A FURTHER GENERALIZATION OF THE CATALAN NUMBERS AND ITS EXPLICIT FORMULA AND INTEGRAL REPRESENTATION

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Abstract. In the paper, motivated by the generating function of the Catalan numbers in combinatorial number theory and with the aid of Cauchy's integral formula in complex analysis, the authors generalize the Catalan numbers and its generating function, establish an explicit formula and an integral representation for the generalization of the Catalan numbers and corresponding generating function, and derive several integral formulas and combinatorial identities.

Contents

1. Backgrounds and motivations

This paper is a continuation of the articles [19, 41].

The Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)} \tag{1.1}$$

form a sequence of integers (see [10, 11, 55]), can be interpreted combinatorially (see [6, 16, 52]), date back to the year 1730 (see [17, 18, 23]), and can be generated (see [26, 52, 56]) by

$$G(x) = \frac{2}{1+\sqrt{1-4x}} = \sum_{n=0}^{\infty} C_n x^n, \tag{1.2}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t, \quad \Re(z) > 0$$

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or

$$\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{\prod_{k=0}^{n}(z+k)},\quad z\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$$

is the classical Euler gamma function (see [1, Chapter 6], [27, Chapter 5], [57, Chapter 3], and [35]).

The Catalan numbers C_n have been combinatorially generalized as the Fuss numbers (see [8] and [16, pp. 377–378]), the Fuss–Catalan numbers (see [3, 5, 7, 54] and [16, Exercise A14, p. 108]), and others (see [2, 13, 15] and [16, pp. 375–376]).

Motivated by the second expression in (1.1), several mathematicians analytically generalized the Catalan numbers C_n, the Fuss numbers, and the Fuss–Catalan numbers and investigated plenty of their properties in the papers [22, 25, 29, 30, 31, 32, 33, 34, 36, 37, 38, 39, 40, 42, 45, 46, 47, 48, 49, 50, 51, 53, 59] and closely related references.

Motivated by the generating function $G(x)$ in (1.2), several mathematicians generalized the Catalan numbers C_n to $\mathcal{C}_n(a,b)$ in [19, 41] by considering the generating function

$$G_{a,b}(x)=\frac{1}{a+\sqrt{b-x}}=\sum_{k=0}^{\infty}\mathcal{C}_n(a,b)x^n \tag{1.3}$$

for real numbers $a\ge0$ and $b>0$. It is clear that $\mathcal{C}_n\left(\frac{1}{2},\frac{1}{4}\right)=C_n$. The main results in [19, 41] are as follows.

(1) The Catalan–Qi numbers of the second kind $\mathcal{C}_n(a,b)$ for $n\ge0$, $a\ge0$, and $b>0$ can be explicitly computed by

$$\mathcal{C}_n(a,b)=\frac{1}{(2n)!!b^{n+1/2}}\sum_{k=0}^{n}\binom{2n-k-1}{2(n-k)}\frac{k![2(n-k)-1]!!}{\left(1+a/\sqrt{b}\right)^{k+1}}, \tag{1.4}$$

where $(-1)!!=1$.

(2) The principal branch of the generating function $G_{a,b}(z)$ for $a\ge0$ and $b>0$ can be represented by

$$G_{a,b}(z)=\frac{1}{a+\sqrt{b-z}}=\frac{1}{\pi}\int_0^\infty\frac{\sqrt{t}}{a^2+t}\frac{1}{b+t-z}\mathrm{d}t,\quad z\in\mathbb{C}\setminus[b,\infty). \tag{1.5}$$

Consequently, the Catalan–Qi numbers of the second kind $\mathcal{C}_n(a,b)$ for $a\ge0$ and $b>0$ can be represented by

$$\mathcal{C}_n(a,b)=\frac{1}{\pi}\int_0^\infty\frac{\sqrt{t}}{a^2+t}\frac{1}{(b+t)^{n+1}}\mathrm{d}t,\quad n\ge0. \tag{1.6}$$

In this paper, motivated by the generating functions in (1.2) and (1.3), we consider the functional sequence $\mathfrak{C}_n(a,b;p)$ which is defined by

$$G_{a,b;p}(x)=\frac{1}{a+(b-x)^p}=\sum_{n=0}^{\infty}\mathfrak{C}_n(a,b;p)x^n \tag{1.7}$$

for $a\ge0$, $b>0$, and $0<p<1$. It is clear that

$$G_{a,b;1/2}(x)=G_{a,b}(x),\quad G_{1/2,1/4}(x)=G(x),\quad G_{1/2,1/4;1/2}(x)=G(x),$$
$$\mathfrak{C}_n\left(a,b;\frac{1}{2}\right)=\mathcal{C}_n(a,b),\quad \mathcal{C}_n\left(\frac{1}{2},\frac{1}{4}\right)=C_n,\qquad \mathfrak{C}_n\left(\frac{1}{2},\frac{1}{4};\frac{1}{2}\right)=C_n,$$

but one can neither express $G_{a,b;p}(x)$ in terms of $G_{a,b}(x)$ or $G(x)$ nor express $\mathfrak{C}_n(a,b;p)$ in terms of $\mathcal{C}_n(a,b)$ or C_n. In this paper, we will establish an explicit formula for $\mathfrak{C}_n(a,b;p)$, present integral representations for $G_{a,b;p}(x)$ and $\mathfrak{C}_n(a,b;p)$, and derive several integral formulas and combinatorial identities.

2. An explicit formula for $\mathfrak{C}_n(a,b;p)$

In this section, we establish an explicit formula for $\mathfrak{C}_n(a,b;p)$, from which we can derive the explicit formula (1.4).

Theorem 2.1. *Let $a \ge 0$, $b > 0$, and $0 < p < 1$ be real numbers. Then the functional sequence $\mathfrak{C}_n(a,b;p)$ for $n \ge 0$ can be explicitly computed by*

$$\mathfrak{C}_n(a,b;p) = \frac{1}{a+b^p}\frac{(-1)^n}{b^n n!}\sum_{k=0}^{n}\frac{1}{(1+a/b^p)^k}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle p\ell\rangle_n, \tag{2.1}$$

where the notation

$$\langle\alpha\rangle_n = \prod_{k=0}^{n-1}(\alpha-k) = \begin{cases}\alpha(\alpha-1)\cdots(\alpha-n+1), & n\ge 1\\ 1, & n=0\end{cases}$$

for $\alpha \in \mathbb{C}$ is called the falling factorial.

Proof. In [6, p. 139, Theorem C], noted Faà di Bruno's formula is described by

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}f\circ h(x) = \sum_{k=0}^{n} f^{(k)}(h(x))\,\mathrm{B}_{n,k}\bigl(h'(x),h''(x),\dots,h^{(n-k+1)}(x)\bigr), \tag{2.2}$$

where $f\circ h$ denotes the composite of the n-time differentiable functions f and h, and the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ for $n\ge k\ge 0$ are defined (see [6, p. 134, Theorem A]) by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}) = \sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1} i\ell_i = n\\ \sum_{i=1}^{n-k+1}\ell_i = k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

Let $h = h_p(x) = (b-x)^p$ for $0<p<1$. Then $h_p^{(k)}(x) = (-1)^k\langle p\rangle_k(b-x)^{p-k}$ for $k\ge 0$ and, in light of the formula (2.2) with $f(x) = \frac{1}{a+x}$,

$$\begin{aligned}
\frac{\mathrm{d}^n G_{a,b;p}(x)}{\mathrm{d}x^n} &= \sum_{k=0}^{n}\frac{\mathrm{d}^k}{\mathrm{d}h^k}\left(\frac{1}{a+h}\right)\mathrm{B}_{n,k}\bigl(h_p'(x),h_p''(x),\dots,h_p^{(n-k+1)}(x)\bigr)\\
&= \sum_{k=0}^{n}(-1)^k\frac{k!}{[a+h_p(x)]^{k+1}}\,\mathrm{B}_{n,k}\bigl(h_p'(x),h_p''(x),\dots,h_p^{(n-k+1)}(x)\bigr)\\
&= \sum_{k=0}^{n}(-1)^k\frac{k!}{[a+(b-x)^p]^{k+1}}\,\mathrm{B}_{n,k}\bigl(-\langle p\rangle_1(b-x)^{p-1},\langle p\rangle_2(b-x)^{p-2},\\
&\quad \dots,(-1)^{n-k+1}\langle p\rangle_{n-k+1}(b-x)^{p-(n-k+1)}\bigr)\\
&= \sum_{k=0}^{n}(-1)^k\frac{k!}{[a+(b-x)^p]^{k+1}}(-1)^n(b-x)^{kp-n}\,\mathrm{B}_{n,k}(\langle p\rangle_1,\langle p\rangle_2,\dots,\langle p\rangle_{n-k+1})\\
&= \sum_{k=0}^{n}(-1)^k\frac{k!}{[a+(b-x)^p]^{k+1}}(-1)^n(b-x)^{kp-n}\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle p\ell\rangle_n\\
&= \frac{(-1)^n}{(b-x)^n}\sum_{k=0}^{n}\frac{(b-x)^{kp}}{[a+(b-x)^p]^{k+1}}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle p\ell\rangle_n
\end{aligned}$$

