The Search for Chaos and Species Response in Lotka-Volterra Systems

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Abstract

An attempt at a computational model built using the programming language Python in order to study predator prey systems, and specifically how chaos in those systems ensues. Varying parameters within a predator prey system can prelude the result of unpredictable chaotic behavior. Using generated plots of population and growth dynamics over time, trends in whether a species rises to survive or vanquish will become visually malleable, while demonstrating how altered parameters may cause one species to affect another. The search for chaos resulted in the search for a point at which chaos might actually be enabled to grow, which proved to be rare and difficult to find and led to the search for oscillations in a multi-species system rather than chaos.

Introduction

The Lotka-Volterra equations were created independently by Alfred Lotka and Vito Volterra in 1925 and 1926 respectively. Alfred Lotka (1880-1949) was a chemist, demographer, ecologist, and mathematician. Vito Volterra (May 3, 1860-October 11, 1940) was a mathematician and physicist. Initially the equations were inspired by Lotka's interest in chemical reactions and how chemical concentrations oscillate. At nearly the same time Volterra derived differential equations in the same form to study predator fish populations in the Adriatic Sea during World War I. Both Lotka and Volterra have inspired applications of mathematics for analyzing predator-prey systems, along with a great amount of literature on the subject [2].

Materials and Methods

Computational exercise of the Lotka-Volterra equations were done by methods of a differential equation solver using the programming language Python on the interface Spyder (Python 3.5). The solver chosen was scipy.integrate.odeint. The requirements to run odeint are some decided initial conditions, a defined function of the interactions between species, and an input of constant parameters within said species functions.

In order for a predator-prey system to become chaotic it is required that at least 4 species be involved in what could be an N-species system, as can be described by the general form of the Lotka-Volterra coupled differential equations [1].

N-species Lotka-Volterra Equation, where r represents the growth rate of a species, x is an initial population number, α is the effect a species feels from another species, and K is the carrying capacity of a species (the limit to how large a species can grow based on a number of natural factors, like available water in a given area), are as follows, with first the general form,

$$\frac{dx_i}{dt} = r_i x_i \left(1 - \frac{\Sigma_{j=1}^{N} \alpha_{ij} x_j}{K_i}\right),$$

and then for a 4 species system,

$$\begin{aligned}
\frac{dx_1}{dt} &= r_1 x_1 \left(1 - \frac{(x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 + \alpha_{14}x_4)}{K_1}\right) \\
\frac{dx_2}{dt} &= r_2 x_2 \left(1 - \frac{(x_2 + \alpha_{21}x_1 + \alpha_{23}x_3 + \alpha_{24}x_4)}{K_2}\right) \\
\frac{dx_3}{dt} &= r_3 x_3 \left(1 - \frac{(x_3 + \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{34}x_4)}{K_3}\right) \\
\frac{dx_4}{dt} &= r_4 x_4 \left(1 - \frac{(x_4 + \alpha_{41}x_1 + \alpha_{42}x_2 + \alpha_{43}x_3)}{K_4}\right).
\end{aligned}$$

Results

After taking a closer look into how to approach the search for the finding initial conditions and their associated parameters, it became reasonable to cut down the number of species from 4 to just 2 species for the sake of having a smaller area to search within. For a 4 species system there would've been 20 total parameters and 4 initial conditions to determine, and after reducing to just a 2 species system the amount of parameters became only 6. After fixing the carrying capacity $\mathrm{K_i}$ and the growth rates $\mathrm{r_i}$, only 2 parameters were left to determine at the very least. Reducing the amount of parameters to search for was important because the methods used to find them involved a series of nested for loops, each loop running a range of values to test independently each parameter in scipy.optimize.root, a root solver in Python. Searching for α_{ij} alone for a 4 species system, testing 10 different values independently for each α_{ij} corresponded to running the root solver 10^{12} times, which is far too much, and would've become increasingly more had other parameters decided to have been varied as well. With just a 2 species system the amount of calculations for the computer were greatly reduced, even while varying the growth rates ($\mathrm{r_i}$), effects species have on one another (α_{ij}), and carrying capacites ($\mathrm{K_i}$), with i = 1, 2, 3, etc. for species # i. The resulting Lotka Volterra equations thus became as follows.

$$\frac{dx_1}{dt} = r_1x_1 - \frac{r_1x_1^2}{K_1} - \frac{r_1x_1x_2\alpha_{12}}{K_1}$$
$$\frac{dx_2}{dt} = r_2x_2 - \frac{r_2x_2^2}{K_2} - \frac{r_2x_1x_2\alpha_{21}}{K_2}$$

Since reducing the amount of species in the system modeled from 4 to 2, the search switched to finding the right parameters that would allow species populations to oscillate around each other instead of determining parameters that would allow for chaos, as determined by many others in the past, 4 species are required for a system of this nature to be chaotic[1]. While solutions for initial populations were found, along with their associated parameters, all solutions tested in odeint only showed populations that would either die, exponentially grow out of control, or stay constant. It appears that finding solutions that will just oscillate continuously around an equilibrium point after a small perturbation is a rare event. It was never resolved as to how to create species that interact with each other just enough to push and pull on one others populations, but after spending many hours searching for parameters and initial population conditions, it became apparent that the real work in modeling lies in coming up with ways to systematically and efficiently cycle through test values that would yield candidates with the correct attributes, while allocating those values in an organized way to further be experimented on through the Lotka Volterra model itself. Creating a model is not impossible, but deciding the right values to pass through that model is where the challenge resides. Further refinement of the parameter search procedure, and a larger scope for which to hunt in may one day yield the correct values to demonstrate an oscillatory system, although inevitably the larger the scope to look within, the more time required for the computer to perform its calculations and typically magnitudes more calculations turns into much, much more time to twiddle thumbs.

References

[1] Chaos in Low-Dimensional Lotka-Volterra Models of Competition. http://sprott.physics.wisc.edu/chaos/lvmodel/chaos.pdf. Accessed on Fri, May 18, 2018.

[2] Predator-prey model - Scholarpedia. http://www.scholarpedia.org/article/Predator-prey$_{m}odel. AccessedonFri, May18, 2018.$