k(\alpha+\beta)}[u_1(s-\tau)+u_2(s-\tau)]\} \end{aligned}$$

(13)

By using method in [4], we choose $u_1 = f(s)y(s)$ to satisfy

$$y(s) = \begin{cases} f(s)\big[B_1^* E_{\alpha,\alpha}(A_1^*(t-s)^{\alpha+\beta})\big]z; & t-\tau \leq s \leq t \\ f(s)[B_1^* E_{\alpha,\alpha}\big[A_1^*(t-s)^{\alpha+\beta}\big] + H_1^* E_{\alpha,\alpha}\big[A_1^*(t-\tau-s)^{\alpha+\beta}\big]z; & 0 \leq s \leq t-\tau \end{cases}$$

$$\begin{aligned} &\int_0^{t_w} E_{\alpha,\alpha}(A_1(t_w-s)^{\alpha+\beta})B_1 f(s)g(s)\ ds + \int_0^{t_w-\tau} E_{\alpha,\alpha}(A_1(t_w-\tau-s)^{\alpha+\beta})H_1 f(s-\tau)g(s-\tau)\ ds \\ &= v_1 - \int_0^{t_w} E_{\alpha,\alpha}(A_1(t_w-s)^{\alpha+\beta})B_1 u_2\ ds + \int_0^{t_w-\tau} E_{\alpha,\alpha}(A_1(t_w-\tau-s)^{\alpha+\beta})H_1 u_2(s-\tau)\ ds \\ &\equiv \widehat{v_1} \end{aligned}$$

(14)

${}^{C}D^{k(\alpha+\beta)}u_1(t) \;=\; {}^{C}D^{k(\alpha+\beta)}u_1(t-\tau)=0,\; for\; k=0,1,2,...n_1-1$. Thus $u_1(s)$ does not affect $x_2(t)$ at $s=t$ and $s=t-\tau$. Here $\widehat{v_1}\in\langle A_1|B_1,H_1\rangle$ from lemma (4.3), there exist a $z\in\mathbb{R}^n$ such that

$$W(f,t)z=\widehat{v_1}$$

Now, to prove the equation (14) holds. Let us consider

then

$$\begin{aligned}
&\int_0^{t-\tau} f(s)\{E_{\alpha,\alpha}[A_1(t-s)^{\alpha+\beta}]B_1B_1^*E_{\alpha,\alpha}[A_1^*(t-s)^{\alpha+\beta}]+E_{\alpha,\alpha}[A_1(t-\tau-s)^{\alpha+\beta}]H_1H_1^*\\
&\times E_{\alpha,\alpha}[A_1^*(t-\tau-s)^{\alpha+\beta}]\}f(s)z\;ds+\int_{t-\tau}^{t} f(s)E_{\alpha,\alpha}[A_1(t-s)^{\alpha+\beta}]B_1B_1^*E_{\alpha,\alpha}[A_1^*(t-s)^{\alpha+\beta}]f(s)z\;ds\\
&=\;W(f,t)z=\widehat{v_1}
\end{aligned}$$

(15)

Therefore, the equation (14) is true. On the other hand, for $v_2\in\langle M|B_2,H_2\rangle$, there exists a_k,b_k such that

$$u_2(s)=\begin{cases} h(s) & ;0\le s\le t\\ 0 & ;-\tau\le s\le 0\end{cases}$$

Then there exists a function $h(s)$ such that ${}^{C}D^{k(\alpha+\beta)}h(0)=0$; ${}^{C}D^{k(\alpha+\beta)}h(s)=a_k$ and ${}^{C}D^{k(\alpha+\beta)}h(s-\tau)=b_k$. Let

Therefore,

$$x_2(t) \;=\; \sum_{k=0}^{n_2-1} M^k\{B_2\;{}^{C}D^{k(\alpha+\beta)}u_2(s)+H_2\;{}^{C}D^{k(\alpha+\beta)}u_2(s-\tau)\}$$

(16)

and all initial conditions are zero. From equations (14) and (16) that $v\in\mathfrak{R}(0,0,0)$. Hence the theorem.