$$\to \frac{(-1)^n}{b^n(a+b^p)} \sum_{k=0}^{n} \frac{1}{(1+a/b^p)^k} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \langle p\ell \rangle_n$$

as $x \to 0$, where we used the formula

$$\mathrm{B}_{n,k}\bigl(\alpha\beta x_1, \alpha\beta^2 x_2, \dots, \alpha\beta^{n-k+1} x_{n-k+1}\bigr) = \alpha^k \beta^n \, \mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{2.3}$$

in [6, p. 135] and the formula

$$\mathrm{B}_{n,k}(\langle\alpha\rangle_1, \langle\alpha\rangle_2, \dots, \langle\alpha\rangle_{n-k+1}) = \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \langle \alpha\ell \rangle_n \tag{2.4}$$

in [43, Theorem 2.1] and [44, Section 1.3] for $\alpha, \beta \in \mathbb{C}$.

The equation (1.7) means that

$$n! \mathfrak{C}_n(a, b; p) = \lim_{x \to 0} \frac{\mathrm{d}^n G_{a,b;p}(x)}{\mathrm{d}x^n}.$$

Consequently, we obtain the explicit formula

$$\mathfrak{C}_n(a, b; p) = \frac{1}{n!} \lim_{x \to 0} \frac{\mathrm{d}^n G_{a,b;p}(x)}{\mathrm{d}x^n} = \frac{(-1)^n}{n! b^n (a+b^p)} \sum_{k=0}^{n} \frac{1}{(1+a/b^p)^k} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \langle p\ell \rangle_n.$$

The proof of Theorem 2.1 is complete. □

Remark 2.1. When taking $p = \frac{1}{2}$ in (2.1), we derive

$$\mathfrak{C}_n\left(a, b; \frac{1}{2}\right) = \mathcal{C}_n(a, b) = \frac{(-1)^n}{n! b^n \bigl(a + \sqrt{b}\bigr)} \sum_{k=0}^{n} \frac{1}{\bigl(1 + a/\sqrt{b}\bigr)^k} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \left\langle \frac{\ell}{2} \right\rangle_n.$$

Further employing the identity

$$\sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \left\langle \frac{\ell}{2} \right\rangle_n = (-1)^n \frac{k! [2(n-k)-1]!!}{2^n} \binom{2n-k-1}{2(n-k)} \tag{2.5}$$

in [43, Theorem 3.2] and [44, Section 1.5], which can be derived from comparing (2.4) for $\alpha = \frac{1}{2}$ with the identity

$$\mathrm{B}_{n,k}((-1)!!, 1!!, 3!!, \dots, [2(n-k)-1]!!) = [2(n-k)-1]!! \binom{2n-k-1}{2(n-k)}$$

in [47, Theorem 1.2] and [44, Section 1.5] or with the identity

$$\mathrm{B}_{n,k}\left(\left\langle \frac{1}{2} \right\rangle_1, \left\langle \frac{1}{2} \right\rangle_2, \dots, \left\langle \frac{1}{2} \right\rangle_{n-k+1}\right) = (-1)^{n+k} [2(n-k)-1]!! \left(\frac{1}{2}\right)^n \binom{2n-k-1}{2(n-k)} \tag{2.6}$$

in [43, p. 169] and [44, Section 1.5], we recover (1.4) straightforwardly.

3. An integral representation of the generating function $G_{a,b;p}(z)$

In this section, we establish an integral representation for the principal branch of the complex generating function $G_{a,b;p}(z)$ by virtue of Cauchy's integral formula in complex analysis.

Theorem 3.1. *Let $a \ge 0$, $b > 0$, and $0 < p < 1$ be real numbers. Then the principal branch of the complex function*

$$G_{a,b;p}(z) = \frac{1}{a + (b-z)^p}, \quad z \in \mathbb{C} \setminus [b, \infty)$$

can be represented by

$$G_{a,b;p}(z)=\frac{\sin(p\pi)}{\pi}\int_0^\infty \frac{t^p}{a^2+2a\cos(p\pi)t^p+t^{2p}}\frac{1}{b+t-z}\mathrm{d}t,\quad \arg(z-b)\in(0,2\pi). \tag{3.1}$$

Proof. In complex analysis, on the cut plane $\mathbb{C}\setminus(-\infty,0]$, the logarithmic function $\ln z=\ln|z|+i\arg z$, where $i=\sqrt{-1}$ is the imaginary unit, and the principal value $\arg z$ of the argument of z satisfies $|\arg z|<\pi$.

For $a\geq 0$ and $0<p<1$, let

$$F_{a;p}(z)=\frac{1}{a+\exp\left[p\ln(-z)\right]},\quad z\in\mathbb{C}\setminus[0,\infty),\quad \arg z\in(0,2\pi).$$

By virtue of Cauchy's integral formula in complex analysis, for any fixed point $z_0=x_0+iy_0\in\mathbb{C}\setminus[0,\infty)$, we have

$$F_{a;p}(z_0)=\frac{1}{2\pi i}\int_{\mathcal{L}(r,R)}\frac{F_{a;p}(\xi)}{\xi-z_0}\mathrm{d}\xi,$$

where $\mathcal{L}(r,R)$ is a positively oriented contour in $\mathbb{C}\setminus[0,\infty)$, as showed in Figure 1, such that

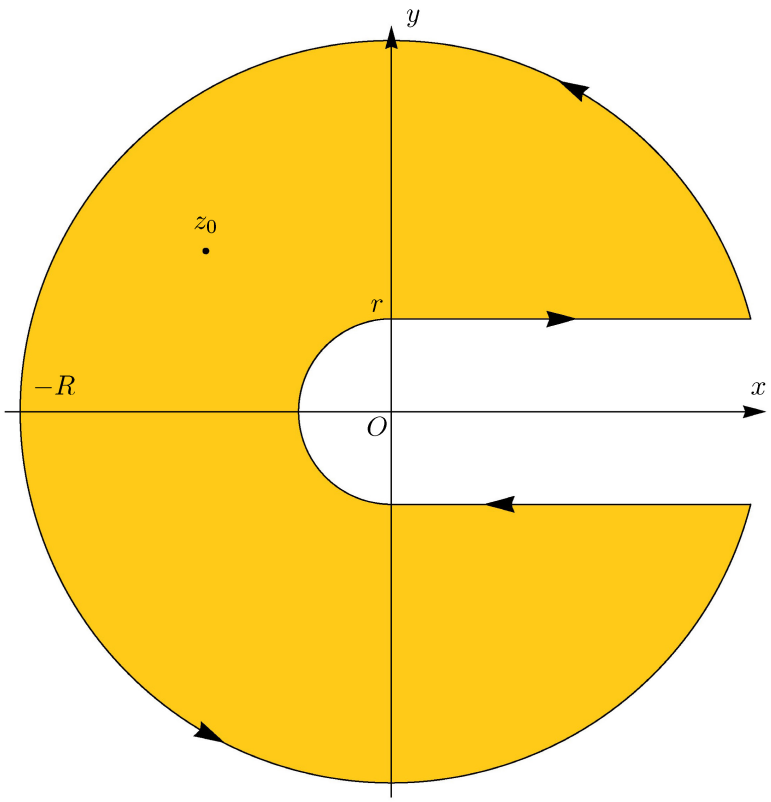

FIGURE 1. The positively oriented contour $\mathcal{L}(r,R)$ in $\mathbb{C}\setminus[0,\infty)$

(1) $0<r<|z_0|<R$;
(2) $L(r,R)$ consists of the half circle $z=re^{i\theta}$ for $\theta\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$;
(3) $L(r,R)$ consists of the line segments $z=x\pm ir$ for $x\in(0,R(r)]$, where $R(r)=\sqrt{R^2-r^2}$;

(4) $L(r,R)$ consists of the circular arc $z=Re^{i\theta}$ for

$$\theta\in\left(\arctan\frac{r}{R(r)},2\pi-\arctan\frac{r}{R(r)}\right)=\left(\arcsin\frac{r}{R},2\pi-\arcsin\frac{r}{R}\right);$$

(5) the line segments $z=x\pm ir$ for $x\in(0,R(r)]$ cut the circle $|z|=R$ at the points $R(r)\pm ir$ and $R(r)\to R$ as $r\to0^+$.

