On the affect of the Laplacian in equlibration dynamics of the Spherical Model

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1 Mathematical Analysis

1.1 Motivation for the Model

Consider N nodes in a undirected network passing around a finite resource. Eventually the resource will equilibrate within the network solely due it its finiteness. We wish to study the equilibration timescales of such a problem, where finiteness of the resource and connectedness of the network may be expressed is a variety of ways, giving rise different flavours of the Sherrington-Kirkpatrick spherical model.

Exact dynamical solutions can be obtained if finiteness of the resource is the expressed as a constant L_2 norm across all the nodes. In addition, if the connectedness of the network is known, it may be used to obtain explicit expressions — at least in some limit — for the dynamics of the resource, and hence timescales can be extracted.

The Wigner ensemble is used to generate the connectivity in an attempt to minimise a priori assumptions on it. Then intorducing the M-dimensional periodic lattice laplacian allows the study of the model as it departs from randomness and gains spatial structure. Is it possible to extract the dimension of the manifold that the nodes lie on simply from observed correlations and equilibration timescales in the resource? This work may have applications in dimensionality reduction methods.

1.2 Spherical Sherrington-Kirkpatrick Model

Let the resource across N nodes be represented by vector $s(t) \in \mathbb{R}^N$, and the weighted undirected connections between them be a Hermitian $N \times N$ matrix $\mathbf{J}$. This matrix represents the rate at which the resource is passed between nodes. Without finiteness, we can solve the linear equations, revealing that the resource at different nodes would diverge or go to zero depending on the sign of the eigenvalues of $\mathbf{J}$. This does not seem reasonable.

Therefore a lagrange multiplier $\mu(t) \in \mathbb{R}$ is introduced which holds the L_2 norm of vector $s(t)$ constant. This implicitly introduces a nonlinearity. The dynamics is cast as a Langevin Equation with white noise $\xi(t) \in \mathbb{R}^N$ which is characterised by its moments $\langle \ldots \rangle$. Eventually the resource will equilibrate due it its finiteness, subject to the amplitude of the noise. The symbol $\mathbb{1}$ represents the diagonal identity.

$$\partial_t s = \big[\mathbf{J} - \mathbb{1}\mu(t)\big] s + \xi \tag{1.1}$$

$$\langle \xi(t) \rangle = \mathbb{0} \qquad \langle \xi(t)\xi(t') \rangle = 2T\mathbb{1}\delta(t-t') \tag{1.2}$$

$$\text{where} \quad \mu(t) = \frac{1}{N} s^\top (\mathbf{J}s + \xi) \quad \text{enforces constraint} \quad s^\top s = N \tag{1.3}$$

1.2.1 Dynamical Properties

It is possible to solve the inhomogenous time-dependent ordinary linear system of differential equations for $s(t)$ using integrating factor method and rotate into the eigenbasis $\sigma_\lambda(t) \in \mathbb{R}$ of matrix $\mathbf{J}$. This does not yield a closed form for $\mu(t)$ however. Following the derivation by Cugliandolo, L. F. and Dean D. S. [] yields the moments explicitly for the uniform initial condition $\sigma_\lambda(0) = 1$. Here the complexity lies in performing the inverse laplace transform $\mathcal{L}^{-1}$ which in turn depends on the eigenvalue distribution $p(\lambda \in \mathbf{J})$.

$$\langle \sigma_\lambda(t) \rangle = \frac{e^{\lambda t}}{\sqrt{\Gamma(t)}} \qquad \begin{aligned} \langle s(t)^\top s(t') \rangle = \frac{N}{\sqrt{\Gamma(t)\Gamma(t')}} \Bigg(& \int_{\mathbb{R}} p(\lambda \in \mathbf{J}) e^{\lambda(t+t')} d\lambda \\ & + 2T \int_{\mathbb{R}} p(\lambda \in \mathbf{J}) \int_0^{\min(t,t')} \Gamma(\tau) e^{\lambda(t+t'-2\tau)} d\tau d\lambda \Bigg) \end{aligned} \tag{1.4}$$

$$\Gamma(t) = \sum_{k=0}^{\infty} (2T)^k \mathcal{L}^{-1}\left[\Phi(s)^k\right] \quad \text{where} \quad \Phi(s) = \int_{\mathbb{R}} \frac{p(\lambda \in \mathbf{J})}{s - 2\lambda} d\lambda \tag{1.5}$$

