Equivalent Conditions of a Hilbert-Type Integral Inequality with the General Nonhomogeneous Kernel in the Whole Plane

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Abstract

By means of the techniques of the real analysis and the weight functions, a few equivalent conditions of a Hilbert-type integral inequality with the general nonhomogeneous kernel in the whole plane are obtained. The constant factor is proved to be the best possible. As applications, a few equivalent conditions of a Hilbert-type integral inequality with the general homogeneous kernel in the whole plane are deduced. We also consider the operator expressions, a few particular cases and some examples.

Key words: Hilbert-type integral inequality; weight function; equivalent form; operator; norm

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Introduction

Assuming that $f(x), g(y) \geq 0$, $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(y)dy < \infty$, we have the following well-known Hilbert's integral inequality (see (missing citation)):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy \right)^{\frac{1}{2}}, \tag{1}$$

with the best possible constant factor π.

Recently, by means of the weight functions, a lot of extensions of (1) were given by two books (see (B. C. Yang & inequalities, n.d.), (B. C. Yang, n.d.)). Some Hilbert-type inequalities with the homogenous kernels and nonhomogenous kernels were provided by (B. C. Yang, n.d.)-(L. Debnath, n.d.). In 2017, Hong (Y. Hong, n.d.) also gave a equivalent condition between a Hilbert-type inequalities with the homogenous kernel and some parameters. Some other kinds of Hilbert-type inequalities were obtained by (M.Th. Rassias, n.d.)-(Q. Liu et al., 2019). Most of them are built in the quarter plane of the first quadrant.

Using the way of real analysis, in 2007, Yang (B. C. Yang, n.d.) gave a Hilbert-type integral inequality in the whole plane as follows:

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$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{(1+e^{x+y})^{\lambda}}dxdy$$
$$< \quad B(\frac{\lambda}{2},\frac{\lambda}{2})\left(\int_{-\infty}^{\infty}e^{-\lambda x}f^2(x)dx\int_{-\infty}^{\infty}e^{-\lambda y}g^2(y)dy\right)^{\frac{1}{2}}, \tag{2}$$

with the best possible constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})(\lambda>0,\ B(u,v)$ is the beta function) (see (missing citation)). He et al. (B. He, n.d.)-(Z.H Gu1, n.d.) proved some new Hilbert-type integral inequalities in the whole plane with the best possible constant factors.

In this paper, by means of the techniques of the real analysis and the weight functions, a few equivalent conditions of a Hilbert-type integral inequality with the general non-homogeneous kernel in the whole plane are obtained in Theorem 1. The constant factor is proved to be the best possible in Theorem 2. As applications, a few equivalent conditions of a Hilbert-type integral inequality with the general homogeneous kernel in the whole plane are deduced in Theorem 3. We also consider the operator expressions in Theorem 4-5. A few particular cases and some examples are obtained in Corollaries 1-4 and Examples 1-2. The lemmas and theorems provide an extensive account of this type of inequalities.

Two lemmas

In what follows, we suppose that $p>1,\frac{1}{p}+\frac{1}{q}=1,\sigma_1,\sigma\in\mathbf{R}=(-\infty,\infty),h(u)$ is a nonnegative measurable function in $\mathbf{R}$, with

$$K^{(1)}(\sigma):=\int_{-1}^{1}h(u)|u|^{\sigma-1}du=\int_{0}^{1}(h(-u)+h(u))u^{\sigma-1}du,$$

$$\begin{aligned}K^{(2)}(\sigma)\quad &: \quad =\int_{\{u;|u|\geq1\}}h(u)|u|^{\sigma-1}du=\int_{1}^{\infty}(h(-u)+h(u))u^{\sigma-1}du,\\ K(\sigma)\quad &: \quad =\int_{-\infty}^{\infty}h(u)|u|^{\sigma-1}du=\int_{0}^{\infty}(h(-u)+h(u))u^{\sigma-1}du\\ &= \quad K^{(1)}(\sigma)+K^{(2)}(\sigma).\end{aligned}$$

(3)

For $n\in\mathbf{N}=\{1,2,\dots\}$, we define the following two expressions:

$$I_1:=\int_{\{y;|y|\geq1\}}\left(\int_{\{x;|x|\leq1\}}h(xy)|x|^{\sigma+\frac{1}{pn}-1}dx\right)|y|^{\sigma_1-\frac{1}{qn}-1}dy, \tag{4}$$

$$I_2:=\int_{\{y;|y|\leq1\}}\left(\int_{\{x;|x|\geq1\}}h(xy)|x|^{\sigma-\frac{1}{pn}-1}dx\right)|y|^{\sigma_1+\frac{1}{qn}-1}dy.$$

(5)