The integral on the circular arc $z=Re^{i\theta}$ with positive orientation of the contour $\mathcal{L}(r,R)$ equals

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\arcsin(r/R)}^{2\pi-\arcsin(r/R)}\frac{Rie^{i\theta}}{Re^{i\theta}-z_0}\frac{1}{a+\exp[p\ln(-Re^{i\theta})]}\mathrm{d}\theta\\=&\frac{1}{2\pi}\int_{\arcsin(r/R)}^{2\pi-\arcsin(r/R)}\frac{Re^{i\theta}}{Re^{i\theta}-z_0}\frac{1}{a+R^p\exp[ip\arg(-Re^{i\theta})]}\mathrm{d}\theta\end{aligned}$$

which tends to 0 uniformly with respect to θ as $R\to\infty$, where we used the limits

$$\lim_{R\to\infty}\frac{Re^{i\theta}}{Re^{i\theta}-z_0}=\lim_{R\to\infty}\frac{e^{i\theta}}{e^{i\theta}-z_0/R}=1$$

and

$$\lim_{R\to\infty}\left|R^p\exp[ip\arg(-Re^{i\theta})]\right|=\lim_{R\to\infty}R^p=\infty.$$

The integral on the half circle $z=re^{i\theta}$ for $\theta\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$ with positive orientation of the contour curve $\mathcal{L}(r,R)$ is

$$\begin{aligned}&\frac{1}{2\pi i}\int_{3\pi/2}^{\pi/2}\frac{rie^{i\theta}}{re^{i\theta}-z_0}\frac{1}{a+\exp[p\ln(-re^{i\theta})]}\mathrm{d}\theta\\=&-\frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}\frac{re^{i\theta}}{re^{i\theta}-z_0}\frac{1}{a+r^p\exp[ip\arg(-re^{i\theta})]}\mathrm{d}\theta\\=&-\frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}\frac{e^{i\theta}}{e^{i\theta}-z_0/r}\frac{1}{a+r^p\exp[ip\arg(-re^{i\theta})]}\mathrm{d}\theta\end{aligned}$$

which tends to 0 uniformly with respect to θ as $r\to0^+$, where we used the limits

$$\lim_{r\to0^+}\frac{e^{i\theta}}{e^{i\theta}-z_0/r}=0\quad\text{and}\quad\lim_{r\to0^+}\left|r^p\exp[ip\arg(-re^{i\theta})]\right|=\lim_{r\to0^+}r^p=0.$$

On the half line $z=x+ir$ for $x\in(0,R(r)]$ and $r>0$, direct computation gives

$$\begin{aligned}F_{a;p}(x+ir)&=\frac{1}{a+\exp[p\ln(-x-ri)]}\\&=\frac{1}{a+\exp\left(p\ln\sqrt{x^2+r^2}+ip[\arctan(r/x)-\pi]\right)}\\&\to\frac{1}{a+x^p\cos(p\pi)-ix^p\sin(p\pi)},\quad r\to0^+\\&=\frac{a+x^p\cos(p\pi)+ix^p\sin(p\pi)}{[a+x^p\cos(p\pi)]^2+[x^p\sin(p\pi)]^2}\\&=\frac{a+x^p\cos(p\pi)+ix^p\sin(p\pi)}{a^2+2ax^p\cos(p\pi)+x^{2p}}.\end{aligned}$$

Since $\overline{e^z}=e^{\bar{z}}$ and $\overline{\ln z}=\ln\bar{z}$, it follows that

$$\begin{aligned}\overline{F_{a;p}(z)}&=\frac{\overline{1}}{\overline{a}+\overline{\exp\left[p\ln(-z)\right]}}=\frac{1}{a+\exp\left[\overline{p\ln(-z)}\right]}=\frac{1}{a+\exp\left[\overline{p}\,\overline{\ln(-z)}\right]}\\&=\frac{1}{a+\exp[p\ln(\overline{-z})]}=\frac{1}{a+\exp[p\ln(-\bar{z})]}=F_{a;p}(\bar{z}).\end{aligned}$$

Accordingly, the integral on the line segments $z=x\pm ir$ for $x>0$ with positive orientation of the contour $\mathcal{L}(r,R)$ is equal to

$$\begin{aligned}&\frac{1}{2\pi i}\left[\int_0^{R(r)}\frac{F_{a;p}(x+ir)}{x+ir-z_0}\mathrm{d}x+\int_{R(r)}^0\frac{F_{a;p}(x-ir)}{x-ir-z_0}\mathrm{d}x\right]\\&=\frac{1}{2\pi i}\int_0^{R(r)}\frac{(x-ir-z_0)F_{a;p}(x+ir)-(x+ir-z_0)F_{a;p}(x-ir)}{(x+ir-z_0)(x-ir-z_0)}\mathrm{d}x\\&=\frac{1}{2\pi i}\int_0^{R(r)}\frac{(x-z_0)[F_{a;p}(x+ir)-F_{a;p}(x-ir)]-ir[F_{a;p}(x+ir)+F_{a;p}(x-ir)]}{(x+ir-z_0)(x-ir-z_0)}\mathrm{d}x\\&=\frac{1}{2\pi i}\int_0^{R(r)}\frac{(x-z_0)\left[F_{a;p}(x+ir)-F\left(\overline{x+ir}\right)\right]-ir\left[F_{a;p}(x+ir)+F\left(\overline{x+ir}\right)\right]}{(x+ir-z_0)(x-ir-z_0)}\mathrm{d}x\\&=\frac{1}{2\pi i}\int_0^{R(r)}\frac{(x-z_0)\left[F_{a;p}(x+ir)-\overline{F_{a;p}(x+ir)}\right]-ir\left[F_{a;p}(x+ir)+\overline{F_{a;p}(x+ir)}\right]}{(x+ir-z_0)(x-ir-z_0)}\mathrm{d}x\\&=\frac{1}{2\pi i}\int_0^{R(r)}\frac{(x-z_0)[2i\Im(F_{a;p}(x+ir))]-ir[2\Re(F_{a;p}(x+ir))]}{(x+ir-z_0)(x-ir-z_0)}\mathrm{d}x\\&\to\frac{1}{2\pi i}\int_0^{\infty}\frac{2i}{x-z_0}\frac{x^p\sin(p\pi)}{a^2+2ax^p\cos(p\pi)+x^{2p}}\mathrm{d}x,\quad r\to0^+,\quad R\to\infty\\&=\frac{1}{\pi}\int_0^{\infty}\frac{1}{x-z_0}\frac{x^p\sin(p\pi)}{a^2+2ax^p\cos(p\pi)+x^{2p}}\mathrm{d}x.\end{aligned}$$

Consequently, it follows that

$$\frac{1}{a+\exp\left[p\ln(-z_0)\right]}=\frac{1}{\pi}\int_0^{\infty}\frac{1}{x-z_0}\frac{x^p\sin(p\pi)}{a^2+2ax^p\cos(p\pi)+x^{2p}}\mathrm{d}x \tag{3.2}$$

for $z_0\in\mathbb{C}\setminus[0,\infty)$ and $\arg z_0\in(0,2\pi)$. Due to the point z_0 in (3.2) being arbitrary, the integral formula (3.2) can be rewritten as

$$F_{a;p}(z)=\frac{1}{a+\exp\left[p\ln(-z)\right]}=\frac{\sin(p\pi)}{\pi}\int_0^{\infty}\frac{1}{t-z}\frac{t^p}{a^2+2a\cos(p\pi)t^p+t^{2p}}\mathrm{d}t, \tag{3.3}$$

where $z\in\mathbb{C}\setminus[0,\infty)$ and $\arg z\in(0,2\pi)$.

