New Blow-up Conditions to p-Laplace Type Nonlinear Parabolic Equations under Nonlinear Boundary Conditions

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Abstract

In this paper, we study blow-up phenomena of the following p-Laplace type nonlinear parabolic equations under nonlinear mixed boundary conditions and $u = 0$ on $\Gamma_2 \times (0, t^*)$ such that $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, where f and h are real-valued C^1-functions. To discuss blow-up solutions, we introduce new conditions:
For each $x \in \Omega$, $z \in \partial\Omega$, $t > 0$, $u > 0$, and $v > 0$, for some constants α, β_1, β_2, γ_1, γ_2, and δ satisfying where $\rho_m := \inf_{w>0} \rho(w)$, $P(v) = \int_0^v \rho(w)dw$, $F(x,t,u) = \int_0^u f(x,t,w)dw$, and $H(x,t,u) = \int_0^u h(x,t,w)dw$. Here, λ_R is the first Robin eigenvalue and λ_S is the first Steklov eigenvalue for the p-Laplace operator, respectively.

Introduction

It is well-known that reaction-diffusion equations can describe lots of natural phenomena such as gravitational potentials, heat flow, and fluid flow (see (2010)). Especially, nonlinear reaction-diffusion equations have been attracted the attention of many researchers. The most famous example of nonlinearity, there are the p-Laplace operator ($\nabla \cdot [|\nabla u|^{p-2}\nabla u]$) as a diffusion operator and autonomous function $f(u)$ as a reaction term. i.e.

$$u_t = \nabla \cdot [|\nabla u|^{p-2}\nabla u] + f(u).$$

(1)

Of course, the equation (1) can express a variety of natural and social phenomena and has been studied by many researchers (see (1985; 1998; 2002) and references therein).

On the other hand, lots of natural and social phenomena can be also affected by external stimuli. Therefore, it is important to consider various boundary conditions such as the Dirichlet boundary condition, the Neumann boundary condition, and the Robin boundary conditions, and so on. Especially, p-Neumann boundary conditions with autonomous function $h(u)$:

$$|\nabla u|^{p-2}\frac{\partial u}{\partial n} = h(u)$$

(2)

have been studied by lots of researchers because of their applicability (see (2014; 2010) and references therein).

In this paper, we deal with blow-up phenomena of the following p-Laplace type nonlinear parabolic equations under mixed nonlinear boundary conditions:
For $p > 1$,

$\rho(|\nabla u|^p)|\nabla u|^{p-2}\frac{\partial u}{\partial n} + \theta(z)\rho(|u|^p)|u|^{p-2}u = h(z,t,u),$ on $\Gamma_1 \times (0,t^*)$,

$$\begin{cases} u_t = \nabla \cdot \left[\rho(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\right] + f(x,t,u), & \text{in } \Omega \times (0,t^*), \\ B_\rho[u] = h(z,t,u), & \text{on } \Gamma_1 \times (0,t^*), \\ u = 0, & \text{on } \Gamma_2 \times (0,t^*), \\ u(\cdot,0) = u_0 \geq 0, & \text{in } \overline{\Omega}, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, Γ_1 and Γ_2 are disjoint open and closed subset of $\partial\Omega$, respectively, such that $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, and t^* is maximal existence time of the solution u. Here, $B_\rho[u] = h(z,t,u)$ on $\Gamma_1 \times [0,t^*)$ stands for the boundary condition

where θ is nonnegative $C^1(\partial\Omega)$-function. Moreover, we assume that the function ρ is a $C^1(\mathbb{R}^+)$-function satisfying $\inf_{w>0}\rho(w) > 0$, the function f is a $C^1(\Omega\times\mathbb{R}^+\times\mathbb{R}^+)$-function, the function h is a $C^1(\partial\Omega\times\mathbb{R}^+\times\mathbb{R}^+)$-function, where $\mathbb{R}^+ = [0,\infty)$. Also, the initial data u_0 is a nonnegative nontrivial $C^1(\overline{\Omega})$-function satisfying the compatible condition $B_\rho[u_0] = h(z,0,u_0)$ on Γ_1 and $u_0 = 0$ on Γ_2.

The equation (**??**) is generalized version of (1)-(2) which is well-known p-Laplacian parabolic equation under the nonlinear Neumann boundary condition. The simple versions of the main equation (**??**) (such as autonomous functions $f(u)$ and $h(u)$ instead of $f(x,t,u)$ and $h(z,t,u)$, $\rho \equiv 1$, and $p = 2$) were studied by lots of researchers with respect to the blow-up theory (see (2016; 2016; 2017; 2018; 2018; 2014; 2010; 2014; 2008; 2009; 2016; 2018; 2019)).

Most of blow-up results which discussed nonlinear parabolic equations under nonlinear boundary conditions considered non-negative functions or non-positive functions in the reaction term and the boundary term (for example, see (2016; 2014; 2019)). However, we consider real-valued functions f and h instead of non-negative functions or non-positive functions. Also, we consider the non-autonomous terms f and h include various types of functions such as $k(t)f(u)$ or $b(x)f(u)$.

Especially, Messaoudi (2002) studied the p-Laplacian parabolic equations with the autonomous reaction $f(u)$:
For $p > 2$,

$$u_t = \nabla \cdot [|\nabla u|^{p-2}\nabla u] + f(u), \quad \text{in } \Omega \times (0,t^*),$$

(4)

under the Dirichlet boundary condition. In this result, the blow-up solutions to the equation (4) were obtained when the function f satisfied

$$(A_p)\ :\ (p+\epsilon)\int_0^u f(s)ds \le uf(u),\ \ u>0,$$

and the appropriate initial data condition was satisfied.