Setting $u = xy$ in (4), by Fubini theorem (cf. (J. C. Kuang, n.d.)), it follows that

$$
\begin{aligned}
I_1 &= \int_{\{y;|y|\geq 1\}} \left[\int_{-1}^{0} h(xy)(-x)^{\sigma + \frac{1}{pn} - 1} dx \right. \\
&\quad \left. + \int_{0}^{1} h(xy) x^{\sigma + \frac{1}{pn} - 1} dx \right] |y|^{\sigma_1 - \frac{1}{qn} - 1} dy \\
&= 2 \int_{1}^{\infty} \left(\int_{0}^{y} (h(-u) + h(u)) (\frac{u}{y})^{\sigma + \frac{1}{pn} - 1} \frac{1}{y} du \right) y^{\sigma_1 - \frac{1}{qn} - 1} dy \\
&= 2 \int_{1}^{\infty} \left(\int_{0}^{y} (h(-u) + h(u)) u^{\sigma + \frac{1}{pn} - 1} du \right) y^{\sigma_1 - \sigma - \frac{1}{n} - 1} dy
\end{aligned} \tag{6}
$$

In the same way, we obtain

$$
\begin{aligned}
I_2 &= \int_{\{x;|x|\geq 1\}} \left(\int_{\{y;|y|\leq 1\}} h(xy) |y|^{\sigma_1 + \frac{1}{qn} - 1} dy \right) |x|^{\sigma - \frac{1}{pn} - 1} dx \\
&= 2 \int_{1}^{\infty} \left(\int_{0}^{x} (h(-u) + h(u)) u^{\sigma_1 + \frac{1}{qn} - 1} du \right) x^{\sigma - \sigma_1 - \frac{1}{n} - 1} dx.
\end{aligned} \tag{7}
$$

Lemma 1. ***Lemma 1.*** *If $K^{(1)}(\sigma) > 0$, there exists a constant M, such that for any nonnegative measurable functions $f(x)$ and $g(y)$ in* $\mathbf{R}$*, the following inequality*

$$
\begin{aligned}
I \quad &: = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(xy) f(x) g(y) dx dy \\
&\leq M \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}
\end{aligned} \tag{8}
$$

holds true, then we have $\sigma_1 = \sigma$.

Proof. Proof. If $\sigma_1 < \sigma$, then for $n > \frac{1}{\sigma - \sigma_1}$ $(n \in \mathbf{N})$, we set functions

$$f_n(x) := \{$$

$$\begin{matrix} 0, |x| < 1 \\ |x|^{\sigma - \frac{1}{pn} - 1}, |x| \geq 1 \end{matrix} \quad , g_n(y) := \{$$

$$\begin{matrix} |y|^{\sigma_1 + \frac{1}{qn} - 1}, |y| \leq 1 \\ 0, |y| > 1 \end{matrix}$$,and then obtain

$$
\begin{aligned}
J_2 \quad &: = \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f_n^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g_n^q(y) dy \right]^{\frac{1}{q}} \\
&= \left(\int_{\{x;|x|\geq 1\}} |x|^{-\frac{1}{n} - 1} dx \right)^{\frac{1}{p}} \left(\int_{\{y;|y|\leq 1\}} |y|^{\frac{1}{n} - 1} dy \right)^{\frac{1}{q}} = 2n.
\end{aligned}
$$

By (7), we have

$$\begin{aligned} & 2\int_1^\infty \left[\int_0^1 (h(-u)+h(u))u^{\sigma_1+\frac{1}{qn}-1}du\right] x^{\sigma-\sigma_1-\frac{1}{n}-1}dx \\ \leq \quad & I_2 = \int_{-\infty}^\infty \int_{-\infty}^\infty h(xy)f_n(x)g_n(y)dxdy \leq MJ_2 = 2Mn. \end{aligned} \tag{9}$$

Since for any $n > \frac{1}{\sigma-\sigma_1}$ $(n \in \mathbf{N}), \sigma - \sigma_1 - \frac{1}{n} > 0$, it follows that $\int_1^\infty x^{\sigma-\sigma_1-\frac{1}{n}-1}dx = \infty$. By (9), in view of

$$\int_0^1 (h(-u)+h(u))u^{\sigma_1+\frac{1}{qn}-1}du \geq \int_0^1 (h(-u)+h(u))u^{\sigma-1} = K^{(1)}(\sigma) > 0,$$

we find that $\infty \leq 2Mn < \infty$, which is a contradiction.

If $\sigma_1 > \sigma$, then for $n > \frac{1}{\sigma_1-\sigma}$ $(n \in \mathbf{N})$, we set functions

$$\widetilde{f}_n(x) := \{$$

$|x|^{\sigma+\frac{1}{pn}-1}, |x| \leq 1$
$0, |x| > 1$ $, \widetilde{g}_n(y) := \{$

$0, |y| < 1$
$|y|^{\sigma_1-\frac{1}{qn}-1}, |y| \geq 1$,and then obtain

$$\begin{aligned} \widetilde{J}_2 \quad : \quad & = \left[\int_{-\infty}^\infty |x|^{p(1-\sigma)-1}\widetilde{f}_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^\infty |y|^{q(1-\sigma_1)-1}\widetilde{g}_n^q(y)dy\right]^{\frac{1}{q}} \\ & = \left(\int_{-1}^1 |x|^{\frac{1}{n}-1}dx\right)^{\frac{1}{p}} \left(\int_{\{y;|y|\geq 1\}} |y|^{-\frac{1}{n}-1}dy\right)^{\frac{1}{q}} = 2n. \end{aligned}$$