1.2.2 Equilibrium Properties

If the Langevin dynamics can be written as the gradient of some potential $V(s)$ then the steady state is distrubted according to a Gibbs measure of inverse temperature β with partition function $Z_N(\beta)$. Indeed it is possible to write down $V(s)$ without the constraint enforced by $\mu(t)$; instead it is included via a static lagrange multiplier μ with an average self-consistency condition.

$$Z_N(\beta) = \int_{\mathbb{R}^N} \mathrm{e}^{-\beta V(s) - \mu\, s^\top s}\, \mathrm{d}s \qquad \text{where} \quad \left\langle s^\top s \right\rangle = -\partial_\mu \ln Z_N(\beta) \qquad V(s) = -\frac{1}{2} s^\top \mathbf{J} s \tag{1.6}$$

Rotating into the eigenbasis σ_λ leaves the integral invariant and allows factorisation

$$Z_N(\beta) = \prod_{\lambda \in \mathbf{J}} \int_{\mathbb{R}} \mathrm{e}^{-(\mu - \frac{\lambda\beta}{2})\sigma_\lambda^2}\, \mathrm{d}\sigma_\lambda \tag{1.7}$$

$$= \mathrm{e}^{-\frac{1}{2}\sum_{\lambda\in\mathbf{J}} \ln\left(\frac{\mu - \lambda\beta/2}{\pi}\right)} \qquad \mu > \frac{\Lambda\beta}{2} \quad \text{where} \quad \Lambda = \max_{\lambda\in\mathbf{J}}\{\lambda\} \tag{1.8}$$

The average self-consistency condition $\left\langle s^\top s \right\rangle = N$ for solves for μ with an integral over the density $p(\lambda \in \mathbf{J})$. For high temperatures $\beta \searrow 0$ the singularity in the integrand $\lambda^* \nearrow \infty$ moves out of the bounds of support of $p(\lambda \in \mathbf{J})$, which suggests that the occupation of modes λ at equilibrium is given only by the eigenvalue distirbution. As the temperature approaches $\beta \nearrow 2\mu/\Lambda$ the singularity reaches the upper bound of the density $\lambda^* \searrow \Lambda$. At this critical value the singularity weights the mode λ^* macroscopically. This phase transition has the flavour of Bose condensation.

$$\left\langle s^\top s \right\rangle = \frac{N}{\beta} \int_{\mathbb{R}} \frac{p(\lambda \in \mathbf{J})}{\frac{2\mu}{\beta} - \lambda}\, \mathrm{d}\lambda \tag{1.9}$$

From this it is possible to determine the occupation at any given mode λ which is the same as the moment $\langle \sigma_\lambda(t) \rangle$ in the infinite time limit.

$$\lim_{t\to\infty} \langle \sigma_\lambda(t) \rangle = N^{1/2} \sqrt{1 - \frac{1}{\beta} \int_{\mathbb{R}} \frac{p(\lambda' \in \mathbf{J})}{\lambda - \lambda'}\, \mathrm{d}\lambda'} \tag{1.10}$$

1.3 Hermitian Wigner Ensemble

Suppose the connections between nodes are given by a random matrix $\mathbf{H}$ taken from the Hermitian Wigner ensemble, whos upper triangle elements $[\mathbf{H}]_{ij}$ are distributed according to a density $\mathbb{P}$ with subexponential tails [] with expectation $\mathbb{E}$ and variance $\mathbb{V}$, such that in the limit $N \to \infty$ the eigenvalue spectrum converges to the semicircle law $p(\lambda \in \mathbf{H}) \to \cap(\lambda|J)$. Below the symbols $\mathbf{0}$ and $\mathbf{1}$ denote constant matrices of zeros and ones respectively.