In 2016, Ding and Shen (2016) studied blow-up phenomena to the p-Laplacian parabolic equations under nonlinear boundary conditions

$$(\mathrm{A}_p)'\ :\ \begin{array}{l} p\displaystyle\int_0^u f(s)ds \le uf(u),\ \ u>0, \\ p\displaystyle\int_0^u h(s)ds \le uh(u),\ \ u>0. \end{array}$$

$$\begin{cases} (b(u))_t = \nabla\cdot[|\nabla u|^{p-2}\nabla u] + k(t)f(u), & \text{in } \Omega\times(0,t^*), \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = h(u), & \text{on } \partial\Omega\times(0,t^*), \\ u(\cdot,0) = u_0 \ge 0, & \text{on } \Omega, \end{cases}$$

where $p>2$. In their blow-up conditions, the functions b and k satisfied the condition (**??**) and the nonnegative functions f and h satisfied

In 2018, Zhang, Wang, and Wang (2018) obtained the blow-up solutions to the p-Laplace type nonlinear parabolic equation:
For $p \ge 2$,

$$(b(u))_t = \nabla\cdot[\rho(|\nabla u|^p)|\nabla u|^{p-2}\nabla u] + a(x)k(t)f(u),\ \ \text{in } \Omega\times(0,t^*),$$

(6)

under the Dirichlet boundary condition, where the functions b, ρ, a, k, and f were some appropriate functions to construct the nonnegative solutions. In their assumptions for the blow-up phenomena, the functions b, a, and k satisfied

$$(A_p) \,:\, (p+\epsilon_1)\int_0^u f(s)ds \le uf(u),\ u>0,$$
$$(B_p) \,:\, v\rho(v) \le (p+\epsilon_2)\int_0^v \rho(s)ds,\ u>0,$$

$$\lim_{s\to 0+} s^2 b'(s) = 0,$$
$$b'(s)>0,\ b''(s)\le 0,\ k(0)>0,\ k'(s)\ge 0,\ a(s)>0,$$

for $s>0$ and the functions ρ and f satisfied

where ϵ_1 and ϵ_2 are positive constants with $\epsilon_1 \ge \frac{\epsilon_2}{p}$.

$$\begin{cases} (b(u))_t = \nabla\cdot[\rho(|\nabla|^p)|\nabla u|^{p-2}\nabla u] + a(x)k(t)f(u), & \text{in } \Omega\times(0,t^*),\\ \rho(|\nabla|^p)|\nabla u|^{p-2}\frac{\partial u}{\partial n} = h(u), & \text{on } \partial\Omega\times(0,t^*),\\ u(\cdot,0)=u_0\ge 0, & \text{on } \Omega, \end{cases}$$

$(\mathrm{B}_p)' \,:\, v\rho(v) \le \int_0^v \rho(s)ds.$

In 2019, Zhang and Tian (2019) obtained the blow-up solutions to the p-Laplace type parabolic equations under nonlinear boundary conditions:
For $p>2$,

when the nonnegative functions b, a, and k satisfy the condition (**??**), the functions f and h satisfy the condition $(A_p)'$, and the function ρ satisfies

$$(D_p\,1) \,:\, \begin{array}{l} \alpha F(x,t,u) \le uf(x,t,u) + \beta_1 u^p + \gamma_1,\\ \alpha H(z,t,u) \le uh(z,t,u) + \beta_2 u^p + \gamma_2, \end{array}$$
$$(D_p\,2) \,:\, \delta v\rho(v) \le P(v),$$

$\alpha > 2,\ \delta > 0,\ \beta_1 + \frac{\lambda_R+1}{\lambda_S}\beta_2 \leq \left(\frac{\alpha\delta}{p} - 1\right)\rho_m\lambda_R,$

$0\leq \beta_2 \leq \left(\frac{\alpha\delta}{p} - 1\right)\rho_m\lambda_S,$

Now, we introduce new blow-up conditions for the functions ρ, f, and h to obtain the solutions to the equation (**??**) as follows:
For each $x \in \Omega$, $z \in \partial\Omega$, $t > 0$, $u > 0$, and $v > 0$,

for some constants α, β_1, β_2, γ_1, γ_2, and δ satisfying

and

where $F(x,t,u) := \int_0^u f(x,t,w)dw,\ H(z,t,u) := \int_0^u h(z,t,w)dw$, and $P(v) = \int_0^w \rho(w)ds$. Here, $\rho_m := \inf_{w>0}\rho(w)$, λ_R is the first eigenvalue of the Robin eigenvalue problem, and λ_S is the first eigenvalue of the Steklov eigenvalue problem (which were introuced in Section 2).

$(\mathrm{C}_p)\ :\ (p+\epsilon)\int_0^u f(s)ds \leq uf(u) + \beta u^p + \gamma,\ u > 0,$

There were several results that used the first eigenvalue to the blow-up conditions (see (2017; 2018; S. -Y. Chung & J. Hwang, 2019)). Especially, the authors (2018) obtained blow-up solutions to the equation (4) for $p \geq 2$, under the Dirichlet boundary condition, by using a condition

for some constants $\epsilon > 0$, $0 < \beta \leq \frac{\epsilon}{p}\lambda_D$, and $\gamma > 0$, where λ_D is the first Dirichlet eigenvalue of the p-Laplace operator.