Controllability criteria

The following theorems and definitions establishes the controllability results for our determined system.

Definition 5.1 : System (**??**) or (**??**) is said to be controllable if for any $t_w > 0$, one can reach any admissible initial data $(x_0, \varphi, g(0, x(0)) \in \mathscr{B}$, any $v \in \mathbb{R}^n$, there exist $u \in C_p^h([0, \infty), \mathbb{R}^m)$ such that $x(t_w) = v$.

Theorem 5.2 : Canonical form of (**??**) is controllable if and only if

$$\begin{aligned} rank[B_1, A_1B_1, A_1^2B_1, ..., A_1^{n_1-1}B_1, H_1, A_1H_1, ..., A_1^{n_1-1}H_1] &= n_1 \ and \qquad (17) \\ rank[B_2, MB_2, M^2B_2, ..., M^{n_2-1}B_2, H_2, MH_2, ..., M^{n_2-1}H_2] &= n_2 \end{aligned}$$

(18)

Proof. We first prove necessary part of this theorem. If the system (**??**) is controllable. For any $w =$
$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^n$, initial state $x_0 = 0$ and the initial function $\varphi(t) \equiv 0$, $\exists\ t_w > 0$ and control $u(t) \in C_p^h([0, \infty), \mathbb{R}^m)$ such that w_1, w_2 can be written as $w_1 \in \langle A_1|B_1, H_1\rangle$ and $w_2 \in \langle M|B_2, H_2\rangle$
(ie)

$$w \in \langle A_1|B_1, H_1\rangle \oplus \langle M|B_2, H_2\rangle$$

is true $\forall\ w \in \mathbb{R}^n$

$$\Rightarrow \mathbb{R}^n \subseteq \langle A_1|B_1, H_1\rangle \oplus \langle M|B_2, H_2\rangle$$

We know that

$$\mathbb{R}^n \supseteq \langle A_1|B_1, H_1\rangle \oplus \langle M|B_2, H_2\rangle$$

Therefore, we have

Hence $\langle A_1|B_1, H_1\rangle = \mathbb{R}^{n_1}$ and $\langle M|B_2, H_2\rangle = \mathbb{R}^{n_2}$. Since $\langle A_1|B_1, H_1\rangle$ is generated by column vectors $[B_1, A_1B_1, A_1^2B_1, ..., A_1^{n_1-1}B_1, H_1, A_1H_1, ..., A_1^{n_2-1}H_1]$ such that

$$rank[B_1, A_1B_1, A_1^2B_1, ..., A_1^{n_1-1}B_1, H_1, A_1H_1, ..., A_1^{n_1-1}H_1] = n_1$$

similarly,

$$rank[B_2, MB_2, M^2B_2, ..., M^{n_2-1}B_2, H_2, MH_2, ..., M^{n_2-1}H_2] = n_2$$

Thus the results (17) and (18) is true.
Conversely, if results in (17) and (18) is true. We have to prove (**??**) is controllable. For any $v \in \mathbb{R}^n$ with any initial state x_0 and initial control function $\varphi(t)$. Let us consider,

$$
\begin{aligned}
m_1 &= v_1 - E_{\alpha+\beta,1}(A_1 t^{\alpha+\beta})x_{01} + t^{\alpha} E_{\alpha+\beta,1+\alpha}(A_1 t^{\alpha+\beta})g(0,x(0)) - \int_0^t (t-s)^{\alpha-1} \\
&\quad \times E_{\alpha+\beta,\alpha}(A_1(t-s)^{\alpha+\beta})g_1(s,x(s))\ ds - \int_0^t E_{\alpha,\alpha}(A_1(t-s)^{\alpha+\beta}) \\
&\quad \times B_1\varphi(0)\ ds - \int_0^{t-\tau} E_{\alpha,\alpha}(A_1(t-\tau-s)^{\alpha+\beta})H_1\varphi(0)\ ds \\
&\quad - \int_{-\tau}^0 E_{\alpha,\alpha}(A_1(t-h-s)^{\alpha+\beta})H_1\varphi(s)\ ds \qquad (19) \\
m_2 &= v_2 - \sum_{k=1}^{n_2-1} M^k\{P_k(t)x_{02} + Q_k(t) + R_k(t)g(0,x(0)) + S_k(t)[B_2\varphi(0) \\
&\quad + H_2\varphi(-\tau))\} - B_2\varphi(0) - H_2\varphi(0)
\end{aligned}
$$