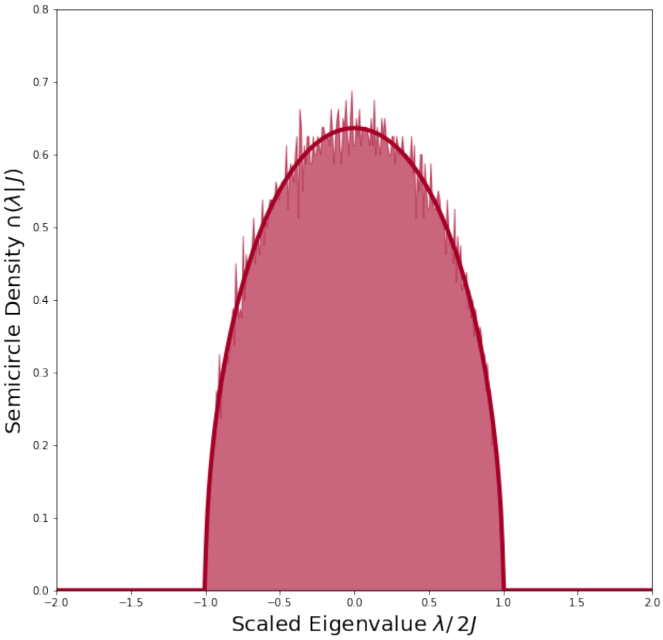

Figure 1: Empirical histogram of wigner ensemble eigenvalues and analytical semicircle distribution $\cap(\lambda|J)$, demonstrating the corresponance of the law in the $N \to \infty$ limit

$$\cap(\lambda|J) = \frac{\Pi(\lambda/2J)}{2\pi J^2}\sqrt{4J^2 - \lambda^2} \qquad \Pi(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geq 1 \end{cases} \tag{1.11}$$

$$\mathbb{E}[\mathbf{H}] = \mathbf{0} \quad \mathbb{V}[\mathbf{H}] = \frac{J^2}{N}(\mathbf{1} + \mathbb{1}) \tag{1.12}$$

$$\mathbb{P}\left(t^\alpha \leq |[\mathbf{H}]_{ij}|\right) \leq e^{-t} \quad \forall t \geq \alpha, \forall i, j \tag{1.13}$$

The Heaviside Pi function $\Pi(x)$ bounds the distirbution. Note the requirement on the distribution of the elements $[\mathbf{H}]_{ij}$ is quite general and allows for the choice of a sparse random matrix. This is useful when optimising the numerical integration of the equations of motion; in particular sparse memory access allows for efficient implementation on graphics processing units [].

If the spherical model has completely random interations the semicircle distribution (1.11) can be integrated in to obtain $\Phi(s)$. To match the terms of the expansion for $\Gamma(t)$ to known inverse laplace transforms some algebraic trickery is required: the infinite sum is evaluated first. Then the result is expanded in terms the denominator part that is a function of s. Only then do the terms match the laplace transforms of modified Bessel functions of the first kind $\frac{I_l(4Jt)}{t}$. These have well-known asymtotics for $t \to \infty$. Finally the moments (1.4) can be numerically evaluated as shown in Figure 2.