By (6), we have

$$\begin{aligned} & 2\int_1^\infty \left[\int_0^1 (h(-u)+h(u))u^{\sigma+\frac{1}{pn}-1}du\right] y^{\sigma_1-\sigma-\frac{1}{n}-1}dy \\ \leq \quad & I_1 = \int_0^\infty \int_0^\infty h(xy)\widetilde{f}_n(x)\widetilde{g}_n(y)dxdy \leq M\widetilde{J}_2 = 2Mn. \end{aligned} \tag{10}$$

Since for $n > \frac{1}{\sigma_1-\sigma}$ $(n \in \mathbf{N})$, $(\sigma_1 - \sigma) - \frac{1}{n} > 0$, it follows that $\int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1}dy = \infty$. By (10), in view of $K^{(1)}(\sigma) > 0$, namely,

$$\int_0^1 (h(-u)+h(u))u^{\sigma+\frac{1}{pn}-1}du > 0,$$

we have $\infty \leq 2Mn < \infty$, which is a contradiction.

Hence, we conclude that $\sigma_1 = \sigma$. The lemma is proved. □

□

Lemma 2. ***Lemma 2.*** *If there exists a constant M, such that for any nonnegative measurable functions $f(x)$ and $g(y)$ in* $\mathbf{R}$, *the following inequality*

$$\begin{aligned} & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(xy)f(x)g(y)dxdy \\ \leq \quad & M\left[\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^{q}(y)dy\right]^{\frac{1}{q}} \end{aligned} \tag{11}$$

holds true, then we have $K(\sigma) \leq M < \infty$.

Proof. *Proof*. For $\sigma_1 = \sigma$, we reduce (6) and then use inequality $I_1 \leq M\widetilde{J}_2$ (when $\sigma_1 = \sigma$) as follows

$$\begin{aligned} \frac{1}{2n}I_1 &= \int_0^1 (h(-u)+h(u))u^{\sigma+\frac{1}{pn}-1}du \\ &+ \int_1^{\infty}(h(-u)+h(u))u^{\sigma-\frac{1}{qn}-1}du \leq M. \end{aligned} \tag{12}$$

By Fatou lemma (cf. (J. C. Kuang, n.d.)) and (12), we have

$$\begin{aligned} K(\sigma) &= \int_0^1 \lim_{n\to\infty}(h(-u)+h(u))u^{\sigma+\frac{1}{pn}-1}du \\ &+ \int_1^{\infty}\lim_{n\to\infty}(h(-u)+h(u))u^{\sigma-\frac{1}{qn}-1}du \\ &\leq \underline{\lim}_{n\to\infty}\frac{1}{2n}I_1 \leq M < \infty. \end{aligned}$$

The lemma is proved. □

□

Main results and a few corollaries

Theorem 0.1. ***Theorem 1.*** *If $K^{(1)}(\sigma) > 0$, then the following conditions are equivalent:*

(i) There exists a constant M, such that for any $f(x) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^{p}(x)dx < \infty$, we have the following inequality:

$$\begin{aligned} J \quad : &= \left[\int_{-\infty}^{\infty}|y|^{p\sigma_1-1}\left(\int_{-\infty}^{\infty}h(xy)f(x)dx\right)^{p}dy\right]^{\frac{1}{p}} \\ &< M\left[\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}. \end{aligned}$$

(13)

(ii) There exists a constant M, such that for any $f(x), g(y) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty$, and $0 < \int_{=\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y)dy < \infty$, we have the following inequality:

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(xy)f(x)g(y)dxdy \\
&< M\left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y)dy\right]^{\frac{1}{q}}.
\end{aligned}
$$

(14)

(iii) $\sigma_1 = \sigma$ and $K(\sigma) < \infty$.

Proof. Proof. $(i) => (ii)$. By Hölder's inequality (see **?**), we have

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} \left(|y|^{\sigma_1 - \frac{1}{p}} \int_{-\infty}^{\infty} h(xy)f(x)dx\right)\left(|y|^{\frac{1}{p}-\sigma_1} g(y)\right) dy \\
&\leq J\left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y)dy\right]^{\frac{1}{q}}.
\end{aligned} \tag{15}
$$

Then by (13), we have (14).

$(ii) => (iii)$. Since $K^{(1)}(\sigma) > 0$, by Lemma 1, we have $\sigma_1 = \sigma$. Then by Lemma 2, we have $K(\sigma) \leq M < \infty$.