Using the blow-up conditions $(D_p\,1)$ and $(D_p\,2)$, we obtain the main theorem as follows:

Suppose that the functions f, h, and ρ satisfy the conditions $(D_p\,1)$ and $(D_p\,2)$. Also, the functions F and H are nondecreasing in t. If the initial data u_0 satisfies

$$-\frac{1}{p}\int_\Omega P(|\nabla u_0(x)|^p)dx + \int_\Omega\left[F(x,0,u_0) - \frac{\gamma_1}{\alpha}\right]dx - \frac{1}{p}\int_{\Gamma_1}\theta(z)P(|u_0(z)|^p)dS + \int_{\Gamma_1}\left[H(z,0,u_0) - \frac{\gamma_2}{\alpha}\right]dS > 0,$$

(8)

then every nonnegative solution u to the equation (**??**) blows up in finite time $0 < t^* \leq T$.

It is worthwhile to notice that the condition $(D_p\,1)$ depends on the domain Ω and the boundary conditions, since β_1 and β_2 depend on the first eigenvalues λ_R and λ_S. In fact, it is natural for blow-up conditions to depend on the domain and the boundary conditions.

It is easy to see that the conditions (A_p) and $(A_p)'$ cannot be unified because of the constant ϵ and the parameter p. From this point of view, our condition $(D_p\,1)$, which includes the conditions (A_p), $(A_p)'$, and (C_p), is the most generalized blow-up condition known so far.

We investigate the case $p > 1$, one of our crucial points. As far as the authors know, the case $1 < p < 2$ wasn't discussed concerning the conditions introduced.

Our main results with $p \geq 2$ improve the results known so far. More precisely, the blow-up conditions $(D_p\,1)$ and $(D_p\,2)$ are the generalized version of the conditions introduced in this section such as (A_p), $(A_p)'$, (B_p), $(B_p)'$, and (C_p). We investigate this in Remark 2.4.

Blow-up phenomena

In this section, we deal with the blow-up phenomena of the equation (**??**). From now on, we assume that the solution u is nonnegative on $\overline{\Omega} \times [0, t^*)$. In order to discuss the blow-up solutions to the equation (**??**), we introduce the definition of blow-up solutions as follows:

$\lim_{t\to t^*-} \int_\Omega u^2(x,t)dx = \infty.$

We say that a solution u to the equation (**??**) blows up in finite time $t^* > 0$, if u satisfies

The blow-up condition $(D_p\,1)$ depends on the first eigenvalues λ_R and λ_S. These eigenvalues were introduced in the following lemmas.

$$\begin{cases} -\nabla\cdot[|\nabla\phi_0|^{p-2}\nabla\phi_0] = \lambda_R|\phi_0|^{p-2}\phi_0, & \text{in } \Omega, \\ |\nabla\phi_0|^{p-2}\frac{\partial\phi_0}{\partial n} + \theta(z)|\phi_0|^{p-2}\phi_0 = 0, & \text{on } \Gamma_1, \\ \phi_0 = 0, & \text{on } \Gamma_2. \end{cases}$$

$\lambda_R := \inf_{\substack{w\in\mathcal{A}\\ w\not\equiv 0}} \frac{\int_\Omega |\nabla w|^p dx + \int_{\Gamma_1}\theta(z)|w|^p dS}{\int_\Omega |w|^p dx},$

[See (2002; 2006)] Let $p > 1$. Then there exist $\lambda_R \geq 0$ and a nonnegative function $\phi_0 \in W^{1,p}(\Omega)$ such that

Moreover, λ_R is given by

where $\mathcal{A} := \{w \in W^{1,p}(\Omega) \,|\, w = 0 \text{ on } \Gamma_2\}$.

$$\begin{cases} \nabla \cdot [|\nabla \phi_0|^{p-2} \nabla \phi_0] = |\phi_0|^{p-2}\phi_0, & \text{in } \Omega, \\ |\nabla \phi_0|^{p-2} \frac{\partial \phi_0}{\partial n} + \theta(z)|\phi_0|^{p-2}\phi_0 = \lambda_S |\phi_0|^{p-2}\phi_0, & \text{on } \Gamma_1, \\ \phi_0 = 0, & \text{on } \Gamma_2. \end{cases}$$

$\lambda_S := \inf_{\substack{w \in \mathcal{A} \\ w \not\equiv 0}} \frac{\int_\Omega [|\nabla w|^p + |w|^p] dx + \int_{\Gamma_1} \theta(z)|w|^p dS}{\int_{\Gamma_1} |w|^p dS}$.

[See (2009; 2006)] Let $\Gamma_1 \neq \emptyset$ and $p > 1$. Then there exist $\lambda_S > 0$ and a nonnegative function $\phi_0 \in W^{1,p}(\Omega)$ such that

Moreover, λ_S is given by

where $\mathcal{A} := \{w \in W^{1,p}(\Omega) \,|\, w = 0 \text{ on } \Gamma_2\}$.

Now, we prove Theorem .