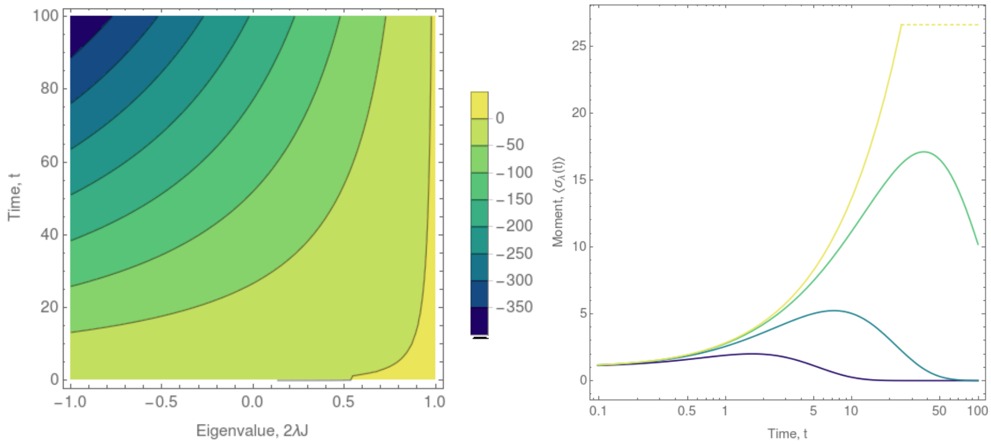

Figure 2: Moments $\langle \sigma_\lambda(t) \rangle$ in the λ, t eigenvalue time plane for a given temperature $T < J$. The contours are on a log scale, revealing that only the maximal eigenvalue component does not decay to zero

$$\Phi(s) = \frac{s - \sqrt{s^2 - 16J^2}}{8J^2} \qquad \Gamma(t) = \frac{1}{2T} \sum_{l=1}^{\infty} l \left(\frac{T}{J}\right)^l \frac{I_l(4Jt)}{t} \tag{1.14}$$

$$\implies \langle \sigma_\lambda(t) \rangle \asymp t^{3/4} \mathbb{e}^{(\lambda - 2J)t} \left(1 - \frac{T}{J}\right) \tag{1.15}$$

Evaluating (1.10) onto the maximal eigenvalue Λ at equilibrium undergoes phase transition at $T = J$. For sufficiently high temperatures the noise washes out any preferred equilibration direction, otherwise the maximal eigenvalue dominates the dynamics as expected.

$$\lim_{t \to \infty} \langle \sigma_\Lambda(t) \rangle = \begin{cases} N^{1/2} \sqrt{1 - \frac{T}{J}} & T < J \\ 0 & T > J \end{cases} \tag{1.16}$$

Equating the asymtotic $t \to \infty$ dynamic moment (1.15) with the value at equilibrium in the thermodynamic limit $N \to \infty$ allows extraction of the equilibration timescale τ. Evidently, this time grows with N so in a large enough system equilibrium is never achieved at any finite time.

$$\tau \asymp \left(\frac{N}{1 - T/J}\right)^{2/3} \tag{1.17}$$

1.4 Discrete Laplacians

Using the Circular Diagonalization Theorem [] one can derive the eigenvalues $\lambda_N(k)$ of an $N \times N$ matrix $\mathbf{X}$ which represents the second-order central difference approximation to the second derivative along N sites of a one dimensional ring.

$$\mathbf{X} := \begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{pmatrix} \tag{1.18}$$

$$\begin{gathered} \lambda_N(k) = 2\left(\cos\left(\tfrac{2\pi k}{N}\right) - 1\right) \\ k \in \{0, 1, \cdots, N-1\} \end{gathered} \tag{1.19}$$

As the number of sites $N \to \infty$ the argument $k/N \in [0, 1]$ and the eigenvalues remain bounded $-2 < \lambda < 0$. By shifting and scaling the index $k \to \frac{k - N\pi}{2\pi}$ the eigenvalues are expressed as the familiar dispersion relation [].

$$\lambda(x) = -2\left(\cos x + 1\right) \quad x \in [-\pi, \pi] \tag{1.20}$$

The discrete M-dimensional laplacian is simply the kronecker sum of one dimensional cases $\mathbf{\Delta} = \mathbf{X} \oplus \mathbf{X} \oplus \cdots \oplus \mathbf{X}$ and thus its eigenvalues is simply the sum one dimensional dispersions [].

