On the completeness of metric spaces with a w-distance

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Abstract

Abstract. A w-distance on a metric space (X,d) is a function $p : X \times X \to [0,\infty)$ which is lower semicontinuous with respect to the second varibale, satisfies the triangle inequality and for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \varepsilon$ for all $x,y,z \in X$. In this short note we prove that a metric space with a w-distance p is complete if and only if every sequence $\{x_i\}$ such that $\sum_{i=1}^{\infty} p(x_i, x_{i+1}) < \infty$ converges.

The notion of w-distance on a metric space was introduced and studied bu Kada *et al.* in [2].

Definition 1. Let (X,d) be a metric space. A w-*distance* on X is a function $p : X \times X \to [0,\infty)$ satisfying the following conditions:

(P1) $p(x,y) \leq p(x,z) + p(z,y)$;

(P2) $p(x,\cdot) : X \to [0,\infty)$ is a lower semicontinuous function for all $x \in X$;

(P3) for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \varepsilon$;

for all $x,y,z \in X$.

We recall that a real-valued function f defined on a metric space (X,d) is lower semicontinuous at a point $x \in X$ if for any sequence $\{x_n\} \subseteq X$ converging to x we have that either $\liminf_{x_n \to x} f(x_n) = +\infty$ or $f(x) \leq \liminf_{x_n \to x} f(x_n)$.

The following lemma shall be used to prove the main result.

Lemma 1. [4] Let (X,d) be a metric space with a w-distance p. If $\{x_n\}$ is a sequence in X such that

$$\lim_{n \to \infty} \sup_{m > n} p(x_n, x_m) = 0$$

then $\{x_n\}$ is Cauchy.

We prove the following statement, which is the main result of this note.

Theorem 1. A metric space (X,d) with a w-distance p is complete if and only if :

(1) every sequence $\{x_i\} \subseteq X$ such that

$$\sum_{i=1}^{\infty} p(x_i, x_{i+1}) < \infty$$

converges to some $x \in X$.

Proof. (⇒:) Let (X,d) be complete, and let $\{x_i\} \subseteq X$ be a sequence such that $\sum_{i=1}^{\infty} p(x_i, x_{i+1}) < \infty$. Then for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $\sum_{i=n}^{\infty} p(x_i, x_{i+1}) < \varepsilon$ for all $n \geq N_\varepsilon$. Hence, for all $m, n \in \mathbb{N}$ such that $m > n \geq N_\varepsilon$ we have

$$p(x_n, x_m) \leq \sum_{i=n}^{m-1} p(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} p(x_i, x_{i+1}) < \varepsilon$$

which implies that $\lim_{n \to \infty} \sup_{m>n} p(x_n, x_m) = 0$, so by Lemma 1, $\{x_i\}$ is Cauchy. Since X is complete, $\{x_i\}$ converges to some $x \in X$.

(⇐:) Now suppose that (1) holds, but X is not complete, so there exists a Cauchy sequence $\{x_i\} \subseteq X$ which is not convergent. Let $F = \{x_i : i \in \mathbb{N}\}$, and let $p : X \times X \to [0, \infty)$ be defined as

$$p(x,y) = \begin{cases} d(x,y), \text{ if } x, y \in F, \\ 2\, diam\, F, \text{ otherwise.} \end{cases}$$

Since $\{x_i\}$ is Cauchy sequence which is not convergent, the set F is closed and bounded, so p is a w-distance on X (see [2, Example 7]). Let i_j be the least natural number such that $p(x_n, x_m) = d(x_n, x_m) \leq \frac{1}{2^j}$ for all $m, n \in$ such that $m > n \geq i_j$. Then we have

$$\sum_{j=1}^{\infty} p(x_{i_j}, x_{i_{j+1}}) \leq \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty$$

which by (1) means that $\{x_i\}$ has a convergent subsequence $\{x_{i_j}\}$ which is impossible (since its limit would be the limit of the whole sequence). □

Remark. In [5] the authors characterized completeness of metric spaces with a w-distance via generalized Banach's contraction, i.e. the weak contraction. In [3] the author of the present paper introduced the functions δ_p and α_p (p-diameter and Kuratowski p-measure of noncompactness) on such spaces and studied the metric completenes via those functions. Hence, we can now formulate our second main result, which represents an analogue of [1, Theorem I.5.1] for metric spaces with a w-distance, and summarizes all known characterizations of completeness for such spaces. For the definition of weak contraction we refer the reader to [5], and for the definitions of δ_p, α_p to [3].

Theorem 2. Let (X,d) be a metric space with a w-distance p. The following conditions are equivalent.

(i) X is complete;

(ii) Every weak contraction on X has a uniqe fixed point;

(iii) Every sequence F_n of nonempty closed subsets in X such that $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \delta_p(F_n) = 0$ has a singleton intersection;

(iv) Every sequence F_n of nonempty closed subsets in X such that $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \alpha_p(F_n) = 0$ has a nonempty compact intersection;

(v) Every sequence $\{x_i\}$ such that $\sum_{i=1}^{\infty} p(x_i, x_{i+1}) < \infty$ converges.

Proof. (i)⇔(ii) is proven in [5, Theorem 4], (i) ⇔ (iii) ⇔ (iv) is proven in [3, Theorem 3.1] and (i) ⇔ (v) is Theorem 1 of the present paper.

References

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