The Double Spring Pendulum

John Stone[1]

[1].

May 16, 2019

Abstract

This article provides information on the construction and analysis of a double Spring Pendulum System and its characteristics. Including the Normal Frequencies and Modes of the coupled system, and the analysis of how it behaves under the change in initial conditions.

Introduction

The double pendulum has been a hallmark of exploring chaos for many years. I have chosen to add on to this classic system by changing the regular arms of the pendulum to springs, to add an even greater amount of complexity to the system. Many have already thoroughly analyzed the Double Pendulum, including full quantification of its chaotic properties [1].

Deriving the Equations

In order to solve for the equations needed to model the double spring pendulum system, the Lagrangian is used. Starting with the Kinetic Energy:

$$K = \frac{1}{2}m_2\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

And the Potential Energy is given by:

$$U \ = \frac{1}{2}\dot{k}_1\left(\sqrt{x_1^2 + y_1^2} - L_1\right)^2 + \frac{1}{2}k_2\left(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} - L_1\right)^2 + m_1 g y_2 + m_2 g y_2$$

Now if we apply the Euler Lagrange Equation to the Lagrangian we can get our differential equations for the accelerations.

$$\ddot{x}_1 = \frac{k_1}{m_1}x_1\left(\frac{L_1}{\sqrt{x_1^2+y_1^2}} - 1\right) + \frac{k_2}{m_1}\left(x_1 - x_2\right)\left(\frac{L_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}} - 1\right)$$

$$\ddot{y}_1 = \frac{k_1}{m_1}y_1\left(\frac{L_1}{\sqrt{x_1^2+y_1^2}} - 1\right) + \frac{k_2}{m_1}\left(y_1 - y_2\right)\left(\frac{L_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}} - 1\right) - g$$

$$\ddot{x}_2 = \frac{k_2}{m_1}\left(x_2 - x_1\right)\left(\frac{L_2}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} - 1\right)$$

$$\ddot{y}_2 = \frac{k_2}{m_1}\left(y_2 - y_1\right)\left(\frac{L_2}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}} - 1\right) - g$$

Analysis

Now that we have the equations to work with we can begin modeling the system. Using Scientific Pythons differential equation solver, odeint, we can solve these equations numerically and graph the results. At this point we should check on how accurate our model is. We can do this by checking a few situations that we already know. For one, at a high spring constant the model should behave as a normal double pendulum. As you can see, the system behaves as a normal double pendulum.