$$\lambda(\bar{\mathbf{x}}) = -2 \sum_{x \in \bar{\mathbf{x}}} \left(\cos x + 1\right) \quad \bar{\mathbf{x}} \in [-\pi, \pi]^M \tag{1.21}$$

The probability density $\Delta_M(\lambda)$ can be expressed as an integral over the M-dimensional hypercube region $\Omega = [-\pi, \pi]^M$ in complete analogue with the density of states.

$$\Delta_M(\lambda') = \frac{1}{Z_M} \int_\Omega \delta(\lambda' - \lambda(\bar{\mathbf{x}}))\, \mathrm{d}\bar{\mathbf{x}} \tag{1.22}$$

We proceed with an element-wise change of variables $\bar{\mathbf{u}} = 2\cos\bar{\mathbf{x}}$ and recognise that the integration region is M-fold symmetric across each component axis, which allows restriction of the domain of integration to a hyperoctant. In coordinates $\bar{\mathbf{u}}$ the region becomes $\Omega' = [-2, 2]^M$.

$$\begin{aligned}
\Delta_M(\lambda) &= \frac{1}{Z_M} \int_{\Omega'} \frac{\delta(\Lambda_M + \sum_{u\in\bar{\mathbf{u}}} u)}{\sqrt{\prod_{u\in\bar{\mathbf{u}}}(1-u^2/4)}}\, \mathrm{d}\bar{\mathbf{u}} \qquad & \Lambda_M = \lambda + 2M \\
& & |\Lambda_M| \leq 2M \\
&= \frac{1}{2\pi Z_M} \int_{-\infty}^{\infty} \int_{\Omega'} \frac{\mathrm{e}^{\Lambda_M \mathrm{i} k} \exp[\sum_{u\in\bar{\mathbf{u}}} u\mathrm{i}k]}{\sqrt{\prod_{u\in\bar{\mathbf{u}}}(1-u^2/4)}}\, \mathrm{d}\bar{\mathbf{u}}\mathrm{d}k \\
&= \frac{1}{2\pi Z_M} \int_{-\infty}^{\infty} \mathrm{e}^{\Lambda_M \mathrm{i}k} \prod_{u\in\bar{\mathbf{u}}} \int_{-2}^{2} \frac{\mathrm{e}^{u\mathrm{i}k}}{\sqrt{1-u^2/4}}\, \mathrm{d}u\mathrm{d}k
\end{aligned}$$

The fourier representation of the delta function allowed the factorisation of the integral. We recognise a repeated Bessel integral and replace it with the Bessel function of the first kind $J_n(k)$, leaving only a fourier transform which we define $\mathcal{F} : f \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k) e^{\mathrm{i}\Lambda k} \mathrm{d}k$. To clean the formula up even further we may use the convolution theorem to deal with the powers of M, leaving only the fourier transform of the Bessel function $J_0(k)$, which is the arcsine distribution $\alpha(\lambda)$. It becomes clear that the eigenvalue density of a kronecker sum of matrices is the convolution of the densities of those matrices.

$$\Delta_M(\lambda) = \underbrace{\alpha(\lambda) * \alpha(\lambda) * \cdots * \alpha(\lambda)}_{M} \tag{1.23}$$

$$\alpha(\lambda) = \frac{\Pi\left(\frac{\lambda+2}{2}\right)}{2\pi\sqrt{1-\left(\frac{\lambda+2}{2}\right)^2}} \qquad \Pi(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geq 1 \end{cases} \tag{1.24}$$