For $a\geq0$, $b>0$, $0<p<1$, and $\arg(z-b)\in(0,2\pi)$, it is easy to see that $G_{a,b;p}(z)=F_{a;p}(z-b)$. Therefore, from (3.3), it follows that

$$G_{a,b;p}(z)=\frac{\sin(p\pi)}{\pi}\int_0^{\infty}\frac{1}{b+t-z}\frac{t^p}{a^2+2a\cos(p\pi)t^p+t^{2p}}\mathrm{d}t.$$

The integral representation (3.1) is thus proved. The proof of Theorem 3.1 is complete. □

Remark 3.1. When taking $z=x\in(-\infty,b)$, the integral representation (3.1) becomes

$$\frac{1}{a+(b-x)^p}=\frac{\sin(p\pi)}{\pi}\int_0^{\infty}\frac{t^p}{a^2+2a\cos(p\pi)t^p+t^{2p}}\frac{1}{b+t-x}\mathrm{d}t. \tag{3.4}$$

When taking $x\to b^-$, the integral in (3.4) converges. Consequently, the integral representation (3.4) is valid on $(-\infty,b]$.

Remark 3.2. Taking $p=\frac{1}{2}$ in the integral representation (3.1) recovers the integral representation (1.5).

Remark 3.3. When taking $p=\frac{1}{3}$ and $p=\frac{1}{4}$ respectively, the integral representation (3.1) becomes

$$\frac{1}{a+\sqrt[3]{b-z}}=\frac{\sqrt{3}}{2\pi}\int_0^\infty \frac{\sqrt[3]{t}}{a^2+a\sqrt[3]{t}+\sqrt[3]{t^2}}\frac{1}{b+t-z}\mathrm{d}t$$

and

$$\frac{1}{a+\sqrt[4]{b-z}}=\frac{\sqrt{2}}{2\pi}\int_0^\infty \frac{\sqrt[4]{t}}{a^2+\sqrt{2}\,a\sqrt[4]{t}+\sqrt{t}}\frac{1}{b+t-z}\mathrm{d}t$$

for $\arg(z-b)\in(0,2\pi)$ respectively.

4. An integral representation for $\mathfrak{C}_n(a,b;p)$

From the integral representation (3.1) or (3.4), we can derive an integral representation for the functional sequence $\mathfrak{C}_n(a,b;p)$ as follows.

Theorem 4.1. *Let $a\ge 0$, $b>0$, and $0<p<1$ be real numbers. Then the functional sequence $\mathfrak{C}_n(a,b;p)$ for $n\ge 0$ can be represented by*

$$\mathfrak{C}_n(a,b;p)=\frac{\sin(p\pi)}{\pi}\int_0^\infty \frac{t^p}{a^2+2a\cos(p\pi)t^p+t^{2p}}\frac{1}{(b+t)^{n+1}}\mathrm{d}t. \tag{4.1}$$

Proof. Differentiating $n\ge 0$ times with respect to z on both sides of (3.1) and taking the limit $z\to 0$ yield

$$\begin{aligned}\lim_{z\to0}[G_{a,b;p}(z)]^{(n)}&=\frac{\sin(p\pi)}{\pi}\lim_{z\to0}\int_0^\infty \frac{t^p}{a^2+2a\cos(p\pi)t^p+t^{2p}}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\biggl(\frac{1}{b+t-z}\biggr)\mathrm{d}t\\&=\frac{\sin(p\pi)}{\pi}\int_0^\infty \frac{t^p}{a^2+2a\cos(p\pi)t^p+t^{2p}}\lim_{z\to0}\frac{n!}{(b+t-z)^{n+1}}\mathrm{d}t\\&=\frac{n!\sin(p\pi)}{\pi}\int_0^\infty \frac{t^p}{a^2+2a\cos(p\pi)t^p+t^{2p}}\frac{1}{(b+t)^{n+1}}\mathrm{d}t.\end{aligned}$$

As a result, by virtue of (1.7), we have

$$\mathfrak{C}_n(a,b;p)=\frac{1}{n!}\lim_{x\to0}\frac{\mathrm{d}^nG_{a,b;p}(x)}{\mathrm{d}x^n}=\frac{\sin(p\pi)}{\pi}\int_0^\infty \frac{t^p}{a^2+2a\cos(p\pi)t^p+t^{2p}}\frac{1}{(b+t)^{n+1}}\mathrm{d}t.$$

The integral representation (4.1) is thus proved. The proof of Theorem 4.1 is complete. □

Remark 4.1. If taking $p=\frac{1}{2}$ in (4.1), one can recover (1.6) immediately.

Remark 4.2. When taking $p=\frac{1}{3}$ and $p=\frac{1}{4}$ respectively, the integral representation (4.1) becomes

$$\mathfrak{C}_n\biggl(a,b;\frac{1}{3}\biggr)=\frac{\sqrt{3}}{2\pi}\int_0^\infty \frac{\sqrt[3]{t}}{a^2+a\sqrt[3]{t}+\sqrt[3]{t^2}}\frac{1}{(b+t)^{n+1}}\mathrm{d}t$$

and

$$\mathfrak{C}_n\biggl(a,b;\frac{1}{4}\biggr)=\frac{\sqrt{2}}{2\pi}\int_0^\infty \frac{\sqrt[4]{t}}{a^2+\sqrt{2}\,a\sqrt[4]{t}+\sqrt{t}}\frac{1}{(b+t)^{n+1}}\mathrm{d}t$$

respectively.

5. Two integral formulas

Combining the explicit formula (2.1) in Theorem 2.1 with the integral representation (4.1) in Theorem 4.1 and combining (1.4) with (1.6), we can conclude two integral formulas.

Theorem 5.1. *Let $a \geq 0$ and $b > 0$ be real numbers and let $n \geq 0$ be integers. If $0 < p < 1$, then*

$$\int_0^\infty \frac{t^p}{a^2 + 2a\cos(p\pi)t^p + t^{2p}} \frac{1}{(b+t)^{n+1}} \mathrm{d}t = \frac{\pi}{(a+b^p)\sin(p\pi)} \frac{(-1)^n}{n! b^n} \sum_{k=0}^{n} \frac{1}{(1+a/b^p)^k} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \langle p\ell \rangle_n. \quad (5.1)$$

In particular, if $p = \frac{1}{2}$, then

$$\int_0^\infty \frac{\sqrt{t}}{a+t} \frac{1}{(b+t)^{n+1}} \mathrm{d}t = \frac{\pi}{(2n)!! b^{n+1/2}} \sum_{k=0}^{n} \binom{2n-k-1}{2(n-k)} \frac{k![2(n-k)-1]!!}{\left(1+\sqrt{a/b}\right)^{k+1}}. \quad (5.2)$$

Alternative proof of the integral formula (5.1) *for $a, b > 0$ and $0 < p < 1$.* The formula of Faà di Bruno and Schlömilch [14, p. 33] states that

$$\frac{\mathrm{d}^n (f \circ g)}{\mathrm{d}x^n} = \sum_{k=1}^{n} \left(f^{(k)} \circ g\right) Y_{n,k}(g) \quad (5.3)$$

where the operator $Y_{n,k}$ has the explicit form

$$Y_{n,k}(g) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} g^{k-j} \left(g^j\right)^{(n)} \quad (5.4)$$

An inductive proof for this formula can also be found in [24].

Consider the case $g(x) = x^p$ with $0 < p < 1$. According to (5.4), we have

$$\begin{aligned} Y_{n,k}(x^p) &= \frac{1}{k!} \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} x^{(k-\ell)p} \left(x^{p\ell}\right)^{(n)} \\ &= \frac{1}{k!} \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} x^{(k-\ell)p} \langle p\ell \rangle_n x^{p\ell-n} \\ &= \frac{x^{kp-n}}{k!} \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \langle p\ell \rangle_n. \end{aligned}$$

When setting $f(x) = \frac{1}{1+x}$, according to (5.3), we have

$$\begin{aligned} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(\frac{1}{1+x^p}\right) &= \sum_{k=1}^{n} \frac{(-1)^k k!}{(1+x^p)^{k+1}} \frac{x^{kp-n}}{k!} \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \langle p\ell \rangle_n \\ &= \frac{1}{x^{n+p}} \sum_{k=1}^{n} \left(\frac{x^p}{1+x^p}\right)^{k+1} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \langle p\ell \rangle_n. \end{aligned} \quad (5.5)$$

Let $0 < p < 1$ and $\lambda > 0$. Consider the meromorphic function

$$F(z) = \frac{1}{(1-\lambda e^{-z})(1+e^{pz})}.$$

Consider also the positively oriented rectangle $\mathcal{R} = \overline{ABCD}$ with vertices $A(-R-\pi i)$, $B(R-\pi i)$, $C(R+\pi i)$, and $D(-R+\pi i)$ for $R > |\ln\lambda|$, as indicated in Figure 2. In the interior of the