A(t):=$\int_\Omega u^2(x,t)dx$

B(t):=-1$\frac{}{p\int_\Omega P(|\nabla u(x,t)|^p)dx + \int_\Omega \left[F(x,t,u(x,t)) - \frac{\gamma_1}{\alpha}\right]dx - \frac{1}{p}\int_{\Gamma_1} \theta(z)P(|u(z,t)|^p)dS + \int_{\Gamma_1}\left[H(z,t,u(z,t)) - \frac{\gamma_2}{\alpha}\right]dS}$

[Proof of Theorem] For a solution $u(x,t)$, we define functions A and B on $[0, t^*)$ by

and

for $t \geq 0$. Firstly, we consider a case $\Gamma_1 \neq \emptyset$. We note that Γ_1 is open subset of $\partial\Omega$. By the boundary condition, we have

$$\begin{aligned}
B'(t) =& \int_\Omega \left[-\rho(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla u_t + f(x,t,u)u_t + \frac{\partial}{\partial t}F(x,t,u)\right] dx \\
&+ \int_{\Gamma_1} \left[-\theta(z)\rho(|u|^p)|u|^{p-2}uu_t + h(z,t,u)u_t + \frac{\partial}{\partial t}H(z,t,u)\right] dS \\
\geq& \int_\Omega -\rho(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\nabla u_t dx + \int_{\partial\Omega} \rho(|\nabla u|^p)|\nabla u|^{p-2}\frac{\partial u}{\partial n}u_t\, dS + \int_\Omega f(x,t,u)u_t dx
\end{aligned} \tag{9}$$

for all $t \in (0, t^*)$. Now, using integration by parts, it follows from (9) and the equation (**??**) that

$$B'(t) \geq \int_\Omega \left[f(x,t,u)u_t + \nabla \cdot \left[\rho(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\right] u_t\right] dx = \int_\Omega u_t^2\, dx \geq 0 \tag{10}$$

for all $t \in (0, t^*)$. On the other hand, we have

$$\begin{aligned}
A'(t) =& 2\int_\Omega [uf(x,t,u) - \rho(|\nabla u|^p)|\nabla u|^p]\, dx + 2\int_{\Gamma_1} [uh(z,t,u) - \theta(z)\rho(|u|^p)|u|^p]\, dS \\
\geq& 2\int_\Omega [\alpha F(x,t,u) - \beta_1 u^p - \gamma_1]\, dx + 2\int_{\Gamma_1} [\alpha H(z,t,u) - \beta_2 u^p - \gamma_2]\, dS \\
&- 2\int_\Omega \rho(|\nabla u|^p)|\nabla u|^p dx - 2\int_{\Gamma_1} \theta(z)\rho(|u|^p)|u|^p\, dS
\end{aligned}$$

$$\begin{aligned}
A'(t) =& 2\int_\Omega uu_t\, dx \\
=& 2\int_\Omega u\left[f(x,t,u) + \nabla \cdot (\rho(|\nabla u|^p)|\nabla u|^{p-2}\nabla u)\right] dx \\
=& 2\int_\Omega [uf(x,t,u) - \rho(|\nabla u|^p)|\nabla u|^p]\, dx + 2\int_{\partial\Omega} u\rho(|\nabla u|^p)|\nabla u|^{p-2}\frac{\partial u}{\partial n}\, dS
\end{aligned}$$

for all $t \in (0, t^*)$. Making use of the condition $(D_p\, 1)$, we obtain from the boundary condition that for all $t \in (0, t^*)$. It follows that

$$A'(t) \geq 2\alpha B(t) + 2\left(\frac{\alpha\delta}{p} - 1\right)\rho_m\left[\int_\Omega |\nabla u|^p dx + \int_{\Gamma_1} \theta(z) u^p dS\right] - 2\beta_1 \int_\Omega u^p dx - 2\beta_2 \int_{\Gamma_1} u^p dS$$

$$\begin{aligned} A'(t) \geq & 2\alpha B(t) + \frac{2\alpha}{p}\int_\Omega P(|\nabla u|^p)dx - 2\int_\Omega \rho(|\nabla u|^p)|\nabla u|^p dx - 2\beta_1 \int_\Omega u^p dx \\ & + \frac{2\alpha}{p}\int_{\Gamma_1} \theta(z) P(|u|^p) dS - 2\int_{\Gamma_1} \theta(z)\rho(|u|^p)|u|^p\, dS - 2\beta_2 \int_{\Gamma_1} u^p dS \end{aligned}$$

Thanks to the condition $(D_p\,2)$, (**??**) implies that

for all $t \in (0, t^*)$, where $\rho_m := \inf_{w>0} \rho(w)$. Applying Lemma and , we obtain that

$\alpha \overline{\scriptstyle 2A'(t)B(t) \leq \frac{1}{4}[A'(t)]^2 \leq \left(\int_\Omega u^2 dx\right)\left(\int_\Omega u_t^2 dx\right) \leq A(t)B'(t)}$

$\mathrm{d}\overline{\scriptstyle dt\left[A^{-\frac{\alpha}{2}}(t)B(t)\right] \geq 0}$

$$\begin{aligned} A'(t) \geq & 2\alpha B(t) - 2\left(\beta_1 + \frac{\beta_2}{\lambda_S}\right)\int_\Omega u^p dx + 2\left[\left(\frac{\alpha\delta}{p} - 1\right)\rho_m - \frac{\beta_2}{\lambda_S}\right]\left[\int_\Omega |\nabla u|^p dx + \int_{\Gamma_1} \theta(z) u^p dS\right] \\ \geq & 2\alpha B(t) + 2\left[\left[\left(\frac{\alpha\delta}{p} - 1\right)\rho_m - \frac{\beta_2}{\lambda_S}\right]\lambda_R - \left(\beta_1 + \frac{\beta_2}{\lambda_S}\right)\right]\int_\Omega u^p dx \\ \geq & 2\alpha B(t) \end{aligned}$$

for all $t \in (0, t^*)$. Considering (10), (**??**), and the initial condition $B(0) > 0$, it is easy to see that $A'(t) > 0$ and $B'(t) > 0$ for all $t \in (0, t^*)$. Therefore, we obtain $A(t) > 0$ and $B(t) > 0$ for all $t \in (0, t^*)$. Now, we use the Schwarz inequality and (**??**) to get

for all $t \in (0, t^*)$. Then it follows that

for all $t \in (0, t^*)$. Then we have

A(t)$\geq \left[\frac{1}{A^{-\frac{\alpha-2}{2}}(0)-\alpha(\alpha-2)A^{-\frac{\alpha}{2}}(0)B(0)t}\right]^{\frac{2}{\alpha-2}}$.