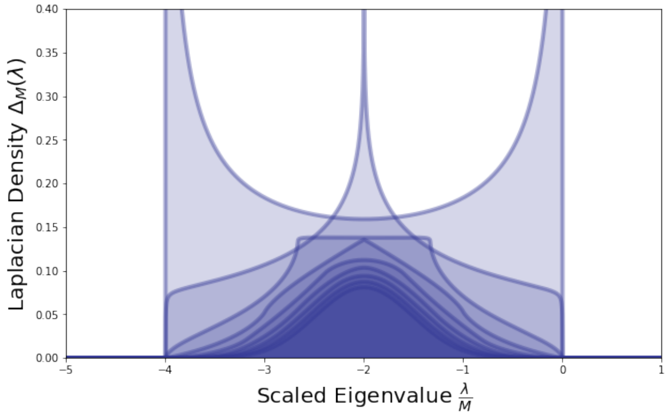

Figure 3: Eigenvalue distributions $\Delta_M(\lambda)$ of the M-dimensional Laplacian

The one dimensional density has two Van Hove singularities at $\lambda = -4, 0$ given by the arcsine law $\alpha(\lambda)$, whereas the two dimensional case has one at $\lambda = -4$ given by the complete elliptic integral of the first kind $K(m)$. Figure 3 reveals that in higher dimensions singularities do not occur; instead there appear to be discontinuities in the higher order derivatives. The density smooths out as repeated convolutions bring it to a normal distrubtion; this is another way to state the Central Limit Theorem [].

In the context of an equation of motion, the laplacian often is scaled by a diagonal matrix, which may represent anisotropic diffusion. To take this scaling into account the parameter D is introduced in the arcsine law.

$$\alpha(\lambda|D) = \frac{\Pi\left(\frac{\lambda/D+2}{2}\right)}{2\pi D\sqrt{1-\left(\frac{\lambda/D+2}{2}\right)^2}} \tag{1.25}$$

Substituting this limiting density (1.25) into the second equation in (1.5) the inverse laplace transform of the powers can be performed immediately, this time giving non-integer order modified Bessel functions of the first kind $I_{\frac{l-2}{2}}(4Dt)$, which have the same asymptotics as in the wigner ensemble case.

$$\Phi(s) = \frac{1}{\sqrt{s}\sqrt{s+8D}} \qquad \Gamma(t) = 2Te^{-4Dt}\sum_{l=1}^{\infty}\frac{(\frac{1}{2}-1)!}{(\frac{l}{2}-1)!}\left(\frac{T^2t}{2D}\right)^{\frac{l-2}{2}} I_{\frac{l-2}{2}}(4Dt) \tag{1.26}$$

$$\implies \langle \sigma_\lambda(t) \rangle \asymp \sqrt{\frac{D}{T^2}} \mathbb{e}^{\left(\lambda - \frac{T^2}{4D}\right)t} \tag{1.27}$$

Since all eigenvalues are negative $\lambda \leq 0$ the moments exponentially decay to zero, regardless of the values of T and D which is what would be expected any spectrum with strictly negative eigenvalues.

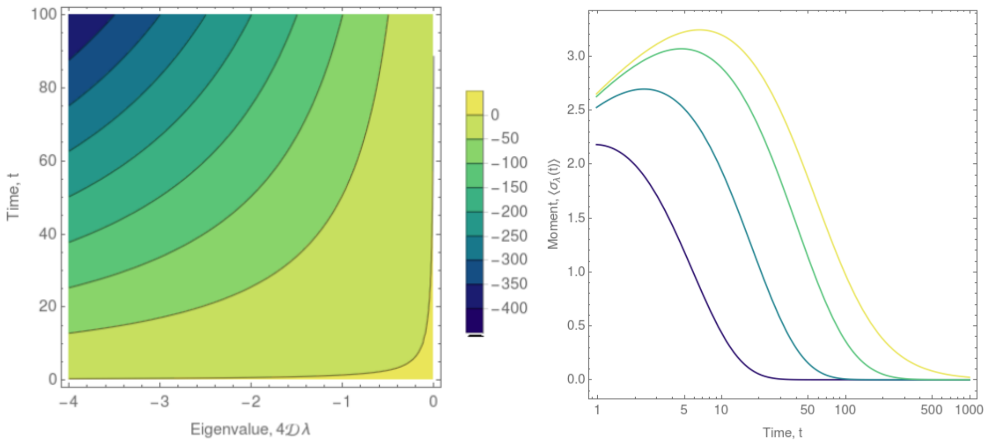

Figure 4: Moments $\langle \sigma_\lambda(t) \rangle$ in the λ, t eigenvalue time plane. The contours are on a log scale, revealing that all eigenvalue components decay to zero