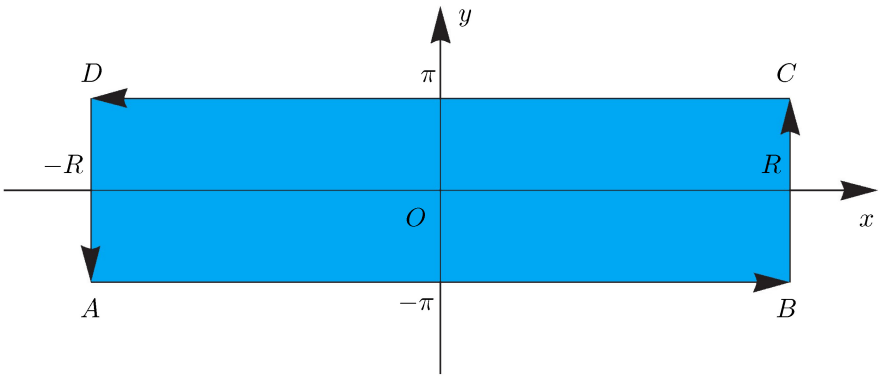

FIGURE 2. The positively oriented rectangle $\mathcal{R} = \overline{ABCD}$

region enclosed by the rectangle $\mathcal{R}$, the function F has a unique pole $\mathfrak{p} = \ln\lambda$ of the residue $\operatorname{Res}(F,\mathfrak{p}) = \frac{1}{1+\lambda^p}$. Therefore, by Cauchy's residue theorem in complex analysis, it follows that

$$\int_{\mathcal{R}} F(z)\mathrm{d}z = \frac{2\pi i}{1+\lambda^p}. \tag{5.6}$$

On the other hand, from

$$\left|\int_{\overline{BC}} F(z)\mathrm{d}z\right| = \left|i\int_{-\pi}^{\pi} F(R+it)\mathrm{d}t\right| \le \frac{\pi}{(1-\lambda e^{-R})(e^{pR}-1)}$$

and

$$\left|\int_{\overline{DA}} F(z)\mathrm{d}z\right| = \left|i\int_{-\pi}^{\pi} F(-R+it)\mathrm{d}t\right| \le \frac{\pi}{(\lambda e^{R}-1)(1-e^{-pR})},$$

we derive

$$\lim_{R\to\infty}\int_{\overline{BC}} F(z)\mathrm{d}z = \lim_{R\to\infty}\int_{\overline{DA}} F(z)\mathrm{d}z = 0.$$

Further, we have

$$\begin{aligned}\int_{\overline{AB}} F(z)\mathrm{d}z + \int_{\overline{CD}} F(z)\mathrm{d}z &= \int_{-R}^{R} F(x-\pi i)\mathrm{d}x - \int_{-R}^{R} F(x+\pi i)\mathrm{d}x\\ &= \int_{-R}^{R}\frac{1}{1+\lambda e^{-x}}\left(\frac{1}{1+e^{px-ip\pi}} - \frac{1}{1+e^{px+ip\pi}}\right)\mathrm{d}x\\ &= 2i\sin(p\pi)\int_{-R}^{R}\frac{1}{1+\lambda e^{-x}}\frac{e^{px}}{1+2\cos(p\pi)e^{px}+e^{2px}}\mathrm{d}x.\end{aligned}$$

Consequently, letting R tend to ∞ in (5.6) and rearranging lead to

$$\int_{-\infty}^{\infty}\frac{1}{1+\lambda e^{-x}}\frac{e^{px}}{1+2\cos(p\pi)e^{px}+e^{2px}}\mathrm{d}x = \frac{\pi}{\sin(p\pi)}\frac{1}{1+\lambda^p}.$$

The change of variables $e^x = t$ yields

$$\int_0^{\infty}\frac{t^p}{1+2\cos(p\pi)t^p+t^{2p}}\frac{\mathrm{d}t}{t+\lambda} = \frac{\pi}{\sin(p\pi)}\frac{1}{1+\lambda^p}.$$

Taking the nth derivative with respect to λ arrives at

$$\int_0^\infty \frac{t^p}{1+2\cos(p\pi)t^p+t^{2p}}\frac{\mathrm{d}t}{(t+\lambda)^{n+1}} = \frac{\pi}{\sin(p\pi)}\frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left(\frac{1}{1+\lambda^p}\right).$$

Combining this with (5.5) reveals

$$\int_0^\infty \frac{t^p}{1+2\cos(p\pi)t^p+t^{2p}}\frac{\mathrm{d}t}{(t+\lambda)^{n+1}} = \frac{\pi}{\sin(p\pi)}\frac{(-1)^n}{n!\lambda^{n+p}}\sum_{k=1}^n\left(\frac{\lambda^p}{1+\lambda^p}\right)^{k+1}\sum_{\ell=0}^k(-1)^\ell\binom{k}{\ell}\langle p\ell\rangle_n.$$

Now the change of variables $t=a^{-1/p}u$ with $a>0$ yields

$$\int_0^\infty \frac{u^p}{a^2+2a\cos(p\pi)u^p+u^{2p}}\frac{a^{1+n/p}\mathrm{d}u}{(u+a^{1/p}\lambda)^{n+1}} = \frac{\pi}{\sin(p\pi)}\frac{(-1)^n}{n!\lambda^{n+p}}\sum_{k=1}^n\left(\frac{\lambda^p}{1+\lambda^p}\right)^{k+1}\sum_{\ell=0}^k(-1)^\ell\binom{k}{\ell}\langle p\ell\rangle_n.$$

Finally, applying this to $\lambda=a^{-1/p}b$ for $b>0$ results in

$$\int_0^\infty \frac{u^p}{a^2+2a\cos(p\pi)u^p+u^{2p}}\frac{\mathrm{d}u}{(u+b)^{n+1}} = \frac{\pi}{\sin(p\pi)}\frac{(-1)^n}{n!\,b^n}\sum_{k=1}^n\frac{b^{pk}}{(a+b^p)^{k+1}}\sum_{\ell=0}^k(-1)^\ell\binom{k}{\ell}\langle p\ell\rangle_n$$

or, equivalently,

$$\int_0^\infty \frac{u^p}{a^2+2a\cos(p\pi)u^p+u^{2p}}\frac{\mathrm{d}u}{(u+b)^{n+1}} = \frac{\pi}{(a+b^p)\sin(p\pi)}\frac{(-1)^n}{n!\,b^n}\sum_{k=1}^n\frac{1}{(1+a/b^p)^k}\sum_{\ell=0}^k(-1)^\ell\binom{k}{\ell}\langle p\ell\rangle_n.$$

The required proof is complete. □

Remark 5.1. By analytic continuation, we claim that the integral formula (5.1) is valid for

$$a\in\{z\in\mathbb{C}:|\arg(z)|<(1-p)\pi\}$$

and $b\in\mathbb{C}\setminus(-\infty,0]$. For example, if $0<p<\frac{1}{2}$, $b=1$, and $a=i$, then

$$\int_0^\infty \frac{u^p}{-1+2i\cos(p\pi)u^p+u^{2p}}\frac{\mathrm{d}u}{(u+1)^{n+1}} = \frac{\pi}{\sin(p\pi)}\frac{(-1)^n}{n!}\sum_{k=0}^n\frac{1}{(1+i)^{k+1}}\sum_{\ell=0}^k(-1)^\ell\binom{k}{\ell}\langle p\ell\rangle_n$$

which is equivalent to

$$\int_0^\infty \frac{u^p[u^{2p}-1-2i\cos(p\pi)u^p]}{1+2\cos(2p\pi)u^{2p}+u^{4p}}\frac{\mathrm{d}u}{(u+1)^{n+1}} = \frac{\pi}{\sin(p\pi)}\frac{(-1)^n}{n!}\sum_{k=0}^n\frac{e^{-i\pi(k+1)/4}}{2^{(k+1)/2}}\sum_{\ell=0}^k(-1)^\ell\binom{k}{\ell}\langle p\ell\rangle_n.$$

Further comparing imaginary parts of this equality gives

$$\int_0^\infty \frac{u^{2p}}{1+2\cos(2p\pi)u^{2p}+u^{4p}}\frac{\mathrm{d}u}{(u+1)^{n+1}} = \frac{\pi}{\sin(2p\pi)}\frac{(-1)^n}{n!}\sum_{k=0}^n\frac{\sin(\pi(k+1)/4)}{2^{(k+1)/2}}\sum_{\ell=0}^k(-1)^\ell\binom{k}{\ell}\langle p\ell\rangle_n.$$

