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# Reimann Sums and the Definite Integrals - Definitions and Derivations

## Circumstances behind the Original Proof

Bernhard Reimann was a student of Carl Frederick Gauss, and worked with him in the field of complex analysis. However, relevant to us is his work on real analysis, or the analysis of real numbers and quantities. Bernhard Reimann developed the idea of Reimann Sums by realizing that an integral could be developed by subdividing the region under a curve into rectangles with an infinitely small width and a height the length of the function and adding each of those rectangles up. Doing this Reimann developed the idea of his Sum, by adding a a multitude of thin rectangles with each of their widths approaching zero.

## Reimann Sums: The Essentials

### What is a Reimann Sum?

$$\sum\_{i=0}^{n}(c\_{i}) ⋅Δx\_{i}$$

Put simply, a Reimann sum is a method of writing rectangular approximation in terms of a summation

### Left and Right Reimann Sums

As Reimann sums are functional representations of approximations, the most basic Reimann sums will correspond to the basic approximations: the left hand and right hand rectangular approximations. This difference is important when considering any finite number of steps, and all rules corresponding to left and right approximations will apply to left and right Reimann sums. But, these differences will become inconsequential when infinite steps are considered, which will return the exact area under the function.

## Reimann Integral: The Essentials

### What is a Reimann Definite Integral?

$$\lim\_{n\to \infty ,Δx\to 0} \sum\_{i=0}^{n}(c\_{i}) ⋅Δx\_{i}=\int\_{a}^{b}f(x)dx$$

The Reimann Integral is defined as the limit of the Reimann Sum of a function $c\_{i}$ as $Δx$ approaches zero, or as an infinite quantity of subsections are calculated under a given function, which would be represented by $n$ approaching infinity.

### Explanation and Justification

$$A(x)=\sum\_{x=0}^{n}((a+i(\frac{b−a}{n})) ⋅\frac{b−a}{n})$$

The Reimann sum we wish to use for our justification has terms $Δx=\frac{b−a}{n}$ and $c\_{i}=a+i(Δx)$. This, fully expanded, can be represented as the given Reimann Sum above. The upper bound $n$ represents the exact amount of subdivisions under the function, not the bounds of integration. The selected bounds determine the $Δx$ term within the Reimann sum, which then determines $c\_{i}$.

### Propagation of Differences in the Reimann Sum in Respective Definite Integrals

Changes in $Δx$ and $c\_{i}$ will result in different definite integrals, but will not necessarily result in different answers. Consider this set of potential $Δx$ and $c\_{i}$. Manipulation is necessary in order to change a certain Reimann Sum to a definite integral you want. It is dependent upon what the values you choose for $Δx$ and $c\_{i}$ are. The changing of the values of $Δx$ and $c\_{i}$ will lead to a different function but it works out in the end as the changed bounds make up for the different function within the integral. They each end up covering the same amount of area and end up equaling the same value.

## Beyond the Reimann Integral: Alternate Integrals and Techniques (or what we actually talked about)

Although extremely versatile and useful, the Reimann Integral fails under certain circumstances where the function possesses infinite periodic discontinuities or irregularities. However, these functions can be integrated, but it requires the introduction of a more advanced integral or other evaluation techniques.

### Differentiation under the Integral

Certain functions exist that, while Riemann integrable, are extremely difficult to integrate through the methods learned in Calculus BC. Consider the following function, integrated between 0 and 1:

$$f(t)=\frac{t^{2}+1}{lnt}$$

This is a very problematic function to integrate, for no u-substitution can successfully simplify this. We have not learned any technique to handle this type of function throughout the year. However, if we take a quick dive into multivariable calculus we can solve this integral.

As was taught earlier in the year, the differential and the integral are inverse functions. Normally, differentiating an integral will simply cancel the other out. For single variable functions, this is true in every case. However, multivariable functions can be “equivalent” to the undifferentiated function, but still be a different function in actuality, by differentiating one variable and integrating another. This concept is evident in classical high school physics, where through various transformations something unitarily equivalent to force, for example, can be produced through various different processes.