Analysis on equilibrium values for $\langle \sigma_\lambda \rangle$ and extraction of timescale τ like in previous section would go here...

1.5 Wigner Laplacian Matrices

A Wigner Laplacian is defined as the sum of a Wigner matrix of scaling J and an M-dimensional laplacian of isotropic scaling D. In the formalism of Free Probability it is possible to express the $N \to \infty$ limiting eigenvalue density ρ of a sum of matrices as the free convolution $\boxplus$ the individual limiting compact densities $\cap(\lambda|J)$ and $\mu(\lambda|D)$ [1].

$$\rho(\lambda|J, D) = \cap(\lambda|J) \boxplus \mu(\lambda|D) \tag{1.28}$$

Unfortunately the general definition of this operation is rather implicit. The expression can be unpacked in terms of an infinum [2] and regular convolution $*$ for the specific case of free convolution of semicircle distrubtion $\cap(\lambda|J)$.

$$\rho(\lambda|J,D) = \frac{1}{\pi J^2} \inf_{\rho\in[0,\infty)} \{\varepsilon(\lambda,\rho|J,D) \geq 0\} \tag{1.29}$$

$$\varepsilon(\lambda,\rho|J,D) = 1 - J^2\mu(\lambda|D) * \gamma(\lambda,\rho) \qquad \text{where} \quad \gamma(\lambda,\rho) = \frac{1}{\lambda^2+\rho^2} \tag{1.30}$$

Substituting the distribution $\mu(\lambda|D)$ for the M-dimensional laplacian (1.23) with isotropic diffusion D yields an expression for the contours $\varepsilon(\lambda,\rho|J,D)$ as the repeated convolution of arcsine laws (1.25) with $\gamma(\lambda,\rho)$. Numerically this is efficiently evaluated in the fourier domain.

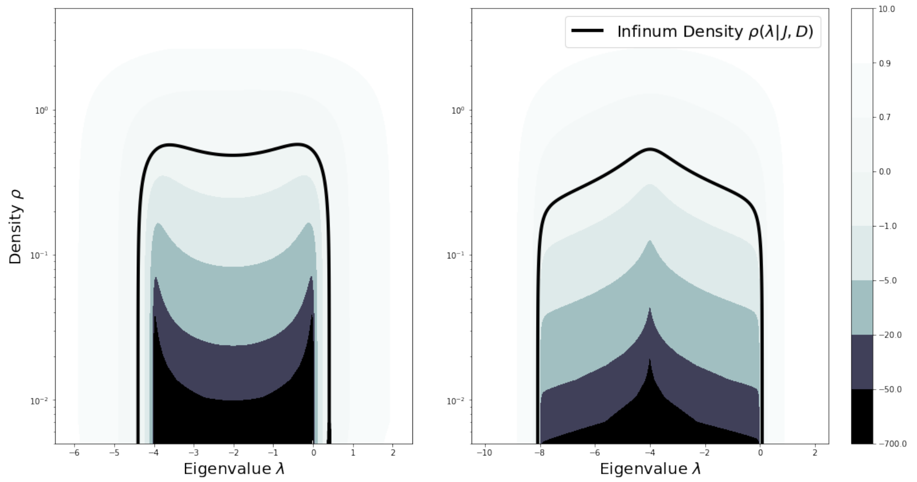

Figure 5: Left/Right: Numerical evaluation of the contours $\varepsilon(\lambda,\rho|J,D)$ for the one/two dimensional laplacian. The infinum picks out the contour at the value zero shown as a black line for given values of $J,D=1$