On the other hand, setting $a = b = 1$ and replacing p by $2p$ in (5.1) result in

$$\int_0^\infty \frac{u^{2p}}{1+2\cos(2p\pi)u^p+u^{2p}}\frac{\mathrm{d}u}{(u+1)^{n+1}} = \frac{\pi}{\sin(2p\pi)}\frac{(-1)^n}{n!}\sum_{k=0}^{n}\frac{1}{2^{k+1}}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle 2p\ell\rangle_n.$$

Accordingly, for $0 < p < \frac{1}{2}$, we obtain

$$\sum_{k=0}^{n}\frac{1}{2^{k+1}}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle 2p\ell\rangle_n = \sum_{k=0}^{n}\frac{\sin(\pi(k+1)/4)}{2^{(k+1)/2}}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle p\ell\rangle_n.$$

Since both sides of this equality are polynomials in the variable p, we acquire

$$\sum_{k=0}^{n}\frac{1}{2^{k+1}}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle 2\ell z\rangle_n = \sum_{k=0}^{n}\frac{\sin(\pi(k+1)/4)}{2^{(k+1)/2}}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle \ell z\rangle_n$$

for all $z \in \mathbb{C}$.

Remark 5.2. From the proof of Theorem 2.1, we conclude that

$$\frac{\mathrm{d}^n G_{a,b;p}(x)}{\mathrm{d}x^n} = \frac{(-1)^n}{(b-x)^n}\sum_{k=0}^{n}\frac{(b-x)^{kp}}{[a+(b-x)^p]^{k+1}}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle p\ell\rangle_n.$$

Combining this with the integral representation (3.1) in Theorem 3.1 derives the integral formula

$$\begin{aligned}\int_0^\infty &\frac{t^p}{a^2+2a\cos(p\pi)t^p+t^{2p}}\frac{1}{(b+t-x)^{n+1}}\mathrm{d}t \\ &= \frac{\pi}{[a+(b-x)^p]\sin(p\pi)}\frac{(-1)^n}{n!(b-x)^n}\sum_{k=0}^{n}\frac{1}{[1+a/(b-x)^p]^k}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle p\ell\rangle_n.\end{aligned}$$

This is essentially same as the integral formula (5.1).

Remark 5.3. Letting $a = 0$ and computing integrals in (5.1) and (5.2) leads to

$$\int_0^\infty \frac{1}{t^p}\frac{1}{(b+t)^{n+1}}\mathrm{d}t = \frac{\Gamma(1-p)\Gamma(n+p)}{n!b^{n+p}} = \frac{\pi}{b^p\sin(p\pi)}\frac{(-1)^n}{n!b^n}\sum_{k=0}^{n}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle p\ell\rangle_n$$

and

$$\int_0^\infty \frac{1}{\sqrt{t}\,(b+t)^{n+1}}\mathrm{d}t = \frac{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}{n!b^{n+1/2}} = \frac{\pi}{(2n)!!b^{n+1/2}}\sum_{k=0}^{n}k![2(n-k)-1]!!\binom{2n-k-1}{2(n-k)}.$$

As a result, we obtain two interesting formulas

$$\Gamma(1-p)\Gamma(n+p) = \frac{\pi}{\sin(p\pi)}(-1)^n\sum_{k=0}^{n}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle p\ell\rangle_n \tag{5.7}$$

and

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n}\sum_{k=0}^{n}k![2(n-k)-1]!!\binom{2n-k-1}{2(n-k)} \tag{5.8}$$

for $0 < p < 1$ and $n \ge 0$.

By the recurrent relation $\Gamma(z+1)=z\Gamma(z)$ and the reflection relation $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(z\pi)}$ for $z\neq 0,\pm1,\dots$ (see [27, Chapter 5]), we can rewrite the identity (5.7) as

$$\begin{aligned}(p)_n&=(-1)^n\sum_{k=0}^{n}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle p\ell\rangle_n\\&=(-1)^n\sum_{\ell=0}^{n}(-1)^\ell\langle p\ell\rangle_n\sum_{k=\ell}^{n}\binom{k}{\ell}\\&=(-1)^n\sum_{\ell=0}^{n}(-1)^\ell\langle p\ell\rangle_n\sum_{k=\ell}^{n}\left[\binom{k+1}{\ell+1}-\binom{k}{\ell+1}\right]\\&=(-1)^n\sum_{\ell=0}^{n}(-1)^\ell\binom{n+1}{\ell+1}\langle p\ell\rangle_n\end{aligned}\tag{5.9}$$

for $0<p<1$ and $n\geq 0$, where the notation

$$(\alpha)_n=\prod_{k=0}^{n-1}(\alpha+k)=\begin{cases}\alpha(\alpha+1)\cdots(\alpha+n-1), & n\geq 1\\ 1, & n=0\end{cases}$$

for $\alpha\in\mathbb{C}$ is called the rising factorial, the Pochhammer symbol, or shifted factorial. Because all sides in (5.9) are polynomials in p for $0<p<1$, all equalities in (5.9) must be valid for all $p\in\mathbb{C}$ and $n\geq 0$. Thus, we obtain the identity

$$(z)_n=(-1)^n\sum_{\ell=0}^{n}(-1)^\ell\binom{n+1}{\ell+1}\langle\ell z\rangle_n,\quad z\in\mathbb{C}.$$

By the recurrent relation $\Gamma(z+1)=z\Gamma(z)$ and the formula $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ (see [27, Chapter 5]), we can rewrite the identity (5.8) as

$$(2n-1)!!=\sum_{k=0}^{n}k![2(n-k)-1]!!\binom{2n-k-1}{2(n-k)},\quad n\geq 0.$$

By the way, we can rearrange [12, Theorem 2.1] as

$$\frac{1}{(2n-1)!!}=\frac{(-1)^{n+1}}{2^{3n-2}(n-1)!}\sum_{k=0}^{2n-1}\frac{(-1)^k}{2n-2k-1}\binom{2n-1}{k},\quad n\geq 1.$$

6. An elementary computation of the integral in (5.2)

In this section, we will compute the integral in (5.2) elementarily.

Lemma 6.1. *For real numbers a,b with $a\neq b$ and any positive integer n, we have*

$$\frac{1}{(a+x)(b+x)^n}=\frac{1}{(b-a)^n(a+x)}-\sum_{\ell=1}^{n}\frac{1}{(b-a)^{n-\ell+1}(b+x)^\ell}.\tag{6.1}$$

Proof. The equality (6.1) can be proved straightforwardly by induction on n. However, we will demonstrate an alternative proof as follow.

The right hand side of the equality (6.1) can be restructured and computed as

$$\begin{aligned}\frac{1}{(b-a)^{n+1}}\left[\frac{b-a}{a+x}-\sum_{\ell=1}^{n}\left(\frac{b-a}{b+x}\right)^\ell\right]&=\frac{1}{(b-a)^{n+1}}\left[\frac{b-a}{a+x}-\frac{b-a}{a+x}\left(1-\left(\frac{b-a}{b+x}\right)^n\right)\right]\\&=\frac{1}{(b-a)^{n+1}}\left[\frac{b-a}{a+x}\left(\frac{b-a}{b+x}\right)^n\right)\Bigg]\end{aligned}$$

$$= \frac{1}{(a+x)(b+x)^n}.$$

The proof of Lemma 6.1 is complete. □

Lemma 6.2. *For real number $b > 0$ and integer $\ell \ge 1$, we have*

$$\int_0^\infty \frac{\sqrt{t}}{(b+t)^{\ell+1}}\mathrm{d}t = \frac{\pi}{b^{\ell-1/2}}\frac{(2\ell-3)!!}{(2\ell)!!}. \tag{6.2}$$

For $a, b \ge 0$, we have

$$\int_0^\infty \sqrt{t}\left(\frac{1}{a+t}-\frac{1}{b+t}\right)\mathrm{d}t = \pi\bigl(\sqrt{b}-\sqrt{a}\bigr). \tag{6.3}$$

Proof. It is straightforward that

$$\begin{aligned}\int_0^\infty \frac{\sqrt{t}}{(b+t)^{\ell+1}}\mathrm{d}t &= \frac{1}{b^{\ell+1}}\int_0^\infty \frac{\sqrt{t}}{(1+t/b)^{\ell+1}}\mathrm{d}t = \frac{1}{b^{\ell-1/2}}\int_0^\infty \frac{\sqrt{u}}{(1+u)^{\ell+1}}\mathrm{d}u\\ &= \frac{1}{b^{\ell-1/2}}B\left(\frac{3}{2},\ell-\frac{1}{2}\right) = \frac{1}{b^{\ell-1/2}}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\ell-\frac{1}{2}\right)}{\Gamma(\ell+1)}\\ &= \frac{1}{b^{\ell-1/2}}\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\left(\ell-\frac{3}{2}\right)\left(\ell-\frac{5}{2}\right)\cdots\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\ell!} = \frac{\pi}{b^{\ell-1/2}}\frac{(2\ell-3)!!}{(2\ell)!!},\end{aligned}$$

where $B(z,w)$ denotes the classical beta function

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\mathrm{d}t = \int_0^\infty \frac{t^{z-1}}{(1+t)^{z+w}}\mathrm{d}t$$

for $\Re(z) > 0$ and $\Re(w) > 0$ and satisfies

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = B(w,z).$$

See [1, p. 258, 6.2.1 and 6.2.2].