T=A(0)$\frac{}{\alpha(\alpha-2)B(0).}$

$$A^{-\frac{\alpha}{2}}(t)A'(t) \geq 2\alpha A^{-\frac{\alpha}{2}}(t)B(t) \geq 2\alpha A^{-\frac{\alpha}{2}}(0)B(0).$$

(14)

Integrating (14) from 0 to t, we finally obtain

Hence, the solution u blows up at finite time $0 < t^* \leq T$. Furthermore, the upper bound T of the blow-up time satisfies

For a case $\Gamma_1 = \emptyset$, we easily obtain the blow-up solution u because all integral with respect to Γ_1 are 0.

$\alpha > 2,\ \beta_1 + \frac{\lambda_R+1}{\lambda_S}\beta_2 \leq \left(\frac{\alpha}{p} - 1\right)\lambda_R,$

0$\leq \beta_2 \leq \left(\frac{\alpha}{p} - 1\right)\lambda_S.$

f$_t(x,t,u) \geq 0,\ x \in \Omega,\ t > 0,\ u > 0,$

(i) Local existence and regularity of the solutions to the equation (**??**) were discussed in (1993) with some conditions for the functions f, h, and ρ.

(ii) If $\alpha = 2$, then we obtain from (14) that $T = \infty$. i.e. the solution u blows up at $t = \infty$.

(iii) The constant ρ depends on the function ρ, but can be any positive number. If we put $\rho \equiv 1$, then we can choose $\delta = 1$. Then the conditions for the constants α, β_1, and β_2 should be

and

These imply that the condition $(D_p\,1)$ and $(D_p\,2)$ are the generalized version of the conditions (A_p), $(A_p)'$, (B_p), $(B_p)'$, and (C_p).

(iv) We assumed that F and H are nondecreasing in t. This condition is an improvement condition than the following condition:

which was assumed in (2016; 2017; 2018; 2016; 2018). We illustrate this fact in Example .

(v) If we can choose the constants γ_1 and γ_2 of negative values while satisfying the conditions $(D_p\,1)$ and $(D_p\,2)$, then blow-up may occur even in small initial data.

Next, we introduce simple versions of the equation (**??**) and corresponding blow-up results. Firstly, we introduce the following p-Laplacian parabolic equations:
For $p > 1$,

$$\begin{cases} u_t = \nabla \cdot \left[|\nabla u|^{p-2}\nabla u\right] + f(x,t,u), & \text{in } \Omega \times (0,t^*), \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} + \theta(z)|u|^{p-2}u = 0, & \text{on } \Gamma_1 \times (0,t^*), \\ u = 0, & \text{on } \Gamma_2 \times (0,t^*), \\ u(\cdot,0) = u_0 \geq 0, & \text{in } \overline{\Omega}. \end{cases}$$
(15)

We obtain blow-up solutions to the equation (15) as follows:

$(\mathrm{D}_p\,1)\,:\,\alpha F(x,t,u) \leq uf(x,t,u) + \beta_1 u^p + \gamma_1,$

-1 $\frac{}{p\int_\Omega |\nabla u_0(x)|^p dx - \frac{1}{p}\int_{\Gamma_1} \theta(z)|u_0(z)|^p dS + \int_\Omega \left[F(x,0,u_0) - \frac{\gamma_1}{\alpha}\right] dx > 0,}$

Suppose that the function f satisfies the condition $(D_p\,1)$:
For each $x \in \Omega,\ t > 0$, and $u > 0$,

for some constants α, β_1, and γ_1 satisfying $\alpha > 2$ and $\beta_1 \leq \left(\frac{\alpha}{p} - 1\right)\lambda_R$. Also, the function F is nondecreasing in t. If the initial data u_0 satisfies

then every nonnegative solution u to the equation (15) blows up in finite time $0 < t^* \leq T$.

Next, we introduce p-Laplace type parabolic equations under mixed boundary conditions:
For $p > 1$,

$$\begin{cases} u_t = \nabla \cdot \left[\rho(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\right] + f(x,t,u), & \text{in } \Omega \times (0,t^*), \\ B_\rho[u] = 0, & \text{on } \Gamma_1 \times (0,t^*), \\ u = 0, & \text{on } \Gamma_2 \times (0,t^*), \\ u(\cdot,0) = u_0 \geq 0, & \text{in } \overline{\Omega}. \end{cases}$$
(16)

We obtain blow-up solutions to the equation (16) as follows:

$(D_p\,1)\ :\ \alpha F(x,t,u) \leq uf(x,t,u) + \beta_1 u^p + \gamma_1,$
$(D_p\,2)\ :\ \delta v\rho(v) \leq P(v),$

-1 $\overline{p \int_\Omega P(|\nabla u_0(x)|^p)dx - \frac{1}{p}\int_{\Gamma_1}\theta(z)P(|u_0(z)|^p)dS + \int_\Omega\left[F(x,0,u_0) - \frac{\gamma_1}{\alpha}\right]dx > 0,}$

Suppose that the functions f and ρ satisfy the conditions $(D_p\,1)$ and $(D_p\,2)$:
For each $x \in \Omega,\ t > 0,\ u > 0$, and $v > 0$,

for some constants α, β_1, γ_1, and δ satisfying $\alpha > 2$, $\delta > 0$, and $\beta_1 \leq \left(\frac{\alpha\delta}{p} - 1\right)\rho_m\lambda_R$. Also, the function F is nondecreasing in t. If the initial data u_0 satisfies

then every nonnegative solution u to the equation (16) blows up in finite time $0 < t^* \leq T$.