In the provided example, we can introduce a new parameter $p$, and set it equal to 2, to produce the function:

$$f(t)=\frac{t^{p}+1}{lnt}|\_{p=2}$$

Then, we can solve for an altered $f(t)$, then differentiate and integrate our function. If we differentiate the function by $p$, then integrate it by $t$, we receive a more manageable function.

$$f(t)=\int\_{0}^{1}\frac{∂}{∂p}\frac{t^{p}+1}{lnt}dt=∫t^{p}\*\frac{lnt}{lnt}=\frac{t^{p+1}}{p+1}|\_{0}^{1}=\frac{1}{p+1}$$

We then integrate this function to $lnp+1+C$, then solve as usual.

This process takes advantage of the partial derivative in order to selectively eliminate problematic elements of an integral. Similarly to u-substitution, it alters the function itself in numerically equivalent but computationally different ways to create a far easier to evaluate function. In other cases, it is sufficient to show that the differentiation produces a result that is the same formula as the target function. However, an equivalence must always be established in order to produce a useful result from differentiation under the integral.

We can outline a few general cases where differentiation under the integral is useful. One, introduced in this example, would be $\int\_{0}^{1}\frac{x^{n}+C}{lnx}=lnn+1+C$. Notably, we have identified which values are ultimately inconsequential aside from the C result from integrating which will be 0 in all cases regardless due to the denominator’s $lnx$; the constant in the numerator was eliminated when the derivative was taken. In our multiple choice section, we will introduce another general form, which will be denoted in the respective solution.

### The Lebesgue Integral

The Lebesgue Integral is a generalization of the Reimann Integral, which considers a function‘s area across its range, measuring its area in horizontal slabs which are not necessarily rectangular. This reformulation of the integral expands the set of integrable functions in an important way,

### The Lebesgue Outer Measure

In his dissertation, Lebesgue introduces the concept of measure that we will utilize to construct the Lebesgue integral. The concept of measure is in essence the reduction of a set into more easily measurable sets which we can utilize to estimate the actual measure of more complex sets. Before we do this, we must show two things about measure

1. A set representing the closed interval $[a,b]$ has measure equal to its length $b−a$
	1. By definition, a Cartesian product of intervals on n-dimensional $R^{n}$ has measure equal to the product of the lengths of the individual intervals. Thus, we can simply consider $R^{1}$ and return to the property that an interval has measure equal to its length. Technically, this is already an implicit property of Lebesgue measure, but it isn’t explicitly stated so we reconstruct this property in this redundant fashion.
2. A point has measure 0
	1. A point can be interpreted as an interval $[a,a]$, which clearly has measure 0. Again, this is a property of Lebesgue integration.

Both of these properties are intuitive and trivial facts that are always assumed when we take the measure of a set.

### Constructing the Lebesgue integral

A few aspects of set theory are required in order to achieve this proof. First, we must recognize that an area can be represented by a set of points. Then, we must as said earlier accept the Axiom of Choice (AC). This axiom is specifically denoted as being accepted due to it being controversial due to how it was derived not from set theory but merely stated then observed, tested, and proved to be true. By taking the AC, we are able to select an arbitrary amount of points from each subset of our area, and collect them in arbitrary ways that are easier to measure than a block of volume. The AC is not problematic in two dimensions, the majority of the time where we integrate, but in dimensions higher than two it could be. But, if we were for instance taking the measure of a volume, we could potentially have a selection of sets which can produce a paradoxical result of not returning a value which we would assume possible. However, the Lebesgue measure is undefined in the cases where paradoxical results such as that arise, and those functions which can be measured are specifically denoted as being encompassed by a $σ$-algebra.

For instance, when we are taking our measures, we consider the Lebesgue $σ$-algebra, for some $B\in A,μ(A)=μ(B)+μ(B^{c})$, which states that any measure of a set cannot exceed what we would intuitively expect the measure of the set to be, for simpler sets such as the interval $\{a,b\}$. Technically, this constitutes the Lebesgue outer measure, but Lebesgue integration will come to measure very simple functions which do not necessarily require this nuance. The most basic elementary measures return the range across which the set’s interval spans, which will be the bulk of the measures we need to take.