When $a, b > 0$ and $a \ne b$, the first proof of the formula (6.3) is straightforward as follows:

$$\begin{aligned}\int_0^\infty \sqrt{t}\left(\frac{1}{a+t}-\frac{1}{b+t}\right)\mathrm{d}t &= 2\int_0^\infty s^2\left(\frac{1}{a+s^2}-\frac{1}{b+s^2}\right)\mathrm{d}s\\ &= 2(b-a)\int_0^\infty \frac{s^2}{(a+s^2)(b+s^2)}\mathrm{d}s\\ &= 2(b-a)\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\sqrt{a}\arctan\frac{s}{\sqrt{a}}-\sqrt{b}\arctan\frac{s}{\sqrt{b}}}{a-b}\right]\mathrm{d}s\\ &= 2(b-a)\frac{\sqrt{a}\arctan\frac{s}{\sqrt{a}}-\sqrt{b}\arctan\frac{s}{\sqrt{b}}}{a-b}\biggm|_{s=0}^{s=\infty}\\ &= \pi\bigl(\sqrt{b}-\sqrt{a}\bigr).\end{aligned}$$

When $a, b > 0$ and $a \ne b$, we can alternatively prove the formula (6.3) as follows:

$$\begin{aligned}\int_0^\infty \sqrt{t}\left(\frac{1}{a+t}-\frac{1}{b+t}\right)\mathrm{d}t &= \int_0^\infty \left[\frac{b}{(b+t)\sqrt{t}}-\frac{a}{(a+t)\sqrt{t}}\right]\mathrm{d}t\\ &= 2\sqrt{b}\arctan\sqrt{\frac{t}{b}}\biggm|_{t=0}^{t=\infty} - 2\sqrt{a}\arctan\sqrt{\frac{t}{a}}\biggm|_{t=0}^{t=\infty}\\ &= \pi\bigl(\sqrt{b}-\sqrt{a}\bigr).\end{aligned}$$

The proof of Lemma 6.2 is complete. □

Theorem 6.1. *If $n \geq 0$ is an integer and $a \geq 0$, $b > 0$, and $a \neq b$ are real umbers, then*

$$\int_0^\infty \frac{\sqrt{t}}{a+t}\frac{1}{(b+t)^{n+1}}\mathrm{d}t = -\frac{\pi}{(b-a)^{n+1}}\left[\sqrt{a} + \sqrt{b}\sum_{\ell=0}^{n}\left(1-\frac{a}{b}\right)^{\ell}\frac{(2\ell-3)!!}{(2\ell)!!}\right], \tag{6.4}$$

where $(-3)!! = -1$ and $(-1)!! = 0!! = 1$.

First proof of Theorem 6.1. From Lemma 6.1, it follows that

$$\frac{1}{(a+t)(b+t)^{n+1}} = \frac{1}{(b-a)^{n+1}}\left(\frac{1}{a+t} - \frac{1}{b+t}\right) - \sum_{\ell=2}^{n+1}\frac{1}{(b-a)^{n-\ell+2}(b+t)^{\ell}}.$$

Further integrating on both sides with respect to t and applying the formulas (6.2) and (6.3) in Lemma 6.2 arrive at

$$\begin{aligned}
\int_0^\infty \frac{\sqrt{t}}{a+t}\frac{1}{(b+t)^{n+1}}\mathrm{d}t &= \frac{1}{(b-a)^{n+1}}\int_0^\infty \sqrt{t}\left(\frac{1}{a+t} - \frac{1}{b+t}\right)\mathrm{d}t \\
&\quad - \int_0^\infty \sum_{\ell=2}^{n+1}\frac{\sqrt{t}}{(b-a)^{n-\ell+2}(b+t)^{\ell}}\mathrm{d}t \\
&= \frac{\pi\left(\sqrt{b}-\sqrt{a}\right)}{(b-a)^{n+1}} - \sum_{\ell=1}^{n}\frac{1}{(b-a)^{n-\ell+1}}\int_0^\infty \frac{\sqrt{t}}{(b+t)^{\ell+1}}\mathrm{d}t \\
&= \frac{\pi\left(\sqrt{b}-\sqrt{a}\right)}{(b-a)^{n+1}} - \sum_{\ell=1}^{n}\frac{1}{(b-a)^{n-\ell+1}}\frac{\pi}{b^{\ell-1/2}}\frac{(2\ell-3)!!}{(2\ell)!!} \\
&= \frac{\pi\left(\sqrt{b}-\sqrt{a}\right)}{(b-a)^{n+1}} - \frac{\pi\sqrt{b}}{(b-a)^{n+1}}\sum_{\ell=1}^{n}\left(1-\frac{a}{b}\right)^{\ell}\frac{(2\ell-3)!!}{(2\ell)!!} \\
&= -\frac{\pi}{(b-a)^{n+1}}\left[\sqrt{a} + \sqrt{b}\sum_{\ell=0}^{n}\left(1-\frac{a}{b}\right)^{\ell}\frac{(2\ell-3)!!}{(2\ell)!!}\right].
\end{aligned}$$

The proof of Theorem 6.1 is complete. □

Second proof of Theorem 6.1. The formula (6.3) can be rewritten as

$$\int_0^\infty \frac{\sqrt{t}}{(a+t)(b+t)}\mathrm{d}t = \pi\left(\frac{\sqrt{b}}{b-a} - \frac{\sqrt{a}}{b-a}\right).$$

Differentiating n times with respect to b on both sides of the above equality and making use of Leibnitz's rule for differentiation yield

$$\begin{aligned}
\int_0^\infty \frac{(-1)^n n!\sqrt{t}}{(a+t)(b+t)^{n+1}}\mathrm{d}t &= \pi\left[\frac{\mathrm{d}^n}{\mathrm{d}b^n}\left(\frac{\sqrt{b}}{b-a}\right) - \frac{\mathrm{d}^n}{\mathrm{d}b^n}\left(\frac{\sqrt{a}}{b-a}\right)\right] \\
&= \pi\left[\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{\mathrm{d}^\ell}{\mathrm{d}b^\ell}(\sqrt{b})\frac{\mathrm{d}^{n-\ell}}{\mathrm{d}b^{n-\ell}}\left(\frac{1}{b-a}\right) - \frac{\sqrt{a}\,(-1)^n n!}{(b-a)^{n+1}}\right] \\
&= \pi\left[\sum_{\ell=0}^{n}\binom{n}{\ell}\left\langle\frac{1}{2}\right\rangle_{\ell} b^{1/2-\ell}\frac{(-1)^{n-\ell}(n-\ell)!}{(b-a)^{n-\ell+1}} - \frac{\sqrt{a}\,(-1)^n n!}{(b-a)^{n+1}}\right] \\
&= -\pi\frac{(-1)^n n!}{(b-a)^{n+1}}\left[\sqrt{a} - \frac{(b-a)^{n+1}}{(-1)^n n!}\sum_{\ell=0}^{n}\binom{n}{\ell}\left\langle\frac{1}{2}\right\rangle_{\ell} b^{1/2-\ell}\frac{(-1)^{n-\ell}(n-\ell)!}{(b-a)^{n-\ell+1}}\right]
\end{aligned}$$

$$= -\pi\frac{(-1)^n n!}{(b-a)^{n+1}}\left[\sqrt{a} - \sqrt{b}\sum_{\ell=0}^{n}\frac{(-1)^\ell}{\ell!}\left\langle\frac{1}{2}\right\rangle_\ell\left(1-\frac{a}{b}\right)^\ell\right]$$

$$= -\pi\frac{(-1)^n n!}{(b-a)^{n+1}}\left[\sqrt{a} - \sqrt{b}\sum_{\ell=0}^{n}\frac{(-1)^\ell}{\ell!}\frac{1}{2}\left(-\frac{1}{2}\right)\cdots\left(\frac{1}{2}-\ell+1\right)\left(1-\frac{a}{b}\right)^\ell\right]$$

$$= -\pi\frac{(-1)^n n!}{(b-a)^{n+1}}\left[\sqrt{a} + \sqrt{b}\sum_{\ell=0}^{n}\frac{(2\ell-3)!!}{(2\ell)!!}\left(1-\frac{a}{b}\right)^\ell\right].$$