In order to understand the nonlinear mixed boundary conditions, we introduce the following p-Laplace type equations under the nonlinear mixed boundary conditions:
For $p > 1$,

$$\begin{cases} u_t = \nabla \cdot \left[\rho(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\right], & \text{in } \Omega \times (0,t^*), \\ B_\rho[u] = h(z,t,u), & \text{on } \Gamma_1 \times (0,t^*), \\ u = 0, & \text{on } \Gamma_2 \times (0,t^*), \\ u(\cdot,0) = u_0 \geq 0, & \text{in } \overline{\Omega}. \end{cases}$$
(17)

We obtain blow-up solutions to the equation (17) as follows:

$$(D_p\,1)\ :\ \alpha H(z,t,u) \leq uh(z,t,u) + \beta_2 u^p + \gamma_2,$$
$$(D_p\,2)\ :\ \delta v\rho(v) \leq P(v),$$

-1 $\frac{}{p\int_\Omega P(|\nabla u_0(x)|^p)dx - \frac{1}{p}\int_{\Gamma_1}\theta(z)P(|u_0(z)|^p)dS + \int_{\Gamma_1}\left[H(z,0,u_0) - \frac{\gamma_2}{\alpha}\right]dS > 0,}$

Let $\Gamma_1 \neq \emptyset$. Suppose that the functions h and ρ satisfy the conditions $(D_p\,1)$ and $(D_p\,2)$:
For each $z \in \partial\Omega,\ t > 0$, and $u > 0$,

for some constants α, β_2, and γ_2 satisfying $\alpha > 2$ and $0 \leq \beta_2 \leq \left(\frac{\alpha\rho}{p} - 1\right)\rho_m \frac{\lambda_R\lambda_S}{\lambda_R + 1}$. Also, the function H is nondecreasing in t. If the initial data u_0 satisfies

then every nonnegative solution u to the equation (17) blows up in finite time $0 < t^* \leq T$.

Now, we consider non-negative functions or non-positive functions, since there were improved blow-up results when $f \leq 0$ and $h \geq 0$ (see (2014)). We also improve the blow-up condition $(D_p\,1)$ when $F \leq 0$ or $H \leq 0$ in Theorem and Theorem .

$$(D_p\,1)'\ :\ \begin{array}{l} \alpha_1 F(x,t,u) \leq uf(x,t,u) + \beta_1 u^p + \gamma_1, \\ \alpha_2 H(z,t,u) \leq uh(z,t,u) + \beta_2 u^p + \gamma_2, \end{array}$$
$$(D_p\,2)\ :\ \delta v\rho(v) \leq P(v),$$

$\beta_1 + \frac{\lambda_R+1}{\lambda_S}\beta_2 \leq \left(\frac{\alpha_2\delta}{p} - 1\right)\rho_m\lambda_R, \;\; 0 \leq \beta_2 \leq \left(\frac{\alpha_2\delta}{p} - 1\right)\rho_m\lambda_S,$

$2 \leq \alpha_1 \leq \alpha_2$ with $\alpha_2 > 2.$

Let the function F be non-positive. Also, we suppose that the functions f, h, and ρ satisfy the following conditions

for all $x \in \Omega$, $z \in \partial\Omega$, $t > 0$, $u > 0$, and $v > 0$, for some constants α_1, α_2, β_1, β_2, γ_1, and γ_2 satisfying

and

Also, the functions F and H are nondecreasing in t. If the initial data u_0 satisfies

$$-\frac{1}{p}\int_\Omega P(|\nabla u_0(x)|^p)dx+\int_\Omega \left[F(x,0,u_0) - \frac{\gamma_1}{\alpha_2}\right]dx-\frac{1}{p}\int_{\Gamma_1}\theta(z)P(|u_0(z)|^p)dS+\int_{\Gamma_1}\left[H(z,0,u_0) - \frac{\gamma_2}{\alpha_2}\right]dS > 0,$$

then every solution u to the equation (**??**) blows up in finite time $0 < t^* \leq T$.

