Bicycle Model

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#     Kinematic Bicycle Model

As a first approximation of the Car model in the Simulation, we will use the kinematic bicycle model derived in (Rajamani 2006).

This kinematic bicycle model allows describing the lateral motion of a vehicle under certain assumptions. It provides a mathematical describtion of the vehicle motion without considering the forces that affect the motion. The eom are purely based on geometric relationships governing the system.

**Assumptions:**

* Veloctiy vectors at point A and B are in the direction of the orientation of the front and rear wheels. i.e. slip angles are equal to zero. Which is a reasonable assumption for low speed motion of vehicle (e.g. for vehicle with less than 5 m/s)
* beta: angle of the current velocity of the center of mass with respect to the longitudinal axis of the car.
* $delta\_{f}$: front steer angle
* acceleration of the center of mass in the same direction as the velocity

An illustration of the model was used in (Kong 2015) which is shown here:



Copied from (Kong et al. 2015)

The resulting continuous time kinematic bicycle model equation are.

$$\begin{matrix}\dot{x}&=vcos(ψ+β)\\\dot{y}&=vsin(ψ+β)\\\dot{ψ}&=\frac{v}{l\_{r}}sin(β)\\\dot{v}&=a=u\_{1}\\β&=arctan(\frac{l\_{r}}{l\_{r}+l\_{f}}tan(δ\_{f}))=arctan(\frac{l\_{r}}{l\_{r}+l\_{f}}tan(u\_{2}))\end{matrix}$$

with the state vector $[x\_{1},x\_{2},x\_{3},x\_{4}]=[X,Y,v,ψ]$ and input vector $[u\_{1},u\_{2}]=[a,δ\_{f}]$.

$$\begin{matrix}\dot{x}\_{1}&=x\_{3}cos(x\_{4}+β)\\\dot{x}\_{2}&=x\_{3}sin(x\_{4}+β)\\\dot{x}\_{3}&=u\_{1}\\\dot{x}\_{4}&=\frac{x\_{3}}{l\_{r}}sin(β)\\β&=arctan(\frac{l\_{r}}{l\_{r}+l\_{f}}tan(u\_{2}))\end{matrix}$$

Using the Euler Discretizatoion the discrete time kinematic bicycle model writes as:

$$\begin{matrix}x\_{1}^{+}&=x\_{1}+Δt⋅x\_{3}cos(x\_{4}+β)\\x\_{2}^{+}&=x\_{2}+Δt⋅x\_{3}sin(x\_{4}+β)\\x\_{3}^{+}&=x\_{3}+Δt⋅u\_{1}\\x\_{4}^{+}&=x\_{4}+Δt⋅\frac{x\_{3}}{l\_{r}}sin(β)\\β&=arctan(\frac{l\_{r}}{l\_{r}+l\_{f}}tan(u\_{2}))\end{matrix}$$

Which defines the nonlinear model:

$$x^{+}=f(x,u)$$

For further calculations we need the derivative with respect to the states and inputs. To do this the chain rule and the following results were used $a=\frac{l\_{r}}{l\_{r}+l\_{f}}$, $tan(x)′=\frac{1}{cos^{2}(x)}$ and $arctan(x)′=\frac{1}{x^{2}+1}$

$$\begin{matrix}∇\_{x}f(x,u)&=\left[\begin{matrix}1&0&Δtcos(x\_{4}+β)&−Δtx\_{3}sin(x\_{4}+β)\\0&1&Δtsin(x\_{4}+β)&Δtx\_{3}cos(x\_{4}+β)\\0&0&1&0\\0&0&\frac{Δt}{l\_{f}}sin(β)&1\end{matrix}\right]\\β&=arctan(\frac{l\_{r}}{l\_{r}+l\_{f}}tan(u\_{2}))\end{matrix}$$

$$\begin{matrix}∇\_{u}f(x,u)&=\left[\begin{matrix}0&−Δtx\_{3}sin(x\_{4}+β)\frac{∂β}{∂u\_{2}}\\0&Δtx\_{3}cos(x\_{4}+β)\frac{∂β}{∂u\_{2}}\\Δt&0\\0&Δt\frac{x\_{3}}{l\_{r}}cos(β)\frac{∂β}{∂u\_{2}}\end{matrix}\right]\\\frac{∂β}{∂u\_{2}}&=\frac{1}{\frac{l\_{r}}{l\_{r}+l\_{f}}sin^{2}(u\_{2})+\frac{l\_{r}+l\_{f}}{l\_{r}}cos^{2}(u\_{2})}\end{matrix}$$

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# Dynamics Model

The coordinates of the inertial frame can be calculated by using:

$$\begin{matrix}\dot{X}&=\dot{x}cos(ψ)−\dot{y}sin(ψ)\\\dot{Y}&=\dot{x}sin(ψ)+\dot{y}cos(ψ)\end{matrix}$$

# References

Rajamani, Rajesh. 2006. *Vehicle Dynamics and Control*. Mechanical Engineering Series. New York: Springer Science.

Kong, Jason, Mark Pfeiffer, Georg Schildbach, and Francesco Borrelli. 2015. “Kinematic and Dynamic Vehicle Models for Autonomous Driving Control Design”. In *2015 IEEE Intelligent Vehicles Symposium (IV)*, 1094–99. Seoul, South Korea: IEEE. doi:10.1109/IVS.2015.7225830.