The integral formula (6.4) is thus proved. □

Remark 6.1. The formula (6.3) can also be rearranged as

$$\int_0^\infty \frac{\sqrt{t}}{(a+t)(b+t)}\mathrm{d}t = \frac{\pi}{\sqrt{a}+\sqrt{b}}.$$

Differentiating n times with respect to b on both sides of the above equality, utilizing Faà di Bruno's formula (2.2), making use of the identity (2.3), and employing the formula (2.6) reveal

$$\int_0^\infty \frac{(-1)^n n!\sqrt{t}}{(a+t)(b+t)^{n+1}}\mathrm{d}t = \frac{\mathrm{d}^n}{\mathrm{d}b^n}\left(\frac{\pi}{\sqrt{a}+\sqrt{b}}\right)$$

$$= \sum_{k=0}^{n}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\frac{\pi}{\sqrt{a}+x}\right)\Bigg|_{x=\sqrt{b}}\mathrm{B}_{n,k}\left(\left(\sqrt{b}\right)', \left(\sqrt{b}\right)'', \dots, \left(\sqrt{b}\right)^{n-k+1}\right)$$

$$= \pi\sum_{k=0}^{n}\frac{(-1)^k k!}{(\sqrt{a}+x)^{k+1}}\Bigg|_{x=\sqrt{b}}\mathrm{B}_{n,k}\left(b^{1/2-1}\left\langle\frac{1}{2}\right\rangle_1, b^{1/2-2}\left\langle\frac{1}{2}\right\rangle_2, \dots, b^{1/2-(n-k+1)}\left\langle\frac{1}{2}\right\rangle_{n-k+1}\right)$$

$$= \pi\sum_{k=0}^{n}\frac{(-1)^k k!}{(\sqrt{a}+\sqrt{b})^{k+1}}b^{k/2-n}\,\mathrm{B}_{n,k}\left(\left\langle\frac{1}{2}\right\rangle_1, \left\langle\frac{1}{2}\right\rangle_2, \dots, \left\langle\frac{1}{2}\right\rangle_{n-k+1}\right)$$

$$= \pi\sum_{k=0}^{n}\frac{(-1)^k k!}{(\sqrt{a}+\sqrt{b})^{k+1}}b^{k/2-n}(-1)^{n+k}[2(n-k)-1]!!\left(\frac{1}{2}\right)^n\binom{2n-k-1}{2(n-k)}$$

which recovers the integral formula (5.2), where $a \ge 0$ and $b > 0$.

Remark 6.2. For $a \ge 0$ and $b > 0$, combining (5.2) and (6.4) results in an identity

$$\sum_{k=0}^{n}\binom{2n-k-1}{2(n-k)}\frac{k![2(n-k)-1]!!}{\left(1+\sqrt{a/b}\right)^{k+1}} = -\frac{(2n)!!}{(1-a/b)^{n+1}}\left[\sqrt{\frac{a}{b}} + \sum_{\ell=0}^{n}\left(1-\frac{a}{b}\right)^\ell\frac{(2\ell-3)!!}{(2\ell)!!}\right]$$

which can be further simplified as

$$\sum_{k=0}^{n}\binom{2n-k-1}{2(n-k)}\frac{k![2(n-k)-1]!!}{(1+x)^{k+1}} = -\frac{(2n)!!}{(1-x^2)^{n+1}}\left[x + \sum_{\ell=0}^{n}\frac{(2\ell-3)!!}{(2\ell)!!}\left(1-x^2\right)^\ell\right] \tag{6.5}$$

for $x \ge 0$ with $(-3)!! = -1$ and $(-1)!! = 0!! = 1$.

More extensively, the identity (6.5) is valid for all $x \in \mathbb{C}\setminus\{-1, 1\}$, because both sides of (6.5) are analytic on the set $\mathbb{C}\setminus\{-1, 1\}$.

Remark 6.3. We note that all proofs in this section are elementary. In other words, Cauchy's integral formula in complex analysis is not employed in all proofs of Lemma 6.1, Lemma 6.2, and Theorem 6.1.

7. More remarks

Finally, we list more remarks on main results of this paper.

Remark 7.1. Let $a \geq 0$, $b > 0$, and $0 < p < 1$. Then, motivated by (4.1), one can consider the function

$$\mathfrak{C}(a,b;p;x) = \frac{\sin(p\pi)}{\pi} \int_0^\infty \frac{t^p}{a^2 + 2a\cos(p\pi)t^p + t^{2p}} \frac{1}{(b+t)^{x+1}} \mathrm{d}t$$

for $x \geq 0$. Can one compute this integral explicitly in terms of closed forms? For more details on so-called closed forms, please refer to [4] and closely related references therein.

Remark 7.2. We do not find integral formulas (5.1), (5.2), and (6.4) in monographs and handbooks [1, 9, 27, 58].

Remark 7.3. Integral representations of the Catalan numbers C_n for $n \geq 0$ have been reviewed and surveyed in [36, Section 2]. The first integral representation

$$C_n = \frac{1}{2\pi} \int_0^4 \sqrt{\frac{4-x}{x}}\, x^n \mathrm{d}x, \quad n \geq 0 \tag{7.1}$$

was discovered in [28] and applied in [51]. An alternative integral representation

$$C_n = \frac{1}{\pi} \int_0^\infty \frac{\sqrt{t}}{(t+1/4)^{n+2}} \mathrm{d}t \tag{7.2}$$

was derived in [46, Theorem 1.3]. The equivalence of integral representations (7.1) and (7.2) was proved in [32, Theorem 1.3]. The integral representation (7.2) can be recovered from (4.1) by setting $a = \frac{1}{2}$, $b = \frac{1}{4}$, and $p = \frac{1}{2}$.

Remark 7.4. For $\alpha \in \mathbb{R}$, we consider the sequence $Z_{n,k}(\alpha)$ generated by

$$\frac{[(1+T)^\alpha - 1]^k}{k!} = \sum_{n \geq k \geq 0} Z_{n,k}(\alpha) \frac{T^n}{n!}.$$

See [6, p. 158]. When $\alpha = \frac{1}{2}$, straightforward computation yields

$$\begin{aligned}
\frac{\left[(1+T)^{1/2} - 1\right]^k}{k!} &= \frac{1}{k!} \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} (1+T)^{\ell/2} \\
&= \frac{1}{k!} \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \sum_{n=0}^{\infty} \binom{\ell/2}{n} T^n \\
&= \sum_{n=0}^{\infty} \left[\frac{1}{k!} \sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} \left\langle \frac{\ell}{2} \right\rangle_n \right] \frac{T^n}{n!}
\end{aligned}$$

which means that

$$Z_{n,k}\left(\frac{1}{2}\right) = \frac{(-1)^k}{k!} \sum_{\ell=0}^{n} (-1)^\ell \binom{k}{\ell} \left\langle \frac{\ell}{2} \right\rangle_n.$$

On the other hand, it is listed in [6, p. 158] that

$$Z_{n,k}\left(\frac{1}{2}\right) = (-1)^{n-k} \frac{(n-1)!}{(k-1)!} \binom{2n-k-2}{n-1} \frac{1}{2^{2n-k}}.$$

Accordingly, we obtain the equality

$$\frac{(-1)^k}{k!} \sum_{\ell=0}^{n} (-1)^\ell \binom{k}{\ell} \left\langle \frac{\ell}{2} \right\rangle_n = (-1)^{n-k} \frac{(n-1)!}{(k-1)!} \binom{2n-k-2}{n-1} \frac{1}{2^{2n-k}}$$

which can be rearranged as (2.5). Consequently, we recover the identity (2.5).

Remark 7.5. This paper is a revised version of the electronic preprint [20, 21].

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