Next, we must redefine our notion of a function into something we can manipulate with set theory and measure theory. We classically know a function to be a process that accepts elements of a domain and returns elements of a range. Fortunately, our classical definition of a function can be easily stated in terms of set theory, bu accepting the domain and range as subsets of the reals. We then state a function to be a process that relates elements of the set of the domain X onto the set of the codomain, or range, Y, by some process. This redefinition then introduces the notion of a set that contains all elements of Y which are a result of inputting some element from X: the image $f:x\rightarrow y$, which can be stated to be the following set: $F[A]=\{y\in Y|y=f(x)forx\in X\}$. This is ultimately the tool that we will use to integrate our function, but in its current state we cannot actually measure anything, for the measures would be implicitly unbounded and infinite.



Illustration as to why the image is problematic for our integral

In order to solve this problem, we introduce the notion of inverse image. The inverse image, or preimage, is simply the image of the codomain on the domain. This is similar to taking the inverse of a function, but is defined in far more cases than the inverse function is, because it isn’t necessarily applying a process, it only applies the same process backwards. In fact, drawing the function by taking the points from the inverse image and graphing them would produce the exact same graph as drawing the function by taking the points from the image. Our inverse image is now a set of points defined of B, some subset of Y, of the domain X: $F^{−1}[B]=\{x\in X|f(x)\in B\}$.



Now we can consider our function a collection of intervals

Once we consider our function as a collection of intervals, it is very simple to formulate the Lebesgue integral. We construct a Riemann sum of the intervals from the x-axis to infinity, knowing that for any definite integral these measures must converge to 0. Thus, we formulate the Lebesgue integral as the limit of the following Riemann sum, bounded by $\{a,b\}$:

$$\lim\_{d\to\infty}\sum\_{0}^{d} \mu{(\{x\in{X}|f(x)\in{\biggr[\small\frac{i}{n},\small\frac{i+1}{n}\biggr]}\}}\cdot \frac{b-a}{n}$$

The taken measures are trivial for the measure of an interval is simply the length of the interval itself by definition.

### Complex Integration and the Cauchy Integral

Integration of functions in the complex plane, unlike integration in the regular 2D plane, is dependent upon the path taken to get from point a to b. Therefore to integrate such functions, another approach must be taken in order to solve these integrals. To integrate an integral dependent upon the path taken sounds sort of like a line integral and, to integrate in the complex plane, a method very similar to that of line integrals must be used.

In our case, to solve most functions we are integrating about a contour. If you integrate along a contour it is then synonymous with integrating about along a certain line and that is where the similarities lies. If you have a curve in the complex plane that is simple and closed, oriented in the positive (counterclockwise) direction, you can consider it a line integral and theorems such as the Greens Theorem or something synonymous to that in the complex plane could be applied.

Green’s theorem, in multi-variable calculus, is a theorem that basically allows us to integrate line integrals more efficiently without much work. To be able to use Green’s Theorem, certain conditions have to be met. It needs to be a simple, closed region. What that means is that the curve can’t cross or intersect but could have sharp corners or edges. It also needs to connect from end to end. Another condition these curves have to meet is that they need to be oriented positively; they need to have a positive orientation. What that means is that if you were to were to walk along the curve, the region inside would always be on your left side. Once these conditions are met, Green’s Theorem can be used.

$$∮\_{C}^{}(Ldx+Mdy)=∬\_{D}^{}(\frac{∂M}{∂x}−\frac{∂L}{∂y})dxdy$$

There is a special theorem that is synonymous to Green’s Theorem in the study of complex integration. Basically, the theorem states that when integrating along a simple, closed, connected contour in the imaginary plane, the closed integral along any imaginary function F(z) is equal to 0. What that is saying is that, for example, if you had a semicircle that included all the positive imaginary numbers with a radius a, the integral of the function along the contour C added on to the integral of the function from -a to a, would be equal to 0. The combination of this integral would end up sweeping out the the entire semicircle and would integrate over the entire region C.

That synonymous theorem has ties to the Cauchy Integration Formula. This formula can later allow us to not only integrate functions that are closed in the imaginary plane, but also allow us to solve integrals like the Dirichlet integral which is seen later on in the document.

### Edge Cases of non-Lebesgue Integral Function

Although the Lebesgue Integral is merely a generalized Reimann Integral, there is at least one example of an improper integral that can be considered as a Reimann Integral but not a Lebesgue Integral: $f(x)=\frac{sin(x)}{x}$.