A(t):=$\int_\Omega u^2(x,t)dx$

$$B(t) := -\frac{1}{p}\int_\Omega P(|\nabla u(x,t)|^p)dx + \int_\Omega \left[F(x,t,u(x,t)) - \frac{\gamma_1}{\alpha_2}\right]dx - \frac{1}{p}\int_{\Gamma_1}\theta(z)P(|u(z,t)|^p)dS + \int_{\Gamma_1}\left[H(z,t,u(z,t)) - \frac{\gamma_2}{\alpha_2}\right]dS$$

B'(t)=$\int_\Omega u_t^2\, dx \geq 0$

$$A'(t) = 2\int_\Omega [uf(x,t,u) - \rho(|\nabla u|^p)|\nabla u|^p]\,dx + 2\int_{\partial\Omega} \rho(|\nabla u|^p)|\nabla u|^{p-2}\frac{\partial u}{\partial n} u\,dS$$

$$\begin{aligned} A'(t) \geq & 2\int_\Omega [uf(x,t,u) - \rho(|\nabla u|^p)|\nabla u|^p]\,dx + 2\int_{\Gamma_1} [uh(z,t,u) - \theta(z)\rho(|u|^p)|u|^p]\;dS \\ \geq & 2\int_\Omega [\alpha_1 F(x,t,u) - \beta_1 u^p - \gamma_1]\,dx + 2\int_{\Gamma_1} [\alpha_2 H(z,t,u) - \beta_2 u^p - \gamma_2]\,dS \\ & - 2\int_\Omega \rho(|\nabla u|^p)|\nabla u|^p dx - 2\int_{\Gamma_1} \theta(z)\rho(|u|^p)|u|^p\,dS \end{aligned}$$

The proof is basically similar to the proof of Theorem . Therefore, we state the main difference of the proof. For a solution $u(x,t)$, we define functions A and B on $[0,t^*)$ by

and

for each $t \geq 0$. Then it follows from (9) and (10) that

for all $t \in (0,t^*)$. On the other hand, we have from (11) that

for all $t \in (0,t^*)$. Making use of the condition $(D_p\,1)$, we obtain from the boundary condition that

for all $t \in (0,t^*)$. Since F is non-positive, we have

$$A'(t) \geq 2\alpha_2 B(t) + 2\left(\frac{\alpha_2\delta}{p} - 1\right)\rho_m \left[\int_\Omega |\nabla u|^p dx + \int_{\Gamma_1} \theta(z)u^p dS\right] - 2\beta_1\int_\Omega u^p dx - 2\beta_2\int_{\Gamma_1} u^p dS$$

$$\begin{aligned} A'(t) \geq & 2\alpha_2 B(t) - 2\left(\beta_1 + \frac{\beta_2}{\lambda_S}\right)\int_\Omega u^p dx + 2\left[\left(\frac{\alpha_2\delta}{p} - 1\right)\rho_m - \frac{\beta_2}{\lambda_S}\right]\left[\int_\Omega |\nabla u|^p dx + \int_{\Gamma_1}\theta(z)u^p dS\right] \\ \geq & 2\alpha_2 B(t) + 2\left[\left[\left(\frac{\alpha_2\delta}{p} - 1\right)\rho_m - \frac{\beta_2}{\lambda_S}\right]\lambda_R - \left(\beta_1 + \frac{\beta_2}{\lambda_S}\right)\right]\int_\Omega u^p dx \\ \geq & 2\alpha_2 B(t) \end{aligned}$$

A(t)$\geq \left[\frac{1}{A^{-\frac{\alpha_2-2}{2}}(0)-\alpha_2(\alpha_2-2)A^{-\frac{\alpha_2}{2}}(0)B(0)t}\right]^{\frac{2}{\alpha_2-2}}$.

T=A(0)$\frac{}{\alpha_2(\alpha_2-2)B(0).}$

$$
\begin{aligned}
A'(t) \geq & 2\alpha_2 B(t) + \frac{2\alpha_2}{p}\int_\Omega P(|\nabla u|^p)dx - 2\int_\Omega \rho(|\nabla u|^p)|\nabla u|^p dx - 2\beta_1\int_\Omega u^p dx \\
& + \frac{2\alpha_2}{p}\int_{\Gamma_1}\theta(z)P(|u|^p)dS - 2\int_{\Gamma_1}\theta(z)\rho(|u|^p)|u|^p\,dS - 2\beta_2\int_{\Gamma_1} u^p dS
\end{aligned}
$$

for all $t \in (0, t^*)$. Thanks to the condition $(D_p\,2)$, (**??**) implies that

for all $t \in (0, t^*)$, where $\rho_m := \inf_{w>0}\rho(w)$. Applying Lemma and , we obtain that

for all $t \in (0, t^*)$. Hence, by similar way to the proof of Theorem , we can easily obtain

Hence, the solution u blows up at finite time $0 < t^* \leq T$. Furthermore, the upper bound T of the blow-up time satisfies

$$
(D_p\,1)' : \begin{array}{l} \alpha_1 F(x,t,u) \leq uf(x,t,u) + \beta_1 u^p + \gamma_1, \\ \alpha_2 H(z,t,u) \leq uh(z,t,u) + \beta_2 u^p + \gamma_2, \end{array}
$$

$$
(D_p\,2) : \delta v\rho(v) \leq P(v),
$$

$\beta_1 + \frac{\lambda_R+1}{\lambda_S}\beta_2 \leq \left(\frac{\alpha_1\delta}{p} - 1\right)\rho_m\lambda_R, \quad 0 \leq \beta_2 \leq \left(\frac{\alpha_1\delta}{p} - 1\right)\rho_m\lambda_S,$

$2 \leq \alpha_2 \leq \alpha_1$ with $\alpha_1 > 2$.