$(iii) => (i)$. Setting $u = xy$, we obtain the following weight function: For $y \in (-\infty, 0) \cup (0, \infty)$,

$$
\begin{aligned}
\omega(\sigma, y) \quad &: \quad = |y|^{\sigma} \int_{-\infty}^{\infty} h(xy)|x|^{\sigma-1}dx \\
&= \int_0^{\infty} (h(-u) + h(u))u^{\sigma-1}du = K(\sigma).
\end{aligned}
$$

(16)

By Hölder's inequality with weight and (16), we have

$$
\begin{aligned}
& \left(\int_{-\infty}^{\infty} h(xy) f(x) dx\right)^p \\
= & \left\{\int_{-\infty}^{\infty} h(xy)\left[\frac{|y|^{(\sigma-1)/p}}{|x|^{(\sigma-1)/q}} f(x)\right]\left[\frac{|x|^{(\sigma-1)/q}}{|y|^{(\sigma-1)/p}}\right] dx\right\}^p \\
\leq & \int_{-\infty}^{\infty} h(xy) \frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}} f^p(x) dx\left[\int_{-\infty}^{\infty} h(xy) \frac{|x|^{\sigma-1}}{|y|^{(\sigma-1)q/p}} dx\right]^{p/q} \\
= & \left[\omega(\sigma, y)|y|^{q(1-\sigma)-1}\right]^{p-1} \int_{-\infty}^{\infty} h(xy) \frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}} f^p(x) dx \\
= & (K(\sigma))^{p-1}|y|^{-p\sigma+1} \int_{-\infty}^{\infty} h(xy) \frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}} f^p(x) dx .
\end{aligned}
$$

(17)

If (17) takes the form of equality for a $y \in (-\infty, 0) \cup (0, \infty)$, then (see (J. C. Kuang, n.d.)), there exists constants A and B, such that they are not all zero, and

$$A \frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}} f^p(x) = B \frac{|x|^{\sigma-1}}{|y|^{(\sigma-1)q/p}} \; a.e. \text{ in } \mathbf{R}.$$

We suppose that $A \neq 0$ (otherwise $B = A = 0$). Then it follows that

$$|x|^{p(1-\sigma)-1} f^p(x) = |y|^{q(1-\sigma)} \frac{B}{A|x|} \; a.e. \text{ in } \mathbf{R},$$

which contradicts the fact that $0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty$. Hence, (17) takes the form of strict inequality.

For $\sigma_1 = \sigma$, by Fubini theorem (see (J. C. Kuang, n.d.)) and (17), we have

$$
\begin{aligned}
J & < (K(\sigma))^{\frac{1}{q}}\left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(xy) \frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}} f^p(x) dx dy\right]^{\frac{1}{p}} \\
& = (K(\sigma))^{\frac{1}{q}}\left\{\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(xy) \frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)(p-1)}} dy\right] f^p(x) dx\right\}^{\frac{1}{p}} \\
& = (K(\sigma))^{\frac{1}{q}}\left[\int_{-\infty}^{\infty} \omega(\sigma, x)|x|^{p(1-\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}} \\
& = K(\sigma)\left[\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}} .
\end{aligned}
$$

For $K(\sigma) \in \mathbf{R}_+$, setting $M \geq K(\sigma)$, we have

$$J < K(\sigma)\left[\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}} \leq M\left[\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}},$$

namely, (13) follows.

Therefore, the conditions (i), (ii) and (iii) are equivalent.

The theorem is proved. □

For $\sigma_1 = \sigma$, we have

□

Theorem 0.2. ***Theorem 2.*** *The following conditions are equivalent:*

(i) There exists a constant M, such that for any $f(x) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty$, we have the following inequality:

$$\begin{aligned} &\left[\int_{-\infty}^{\infty} |y|^{p\sigma-1} \left(\int_{-\infty}^{\infty} h(xy)f(x)dx\right)^p dy\right]^{\frac{1}{p}} \\ < \;& M\left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \end{aligned}$$

(18)

(ii) There exists a constant M, such that for any $f(x), g(y) \geq 0, 0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty$, and $0 < \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy < \infty$, we have the following inequality:

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(xy)f(x)g(y)dxdy \\ < \;& M\left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy\right]^{\frac{1}{q}}. \end{aligned}$$

(19)

(iii) $K(\sigma) < \infty$.

Moreover, if (iii) follows and $K(\sigma) > 0$, then the constant factor $M = K(\sigma) \in \mathbf{R}_+$ in (18) and (19) is the best possible.

Proof. Proof. For $\sigma_1 = \sigma$ in Theorem 1, we still can conclude that the conditions (i), (ii) and (iii) in Theorem 2 are equivalent.

When Condition (iii) follows and $K(\sigma) > 0$, if there exists a constant $M \leq K(\sigma)$, such that (19) is valid, then by Lemma 2, we have $K(\sigma) \leq M$. Hence, the constant factor $M = K(\sigma) \in \mathbf{R}_+$ in (19) is the best possible.

The constant factor $M = K(\sigma)$ in (18) is still the best possible. Otherwise, by (15) (for $\sigma_1 = \sigma$), we would reach a contradition that the constant factor $M = K(\sigma)$ in (19) is not the best possible.