(20)

For $m = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \in \mathbb{R}^n = \langle A_1|B_1,H_1\rangle \oplus \langle M|B_2,H_2\rangle$ and we have $m \in \mathfrak{R}(0,0,0)$, then there exists a control $\widehat{u}(s)$ such that

$$
\begin{aligned}
m_1 &= \int_0^t E_{\alpha,\alpha}(A_1(t-s)^{\alpha+\beta})B_1\widehat{u}(s)\ ds \\
&\quad + \int_0^{t-\tau} E_{\alpha,\alpha}(A_1(t-\tau-s)^{\alpha+\beta})H_1\widehat{u}(s)\ ds \qquad (21) \\
m_2 &= \sum_{k=0}^{n_2-1} M^k[B_2{}^C D^{k(\alpha+\beta)}\widehat{u}(s) + H_2{}^C D^{k(\alpha+\beta)}\widehat{u}(s-\tau)]
\end{aligned}
$$

(22)

Now, let us take $u(s) = \widehat{u}(s) + \varphi(0)$. From equations (**??**),(**??**),(21) and (22) as follows

$$
\begin{aligned}
v_1 &= E_{\alpha+\beta,1}(A_1 t^{\alpha+\beta})x_{01} - t^{\alpha} E_{\alpha+\beta,1+\alpha}(A_1 t^{\alpha+\beta})g(0,x(0)) + \int_0^t (t-s)^{\alpha-1} \\
&\quad \times E_{\alpha+\beta,\alpha}(A_1(t-s)^{\alpha+\beta})g_1(s,x(s))\ ds + \int_0^t E_{\alpha,\alpha}(A_1(t-s)^{\alpha+\beta}) \\
&\quad \times (B_1u(t)) + H_1u(t-\tau))\ ds + \int_{-\tau}^0 E_{\alpha,\alpha}(A_1(t-h-s)^{\alpha+\beta})H_1\varphi(s)\ ds
\end{aligned}
$$

$$
\begin{aligned}
v_2(t) &= \sum_{k=1}^{n_2-1} M^k \{P_k(t)x_{02} + Q_k(t) + R_k(t)g(0,x(0)) + S_k(t)[B_2\varphi(0) \\
&\quad + H_2\varphi(-\tau))\} + \sum_{k=1}^{n_2-1} M^k\ {}^C D^{k(\alpha+\beta)} u(t) + \sum_{k=1}^{n_2-1} M^k\ {}^C D^{k(\alpha+\beta)} u(t-\tau)
\end{aligned}
$$

Therefore by the definition (5.1), our system (**??**) is controllable. Hence the theorem.

Theorem 5.3 : The Canonical form of equations (**??**) is controllable if and only if $rank[sI - A_1, B_1, H_1] = n_1$, $\forall s \in \mathbb{C}$, s is finite and $rank[M, B_2, H_2] = n_2$.

Proof. According to Cayley-Hamilton theorem

$$
\begin{aligned}
E_{\alpha,\alpha}(A_1(t-s)^{\alpha+\beta}) &= \sum_{l=0}^{\infty} \frac{(t-s)^{(\alpha+\beta)l}}{\Gamma(l\alpha+\alpha)} (A_1)^l \\
&= \sum_{l=0}^{n_1-1} \eta_l(t-s)(A_1)^l
\end{aligned}
$$

(23)

Let

$$
\begin{aligned}
\chi &= E_{\alpha+\beta,1}(A_1 w^{\alpha+\beta})x_{01} + \int_0^w (w-s)^{\alpha-1} E_{\alpha+\beta,\alpha}(A_1(w-s)^{\alpha+\beta}) g_1(s,x(s))\ ds \\
&\quad + \int_{-h}^0 E_{\alpha,\alpha}(A_1(w-\tau-s)^{\alpha+\beta}) - w^\alpha E_{\alpha+\beta,1+\alpha}(A_1 w^{\alpha+\beta}) g(0,x(0)) H_1 u(s)\ ds
\end{aligned}
$$