Let the function H be non-positive. Also, we suppose that the functions f, h, and ρ satisfy the following conditions

for all $x \in \Omega$, $z \in \partial\Omega$, $t > 0$, $u > 0$, and $v > 0$, for some constants α_1, α_2, β_1, β_2, γ_1, and γ_2 satisfying

and

Also, the functions F and H are nondecreasing in t. If the initial data u_0 satisfies

$$
\begin{aligned}
&-\frac{1}{p}\int_{\Omega} P(|\nabla u_0(x)|^p)dx + \int_{\Omega}\left[F(x,0,u_0) - \frac{\gamma_1}{\alpha_1}\right]dx \\
&-\frac{1}{p}\int_{\Gamma_1} \theta(z)P(|u_0(z)|^p)dS + \int_{\Gamma_1}\left[H(z,0,u_0) - \frac{\gamma_2}{\alpha_1}\right]dS > 0,
\end{aligned}
$$

then every solution u to the equation (**??**) blows up in finite time $0 < t^* \leq T$.

The proof is basically similar to the proof of Theorem and Theorem . Therefore, one can easily complete this proof by following the proof of Theorem and Theorem .

It is trivial that if $f \leq 0$, then $F \leq 0$. However, $F \leq 0$ does not imply $f \leq 0$.

The following example is given to demonstrate the application of Theorem .

Let a function u be a nonnegative solution to the equation

$$
\begin{cases}
u_t = \nabla\cdot\left(\left(\frac{1}{|\nabla u|}+1\right)\nabla u\right) + 128u^3 + 6u^2|x|t & \\
\quad +(\pi^2-1)u + 3u^2 e^{\max\{0,-(t-9)^3(t-11)^3\}}, & \text{in } \Omega\times(0,t^*),\\
u = 0, & \text{on } \partial\Omega\times[0,t^*),\\
u(x,0) = 1-\sum_{i=1}^4 x_i^2, & \text{in } \Omega.
\end{cases}
\tag{19}
$$

Here, the domain Ω is $\left\{x = (x_1,x_2,x_3,x_4) \mid \sum_{i=1}^4 x_i^2 < 1\right\}$ which is a unit ball of $\mathbb{R}^4$. Let us consider $p = 2$, then it is known that the first eigenvalue $\lambda_{2,0}$ is $\pi^2 - 1$ when the dimension of the unit ball is 4, under the Dirichlet boundary condition (see (1976; 1987)). It follows that $0 \leq \beta_1 \leq \left(\frac{\alpha\delta}{2} - 1\right)(\pi^2-1)\rho_m$. From the equation (19), we have

$$\rho(v) = \frac{1}{v^{\frac{1}{2}}} + 1,$$
$$f(x,t,u) = 128u^3 + (\pi^2 - 1)u + 6u^2|x|t + 3u^2 e^{\max\{0, -(t-9)^3(t-11)^3\}},$$
$$h(z,t,u) = 0.$$

Moreover, we can easily see that the functions ρ and f satisfy the conditions $(D_p\,1)$ and $(D_p\,2)$, by choosing $\alpha = 3$, $\beta_1 = \frac{\pi^2-1}{2}$, $\gamma = 0$, and $\delta = 1$. Also, the functions b and ρ satisfy the conditions which we assumed. Now, we obtain by simple calculations that

$$\begin{aligned} A(0) =& \int_\Omega u_0^2 dx \\ =& 2\pi^2 \int_0^1 \left[(1-r^2)^2\right] r^3\,dr \\ \doteqdot& 0.822 \end{aligned}$$

and

$$\begin{aligned} & B(0) \\ =& -\frac{1}{2}\int_\Omega \left[|\nabla u_0|^2 + 2|\nabla u_0|\right] dx + \int_\Omega \left[32u_0^4 + u_0^3 + \frac{\pi^2-1}{2}u_0^2\right] dx \\ =& -\pi^2 \int_0^1 \left(4r^2 + 4r\right) r^3\,dr \\ & + 2\pi^2 \int_0^1 \left(32(1-r^2)^4 + (1-r^2)^3 + \frac{\pi^2-1}{2}(1-r^2)^2\right) r^3\,dr \\ \doteqdot& 0.193, \end{aligned}$$

since we have from the functions b, ρ, and f that

$$F(x,t,u) = 32u^4 + u^3\left[2|x|t + e^{\max\{0,-(t-9)^3(t-11)^3\}}\right] + \frac{\pi^2-1}{2}u^2,$$
$$P(v) = 2v^{\frac{1}{2}} + v.$$

It follows from Theorem that u blows up in finite time $0 < t^* \leq T$ and

$$T = \frac{A(0)}{3B(0)} \doteqdot 1.420.$$

Differentiating the reaction term $f(x,t,u)$ in Example with respect to u, we can obtain by simple calculation that

$$f_t(x,t,u) = 6u^2|x|$$

for $0 < t \leq 9$ or $t \geq 11$, and

$$f_t(x,t,u) = 3u^2\left[2|x| - 6(t-9)^2(t-10)(t-11)^2 e^{-(t-9)^3(t-11)^3}\right]$$

(20)

for $9 < t < 11$. Then (20) follows that $f_t(x,t,u)$ is negative when $u > 0$ and t satisfies

$$(t-9)^2(t-10)(t-11)^2 e^{-(t-9)^3(t-11)^3} > \frac{|x|}{3}.$$

In fact, if we put $t = 10.5$, then we have from the fact $|x| \leq 1$ that

$$(t-9)^2(t-10)(t-11)^2 e^{-(t-9)^3(t-11)^3}|_{t=10.5} \doteqdot 0.428 \geq \frac{|x|}{3},$$

which implies that $f_t(x,t,u)$ is not nonnegative for all $x \in \Omega$, $t > 0$, $u > 0$.

Conflict of Interests

The authors declares that there is no conflict of interests regarding the publication of this paper.

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