This function is undefined under Lebesgue Integration, and will return the incorrect quantity of infinity. However, we can derive $\int\_{0}^{\infty }\frac{sin(x)}{x}dx$ through Euler’s equation, as asked and solved in Questions and Solutions

# Reimann Sums and the Definite Integral: Questions

## Multiple Choice

### 1: Which of the following is a correctly-formed definite integral as a Reimann sum?

1. $\int\_{0}^{2}x^{4}dx$
2. $lim\_{n\rightarrow \infty }\sum\_{i=0}^{n}\frac{i}{6}\*\frac{1}{n}$
3. $lim\_{n\rightarrow \infty }\sum\_{i=0}^{n}\frac{i}{n}^{4}\*\frac{1}{n}$
4. $\sum\_{i=0}^{\infty }\frac{i}{\infty }\*\frac{1}{\infty }$

### 2: Which of the following is not Riemann integrable on the real axis?

1. $\frac{1}{x}$
2. A horizontal line with infinite discontinuities every .01 units along the line
3. $tan(x)$
4. None of the above
5. All of the above

### 3: $\int\_{0}^{1}(xlnx)^{20}dx=$

1. $\frac{20!}{20^{20}}$
2. $\frac{19!}{19^{20}}$
3. $\frac{20!}{21^{21}}$
4. $\frac{21!}{21^{21}}$

### 4: If the definite integral $\int\_{0}^{2}e^{x^{2}}dx$ is first approximated using two inscribed rectangles of equal width and then approximated using the trapezoidal rule with n=2, the difference between the two approximations is

1. 53.60
2. 30.51
3. 27.81
4. 12.78

### 5: $\int\_{1}^{2}\left(4x^{3}−6x\right)dx$

1. 2
2. 4
3. 6
4. 36
5. 42

### 6: $\int\_{0}^{8}\frac{dx}{\sqrt{1+x}}$

1. 1
2. 3/2
3. 2
4. 4
5. 6

## Free Response

### 1: Use a Riemann sum with 4 subdivisions to estimate the value of the following:

$$\int\_{0}^{2}x^{2}dx$$

### 2: Integrate the following function D(x):

$$\left\{\begin{matrix}1&if x is rational\\0&if x is irrational\end{matrix}\right.$$

### 3: Solve using differentiation under the integral

$$\int\_{0}^{\infty }\frac{sin(x)}{x}dx$$

### 4: Solve the previous integral with complex integration

# Reimann Sums and the Definite Integral: Solutions

## Multiple Choice

1) 3. The definite Riemann integral on the interval $[a,b]$ is defined as the Reimann sum: $lim\_{n\rightarrow \infty }\sum\_{i=0}^{\infty }f(x\_{i})Δx$, where $x\_{i}=\frac{i}{n}$ and $Δx=\frac{b−a}{n}$

Only choice 3 accurately conforms to this. Choice 1 is simply a definite integral. Choice 2 lacks the $\frac{b−a}{n}$ term, and choice 4 is not the limit of a Riemann sum as its upper bound approaches infinity, and is therefore not correctly-formed.

2) 4. For choice 1, at 0, the integral for $\frac{1}{x}$ is divergent, only the limit of the integral as it approaches that point exists. However, this is a countable amount of discontinuities and therefore can be ultimately ignored. Choice 2 again represents a countable amount of discontinuities. We will later show that the set of rational numbers is countably infinite, and the discontinuities in 3 are merely a subset of the rationals. Therefore, the discontinuities do not affect the integrability of the function. Similarly, we can evaluate the set of discontinuities of $tanx$, and observe that they differ by integer multiples of $π$. Therefore, we can state that the set of discontinuities of $tanx$ is in fact countably infinite, and therefore does not affect the integrability of the function. Intuitively, all of these functions can theoretically be formulated to be the sum of Riemann integrable parts. Therefore, the only correct choice is 4.

3) 3. We now formulate another standard case of differentiation under the integral, for any $(xlnx)^{n}$.