The theorem is proved. □

□

In particular, for $\sigma_1 = \sigma = \frac{1}{p}$ in Theorem 2, we have
Corollary 1. ***Corollary 1.*** *The following conditions are equivalent:*

(i) There exists a constant M, such that for any $f(x) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty} |x|^{p-2} f^p(x)dx < \infty$, we have the following inequality:

$$\left[\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(xy)f(x)dx\right)^p dy\right]^{\frac{1}{p}} < M \left(\int_{-\infty}^{\infty} |x|^{p-2} f^p(x)dx\right)^{\frac{1}{p}}.$$

(20)

(ii) There exists a constant M, such that for any $f(x), g(y) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty} |x|^{p-2} f^p(x)dx < \infty$, and $0 < \int_{-\infty}^{\infty} g^q(y)dy < \infty$, we have the following inequality:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(xy)f(x)g(y)dxdy < M \left(\int_{-\infty}^{\infty} |x|^{p-2} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} g^q(y)dy\right)^{\frac{1}{q}}.$$

(21)

(iii) $K(\frac{1}{p}) < \infty$.

If Condition (iii) follows and $K(\frac{1}{p}) > 0$, then the constant factor $M = K(\frac{1}{p}) (\in \mathbf{R}_+)$ in (20) and (21) is the best possible.

Setting $y = \frac{1}{Y}$, $G(Y) = g(\frac{1}{Y})\frac{1}{Y^2}$ in Theorem 1-2, then replacing Y by y, we have
Corollary 2. ***Corollary 2.*** *If $K^{(1)}(\sigma) > 0$, then the following conditions are equivalent:*

(i) There exists a constant M, such that for any $f(x) \geq 0$, satisfying $0 < \int_0^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty$, we have the following inequality:

$$\begin{aligned} &\left[\int_{-\infty}^{\infty} |y|^{-p\sigma_1 - 1} \left(\int_{-\infty}^{\infty} h(\frac{x}{y})f(x)dx\right)^p dy\right]^{\frac{1}{p}} \\ < \quad & M \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx\right]. \end{aligned}$$

(22)

(ii) There exists a constant M, such that for any $f(x), G(y) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty$, and $0 < \int_{-\infty}^{\infty} |y|^{q(1+\sigma_1)-1} G^q(y)dy < \infty$, we have the following inequality:

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\frac{x}{y})f(x)G(y)dxdy \\ < \quad & M \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} y^{q(1+\sigma_1)-1} G^q(y)dy\right]^{\frac{1}{q}}. \end{aligned}$$

(23)

(iii) $\sigma_1 = \sigma$, and $K(\sigma) < \infty$.

If Condition (iii) follows, then the constant $M = K(\sigma)\,(\in \mathbf{R}_+)$ in (22) and (23) (for $\sigma_1 = \sigma$) is the best possible.

Note. $h(\frac{x}{y})$ is a homogeneous function of degree 0, namely, $h(\frac{x}{y}) = k_0(x,y)$.

Setting $h(u) = k_\lambda(u,1)$, where $k_\lambda(x,y)$ $(x,y \in \mathbf{R})$ is the homogeneous function of degree $-\lambda \in \mathbf{R}$, with

$$
\begin{aligned}
K_\lambda^{(1)}(\sigma) &:= \int_{-1}^{1} k_\lambda(u,1)|u|^{\sigma-1}du\,,\\
K_\lambda^{(2)}(\sigma) &:= \int_{\{u;|u|\geq 1\}} k_\lambda(u,1)|u|^{\sigma-1}du\,,\\
K_\lambda(\sigma) &:= \int_{-\infty}^{\infty} k_\lambda(u,1)|u|^{\sigma-1}du = K_\lambda^{(1)}(\sigma) + K_\lambda^{(2)}(\sigma),
\end{aligned}
$$

then for $g(y) = |y|^\lambda G(y)$ and $\mu = \lambda - \sigma_1$ in Corollary 2, we have

Theorem 0.3. ***Theorem 3.*** *If $K_\lambda^{(1)}(\sigma) > 0$, then the following conditions are equivalent:*

(i) There exists a constant M, such that for any $f(x) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty$, we have the following inequality:

$$
\begin{aligned}
&\left[\int_{-\infty}^{\infty} |y|^{p\mu-1}\left(\int_{-\infty}^{\infty} k_\lambda(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}}\\
<\ & M\left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}.
\end{aligned}
$$

(24)

(ii) There exists a constant M, such that for any $f(x), g(y) \geq 0$, satisfying $0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty$, and $0 < \int_{-\infty}^{\infty} |y|^{q(1-\mu)-1} g^q(y)dy < \infty$, we have the following inequality:

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_\lambda(x,y)f(x)g(y)dxdy\\
<\ & M\left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} |y|^{q(1-\mu)-1} g^q(y)dy\right]^{\frac{1}{q}}.
\end{aligned}
$$

(25)

(iii) $\mu+\sigma=\lambda$, and $K_\lambda(\sigma)<\infty$.