The infinum minimises the value of the contour function $\varepsilon(\lambda,\rho|J,D)$ in the domain $\rho\in[0,\infty)$ and $\varepsilon\geq 0$ and thus picks out the contour at $\varepsilon(\lambda,\rho|J,D)=0$. This becomes a transcendental equation for ρ, which can be expressed as a single integral containing the fourier transfrom of $\gamma(\lambda,\rho)$ and the M-th power of the zeroth order Bessel function of the first kind $J_0(k)$.

$$\int_{-\infty}^{\infty} \left(J_0(k)\mathrm{e}^{\mathrm{i}k}\right)^M \mathrm{e}^{\frac{\mathrm{i}k\lambda-|k|\rho}{2D}}\,\mathrm{d}k = \frac{2\rho}{J^2} \tag{1.31}$$

Looking at the functional form of the above equation indicates that different choices for J,D amount to shifting and scaling such that the infinum picks out neighbouring contours. Indeed in Figure 5 an approach to the semicircle and convoluted arcsine laws can be seen for positive and negative contours respectively.

Figure 6 reveals the corresponance between the density given by the transcendental equation (1.31) and empirical eigenvalue histograms of Wigner Laplacians. The analytical density had to be renormalised such that its integral is one. While the shifting of the bounds of the distribution towards negative eigenvalues matches well, it appears that there is an error in scaling which manifests as a discrepency at intermediate values of J and D between empirical and theoretical densities.

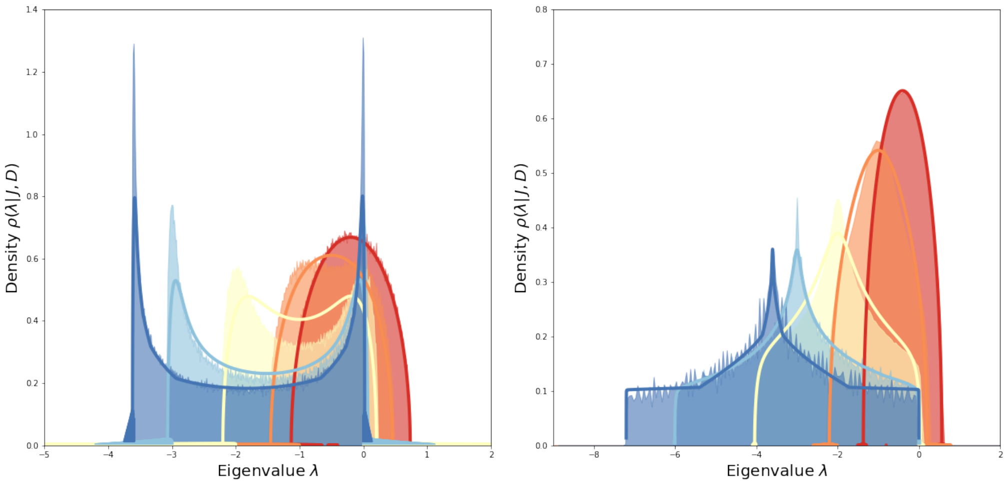

Figure 6: Left/Right: Eigenvalue density of one/two dimensional Wigner Laplacian $\rho(\lambda|\, J, D)$ for parameter values $D \in (0, 1)$ and $J = 1 - D$

Numerical evaluation of integrals in (1.5) should follow here to determine moments. Expecting to observe crossover region where equilibrium is reached in finite time for sufficiently large D...

References

[1] J. Novak, "Three Lectures on Free Probability," 2012.

[2] P. Biane, "On the free convolution with a semi-circular distribution," *Indiana University Mathematics Journal*, vol. 46, no. 3, pp. 705–718, 1997.