Consider the function $f(m)=\int\_{0}^{1}x^{m}dx=\frac{1}{m+1}$, at m = n. If we take the partial derivative n times by m, we can produce the following:

$$f^{(n)}(m)=\int\_{0}^{1}ln^{n}(x)x^{m}dx=\frac{(−1)^{n}\*n!}{(m+1)^{n+1}}$$

From this it is evident that our general term is simply the nth derivative of $\frac{1}{n+1}$

Then, we can definitively say that is our integral, as we have shown through differentiation under the integral that the two are equivalent. Only choice 3 conforms to this general form, for $n=20$, therefore it is the correct answer.

Note: This is also a proof by induction of the general solution for that family of definite integrals.

4) Using 2 inscribed rectangles, the area is:

$A\_{rectangle}=∑f(x)\*Δx$

$A\_{rectangle}=f(0)\*1+f(1)\*1$

$A\_{rectangle}=1\*1+e\*1$

$A\_{rectangle}=1+e$

Using 2 trapezoids:

$A\_{trapezoid}=∑[f(x)+f(x+Δx)]\*Δx/2$

$A\_{trapezoid}=[f(0)+f(1)]\*.5+[f(1)+f(2)]\*.5$

$A\_{trapezoid}=[1+e]\*.5+[e+e^{4}]\*.5$

$A\_{trapezoid}=\frac{(1+2e+e^{4})}{2}$

So the difference between these methods is:

$A\_{trapezoid}−A\_{rectangle}$

$\frac{(1+2e+e^{4})}{2}−(1+e)$

$\frac{(1+2e+e^{4})}{2}−\frac{(2+2e)}{2}$

$(1+2e+e^{4}−2−2e)/2$

$(e^{4}−1)/2$

$≈53.6\*.5$

$≈27.8$

5) Using the reverse power rule we get the equation $(x^{4}−3x^{2})$. We then plug in the values defined by the integral and subtract to get our final answer 6.

6) If we let the equation within the radical equal u and do u-substitution, we can change the bounds to make it from 0 to 8 to 1 to 9. Solving the integral and plugging in our new bounds gives us the answer of 4.

## Free Response

1) Simply estimate the sum as follows: $\frac{1}{2}(.5)^{2}+\frac{1}{2}(1)^{2}+\frac{1}{2}(1.5)^{2}+\frac{1}{2}(2)^{2}=\frac{1}{8}+\frac{1}{2}+\frac{9}{8}+2=3.75$

This question should be trivial to solve, and is given only to solicit the actual subdivisions from the answerer. The full solution to this would be exactly what is written, and nothing less.

2) The inverse image of the Dirichlet function, an indicator function which returns 1 for any element in the set $Q$ is the set $Q$. This function is effectively a collection of disjoint, noncontinuous points, which will measure to 0 as the givens for measure state that a point has measure 0 and the measure of a set is equal to the sum of the measures of its subsets. Therefore, the Dirichlet function integrates to 0.

3) This is known as the Dirichlet integral, and can be evaluated in a few different ways, which are mostly years beyond high school calculus. However, Richard Feynman introduced a trick to solving this integral in a simple manner, utilizing Euler’s formula to replace the $sin(x)$ term and produce a more manageable integral.

We introduce a new element to our function, $e^{−ax}$ taken at $a=0$, which does not change the convergence or value of our function at any point but allows us to eliminate the bottom x. By taking the partial derivative of $e^{−ax}\frac{sinx}{x}$ we receive $−e^{−ax}sinx$, which can be integrated by parts to produce $\frac{1}{a^{2}+1}$. This is the derivative of what we want, so we then integrate by a to get $(−tanx^{−1})|\_{0}^{\infty }$. $tan0^{−1}=f(x)$ and $f(x)=tan\infty ^{−1}$, which produces the value $\frac{π}{2}$, our expected result.

Alternatively, we consider $sinx$ to be the imaginary coefficient of $e^{ix}$, and then take the partial derivative and integrate $\frac{e^{x(−a+i)}}{x}$, which leaves only the numerator, which we integrate by x to receive $\frac{1}{−a+i}⋅(e^{0}−e^{−\infty })$. We then multiply the numerator and denominator by $−a−i$, so that we have $\frac{−a−i}{a^{2}+1}$. We then extract the imaginary coefficient, which is the same thing as what we would have found by working through the whole integration by parts. These processes are mathematically equivalent, as proved by induction.

Some trivia: Because this function is not absolutely convergent, it is actually not defined under Lebesgue integration, but is under Riemann integration.

4)

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