If Condition (iii) follows, then the constant $M=K_\lambda(\sigma)\,(\in\mathbf{R}_+)$ in (24) and (25) is the best possible.
Remark. **Remark 2** . If $\lambda=0,\mu=-\sigma_1$, $k_0(x,y)=h(\frac{x}{y})$, then Theorem 3 reduces to Corollary 2.

In particular, for $\lambda=1,\sigma=\frac{1}{q},\mu=\frac{1}{p}$ in Theorem 3 (also refer to Theorem 2), we have
Corollary 3. ***Corollary 3.*** *The following conditions are equivalent:*

(i) There exists a constant M, such that for any $f(x)\geq 0$, satisfying $0<\int_{-\infty}^{\infty}f^p(x)dx<\infty$, we have the following inequality:

$$\left[\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}k_1(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}}<M\left(\int_{-\infty}^{\infty}f^p(x)dx\right).$$

(26)

(ii) There exists a constant M, such that for any $f(x),g(y)\geq 0$, satisfying $0<\int_{-\infty}^{\infty}f^p(x)dx<\infty$, and $0<\int_{-\infty}^{\infty}g^q(y)dy<\infty$, we have the following inequality:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}k_1(x,y)f(x)g(y)dxdy<M\left(\int_{-\infty}^{\infty}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_{-\infty}^{\infty}g^q(y)dy\right)^{\frac{1}{q}}.$$

(27)

(iii) $K_1(\frac{1}{q})<\infty$.

If Condition (iii) follows and $K_1(\frac{1}{q})>0$, then the constant $M=K_1(\frac{1}{q})(\in\mathbf{R}_+)$ in (26) and (27) is the best possible.

For $\lambda=1,\sigma=\frac{1}{p},\mu=\frac{1}{q}$ in Theorem 3 (also refer to Theorem 2), we have
Corollary 4. ***Corollary 4.*** *The following conditions are equivalent:*

(i) There exists a constant M, such that for any $f(x)\geq 0$, satisfying $0<\int_{-\infty}^{\infty}|x|^{p-2}f^p(x)dx<\infty$, we have the following inequality:

$$\left[\int_{-\infty}^{\infty}|y|^{p-2}\left(\int_{-\infty}^{\infty}k_1(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}}<M\left(\int_{-\infty}^{\infty}|x|^{p-2}f^p(x)dx\right)^{\frac{1}{p}}.$$

(28)

(ii) There exists a constant M, such that for any $f(x),g(y)\geq 0$, satisfying $0<\int_{-\infty}^{\infty}|x|^{p-2}f^p(x)dx<\infty$, and $0<\int_{-\infty}^{\infty}|y|^{q-2}g^q(y)dy<\infty$, we have the following inequality:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}k_1(x,y)f(x)g(y)dxdy$$
$$< \quad M\left(\int_{-\infty}^{\infty}|x|^{p-2}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_{-\infty}^{\infty}|y|^{q-2}g^q(y)dy\right)^{\frac{1}{q}}.$$

(29)

(iii) $K_1(\frac{1}{p}) < \infty$.

If Condition (iii) follows and $K_1(\frac{1}{p}) > 0$, then the constant factor $M = K_1(\frac{1}{p})\,(\in \mathbf{R}_+)$ in (28) and (29) is the best possible.

Operator expressions and examples

For $\mu + \sigma = \lambda$, we set the following functions: $\varphi(x) := |x|^{p(1-\sigma)-1}$, $\psi(y) := |y|^{q(1-\sigma)-1}$, $\phi(y) := |y|^{q(1-\mu)-1}$, wherefrom, $\psi^{1-p}(y) = |y|^{p\sigma-1}$, $\phi^{1-p}(y) = |y|^{p\mu-1}$ $(x, y \in \mathbf{R})$. Define the following real normed linear spaces:

$$L_{p,\varphi}(\mathbf{R}) := \left\{f : ||f||_{p,\varphi} := \left(\int_{-\infty}^{\infty}\varphi(x)|f(x)|^p dx\right)^{\frac{1}{p}} < \infty\right\},$$

wherefrom,

$$\begin{aligned}
L_{q,\psi}(\mathbf{R}) &= \left\{g : ||g||_{q,\psi} := \left(\int_{-\infty}^{\infty}\psi(y)|g(y)|^q dy\right)^{\frac{1}{q}} < \infty\right\},\\
L_{q,\phi}(\mathbf{R}) &= \left\{g : ||g||_{q,\phi} := \left(\int_{-}^{\infty}\phi(y)|g(y)|^q dy\right)^{\frac{1}{q}} < \infty\right\},\\
L_{p,\psi^{1-p}}(\mathbf{R}) &= \left\{h : ||h||_{p,\psi^{1-p}} = \left(\int_{-\infty}^{\infty}\psi^{1-p}(y)|h(y)|^p dy\right)^{\frac{1}{p}} < \infty\right\},\\
L_{q,\phi^{1-p}}(\mathbf{R}) &= \left\{h : ||h||_{p,\phi^{1-p}} = \left(\int_{-\infty}^{\infty}\phi^{1-p}(y)|h(y)|^p dy\right)^{\frac{1}{p}} < \infty\right\}.
\end{aligned}$$