(24)

The following results from equations (5) ,(23) and (24) as,

$$
x_1(w) - \chi = \sum_{k=0}^{n_1-1} \int_0^w \eta_k(w-s) A_1^k B_1 u(s)\ ds + \sum_{k=0}^{n_1-1} \int_0^{w-\tau} \eta_k(w-\tau-s) A_1^k H_1 u(s)\ ds
$$

(25)

For any $x_{01} \in \mathbb{R}^n$ and $x(w) \in \mathbb{R}^n$ the necessary and sufficient condition to have a control input $u(t)$ satisfies (25) is that

$$rank[B_1|A_1B_1|...|A_1^{n_1-1}B_1|H_1|...|A_1^{n_1-1}H_1] = n_1$$

On the other hand, apply similar arguments we get

$$rank[B_2|MB_2|...|M^{n_2-1}B_2|H_2|...|M^{n_2-1}H_2] = n_2$$

Hence the proof.

Corollary 5.4 : System (**??**) is controllable if and only if $rank[sE - A, B, C] = n$, $\forall s \in \mathbb{C}$ is finite and $rank[E, B, C] = n$.

Examples

Example 6.1 : Let us consider the linear fractional singular system

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix} \; {}^CD^{4/5}({}^CD^{2/5}x(t) - tx^2(t))$$

=

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & -1 \end{bmatrix} \text{x(t)} +$$

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 2 \end{bmatrix} \text{u(t)} +$$

$$\begin{bmatrix} 0 & 0 \\ 0 & -3 \\ 1 & 2 \end{bmatrix} \text{u(t - } 3\pi\overline{\text{2)}}$$

For $t \in [0, 2]$ *and* $u(t) = 0$; $\frac{-3\pi}{2} < t \leq 0$ and the inital conditions are $x_1(0) = 0$; $x_2(0) = 0$; ${}^CD^{\alpha}x(t)|_{t=0} = 0$

According to the results in theorem (5.4) to show that the system (**??**) is controllable. Let us take $\alpha = \frac{4}{5}$; $\beta = \frac{2}{5}$; $E =$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix} ; \quad \text{A} =$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & -1 \end{bmatrix} ; \quad \text{B} =$$

$$\begin{bmatrix}1 & 0\\ 1 & -1\\ 0 & 2\end{bmatrix};$$

andH=

$$\begin{bmatrix}0 & 0\\ 0 & -3\\ 1 & 2\end{bmatrix}$$

$If we take \lambda = 2$ then

$$\lambda E + A = 2$$

$$\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 3 & 0\end{bmatrix} +$$

$$\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 2 & 3 & -1\end{bmatrix} =$$

$$\begin{bmatrix}1 & 3 & 0\\ 2 & 0 & 3\\ 2 & 9 & -1\end{bmatrix} = \text{-3} \neq 0$$

Therefore we conclude from definition (5.1) is that (E, A) is regular pencil. By using elementary operation in matrix, we get that results as follows

$$rank[sE - A, B, C] = rank$$

$$\begin{bmatrix}1 & 0 & 0 & * & * & *\\ 0 & 0 & -1 & * & * & *\\ 0 & 1 & 2 & * & * & *\end{bmatrix} = 3$$

rank[E,B,C] = rank

$$\begin{bmatrix}1 & 0 & 1 & * & * & *\\ 0 & 1 & 0 & * & * & *\\ 0 & 0 & 1 & * & * & *\end{bmatrix} = 3$$

Therefore from the theorem (5.4), system (**??**) is controllable. Hence the proof.

Conclusion

The first contribution of this work is to introduced class of equations of fractional singular systems with two orders. The second contribution of this paper is to establish the rechability and controllability analysis of our considered model. Necessary and sufficient conditions for reachability and controllability of addressed systems are exposed The proposed model allows us to analyze the behavior of the subsystems given in section 3. This extended can be easily adopted to real world problems whereas the variance of dynamical systems.

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