(a) In view of Theorem 2, for $f \in L_{p,\varphi}(\mathbf{R})$, setting $h_1(y) := \int_{-\infty}^{\infty} h(xy)\ f(x)dx\ (y \in \mathbf{R})$, by (18), we have

$$||h_1||_{p,\psi^{1-p}} = \left[\int_{-\infty}^{\infty}\psi^{1-p}(y)h_1^p(y)dy\right]^{\frac{1}{p}} < M||f||_{p,\varphi} < \infty.$$

(30)

Definition 1. ***Definition 1.*** *Define a Hilbert-type integral operator with the nonhomogeneous kernel* $T^{(1)}$: $L_{p,\varphi}(\mathbf{R}) \to L_{p,\psi^{1-p}}(\mathbf{R})$ *as follows: For any* $f \in L_{p,\varphi}(\mathbf{R})$, *there exists a unique representation* $T^{(1)}f = h_1 \in L_{p,\psi^{1-p}}(\mathbf{R})$, *satisfying for any* $y \in \mathbf{R}, T^{(1)}f(y) = h_1(y)$.

In view of (30), it follows that

$$||T^{(1)}f||_{p,\psi^{1-p}} = ||h_1||_{p,\psi^{1-p}} \leq M||f||_{p,\varphi},$$

and then the operator $T^{(1)}$ is bounded satisfying

$$||T^{(1)}|| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R})} \frac{||T^{(1)}f||_{p,\psi^{1-p}}}{||f||_{p,\varphi}} \leq M.$$

If we define the formal inner product of $T^{(1)}f$ and g as follows:

$$(T^{(1)}f, g) := \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(xy)f(x)dx \right) g(y)dy,$$

then we can rewrite Theorem 2 as follows:

Theorem 0.4. ***Theorem 4.*** *The following conditions are equivalent:*

(i) There exists a constant M, such that for any $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R})$, $||f||_{p,\varphi} > 0$, we have the following inequality:

$$||T^{(1)}f||_{p,\psi^{1-p}} < M||f||_{p,\varphi}.$$

(31)

(ii) There exists a constant M, such that for any $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}), g \in L_{q,\psi}(\mathbf{R}), ||f||_{p,\varphi}, ||g||_{q,\psi} > 0$, we have the following inequality:

$$(T^{(1)}f, g) < M||f||_{p,\varphi}||g||_{q,\psi}.$$

(32)

(iii) $K(\sigma) < \infty$.

Moreover, if (iii) follows and $K(\sigma) > 0$, then the constant factor $M = K(\sigma)(\in \mathbf{R}_+)$ in (31) and (32) is the best possible, namely, $0 < ||T^{(1)}|| = K(\sigma) \leq M$.

(b) In view of Theorem 3 ($\sigma + \mu = \lambda$), for $f \in L_{p,\varphi}(\mathbf{R})$, setting $h_2(y) := \int_{-\infty}^{\infty} k_\lambda(x,y)f(x)dx$ ($y \in \mathbf{R}$), by (24), we have

$$||h_2||_{p,\phi^{1-p}} = \left[\int_{-\infty}^{\infty} \phi^{1-p}(y)h_2^p(y)dy \right]^{\frac{1}{p}} < M||f||_{p,\varphi} < \infty.$$

(33)

Definition 2. ***Definition 2.*** *Define a Hilbert-type integral operator with the homogeneous kernel* $T^{(2)} : L_{p,\varphi}(\mathbf{R}) \to L_{p,\phi^{1-p}}(\mathbf{R})$ *as follows: For any* $f \in L_{p,\varphi}(\mathbf{R})$, *there exists a unique representation* $T^{(2)}f = h_2 \in L_{p,\phi^{1-p}}(\mathbf{R})$, *satisfying for any* $y \in \mathbf{R}, T^{(2)}f(y) = h_2(y)$.

In view of (33), it follows that

$$||T^{(2)}f||_{p,\phi^{1-p}} = ||h_2||_{p,\phi^{1-p}} \leq M||f||_{p,\varphi},$$

and then the operator $T^{(2)}$ is bounded satisfying

$$||T^{(2)}|| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R})} \frac{||T^{(2)}f||_{p,\phi^{1-p}}}{||f||_{p,\varphi}} \leq M.$$

If we define the formal inner product of $T^{(2)}f$ and g as follows:

$$(T^{(2)}f, g) := \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} k_\lambda(x,y) f(x) dx \right) g(y) dy,$$

then we can rewrite Theorem 3 (for $\mu + \sigma = \lambda$) as follows:

Theorem 0.5. ***Theorem 5.*** *The following conditions are equivalent:*

(i) There exists a constant M, such that for any $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R})$, $||f||_{p,\varphi} > 0$, we have the following inequality:

$$||T^{(2)}f||_{p,\phi^{1-p}} < M||f||_{p,\varphi}.$$

(34)

(ii) There exists a constant M, such that for any $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}), g \in L_{q,\phi}(\mathbf{R}), ||f||_{p,\varphi}, ||g||_{q,\phi} > 0$, we have the following inequality:

$$(T^{(2)}f, g) < M||f||_{p,\varphi}||g||_{q,\phi}.$$

(35)

(iii) $K_\lambda(\sigma) < \infty$.

If Condition (iii) follows and $K_\lambda(\sigma) > 0$, then the constant factor $M = K_\lambda(\sigma)(\in \mathbf{R}_+)$ in (34) and (35) is the best possible, namely, $0 < ||T^{(2)}|| = K_\lambda(\sigma) \leq M$.

Example 1. ***Example 1.*** *Setting* $h(xy) = \frac{|\ln|xy||^{\beta}}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|}$*, and*

$$k_{\lambda}(x,y) = \frac{|\ln|x/y||^{\beta}}{(\max\{|x|,|y|\})^{\lambda-1}|x-y|}\ (x,y \in \mathbf{R}),$$

for $\beta > 0, \sigma, \mu > 0, \sigma + \mu = \lambda$*, it follows that*

$$K(\sigma) = K_{\lambda}(\sigma) = \int_0^{\infty} \frac{|\ln u|^{\beta} u^{\sigma-1}}{(\max\{u,1\})^{\lambda-1}} \left(\frac{1}{u+1} + \frac{1}{|u-1|}\right) du$$

$$\begin{aligned}
&= \int_0^1 (-\ln u)^{\beta} \left(\frac{1}{u+1} + \frac{1}{1-u}\right)(u^{\sigma-1} + u^{\mu-1})du \\
&= 2\int_0^1 (-\ln u)^{\beta} \frac{1}{1-u^2}(u^{\sigma-1} + u^{\mu-1})du \\
&= 2\int_0^1 (-\ln u)^{\beta} \sum_{k=0}^{\infty} u^{2k}(u^{\sigma-1} + u^{\mu-1})du.
\end{aligned}$$

By Lebesgue term by term integration theorem (cf. (J. C. Kuang, n.d.)), we have

$$\begin{aligned}
K(\sigma) &= K_{\lambda}(\sigma) = 2\sum_{k=0}^{\infty} \int_0^1 (-\ln u)^{\beta}(u^{2k+\sigma-1} + u^{2k+\mu-1}du \\
&= 2\sum_{k=0}^{\infty} \left[\frac{1}{(2k+\sigma)^{\beta+1}} + \frac{1}{(2k+\mu)^{\beta+1}}\right] \int_0^{\infty} v^{\beta} e^{-v} dv \\
&= \frac{\Gamma(\beta+1)}{2^{\beta}} \left(\zeta(\beta+1, \frac{\sigma}{2}) + \zeta(\beta+1, \frac{\mu}{2})\right) \in \mathbf{R}_+,
\end{aligned}$$

where, $\zeta(s,a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} Res > 1; a > 0)$ *is the extended Riemann zeta function* ($\zeta(s,1) = \sum_{k=1}^{\infty} \frac{1}{k^s}(Res > 1)$ *is the Riemann zeta function) (cf. (missing citation)). Then by Theorem 4 and Theorem 5, we have*

$$||T^{(1)}|| = ||T^{(2)}|| = \frac{\Gamma(\beta+1)}{2^{\beta}} \left(\zeta(\beta+1, \frac{\sigma}{2}) + \zeta(\beta+1, \frac{\mu}{2})\right).$$

(36)

Example 2. ***Example 2.*** *Setting* $h(xy) = \frac{1}{|xy-1|^{\lambda}}, k_{\lambda}(x,y) = \frac{1}{|x-y|^{\lambda}}\ (x,y \in \mathbf{R})$*, for* $\sigma, \mu > 0, \sigma+\mu = \lambda < 1$*, it follows that*

$$\begin{aligned}
K(\sigma) &= K_{\lambda}(\sigma) = \int_0^{\infty} \left(\frac{1}{(u+1)^{\lambda}} + \frac{1}{|u-1|^{\lambda}}\right) u^{\sigma-1} du \\
&= \int_0^{\infty} \frac{u^{\sigma-1}}{(u+1)^{\lambda}} du + \int_0^1 \frac{1}{(1-u)^{\lambda}}(u^{\sigma-1} + u^{\mu-1})du \\
&= B(\sigma,\mu) + B(1-\lambda,\sigma) + B(1-\lambda,\mu) \in \mathbf{R}_+.
\end{aligned}$$

Then by Theorem 4 and Theorem 5, we have

$$||T^{(1)}|| = ||T^{(2)}|| = B(\sigma,\mu) + B(1-\lambda,\sigma) + B(1-\lambda,\mu). \tag{37}$$

References

(2019).