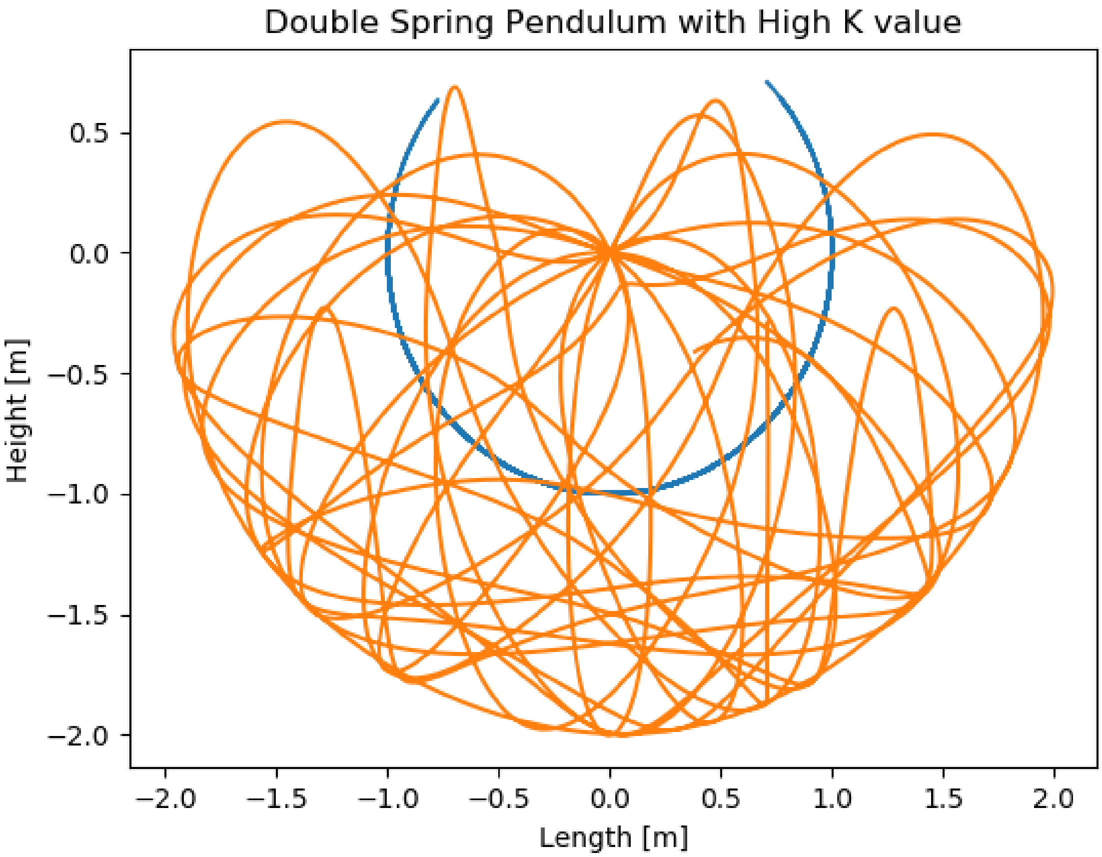

Figure 1: Double Spring Pendulum trajectory with high k value

Next we should make sure that the system conserves energy. By using our equations for Kinetic and Potential Energy shown previously, we can graph the change in total energy over time as the system advances. Comparing the initial value of energy to the final value of energy we can see that there is only a very small change in total energy.

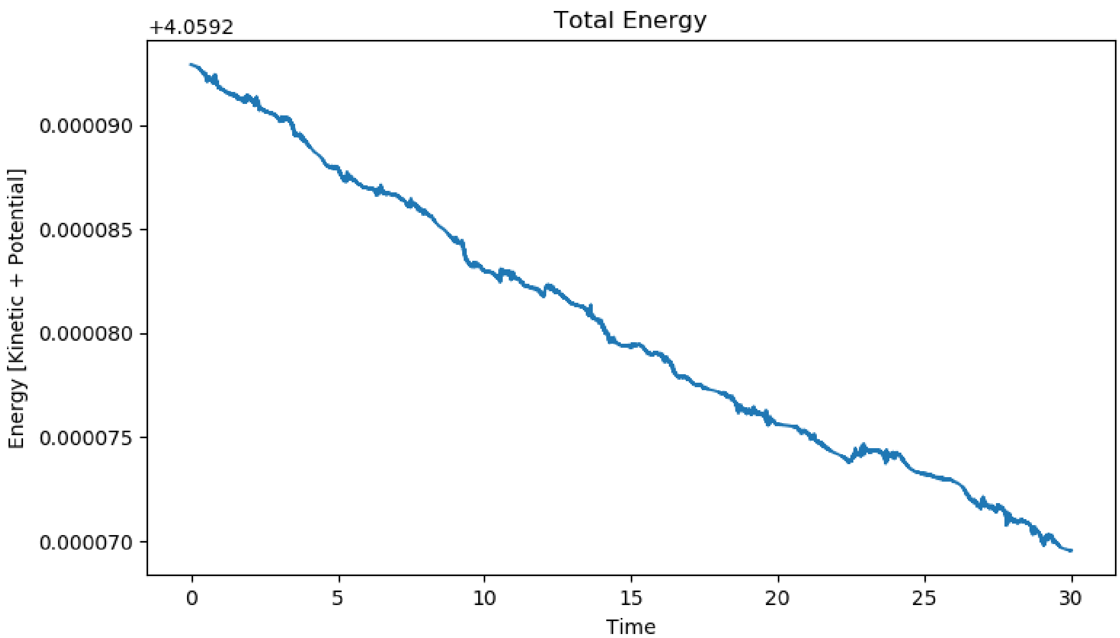

Figure 2: Total Energy over time

Now that we know we have an accurate model, we can actually test it. The first thing that may be helpful is to find the normal frequencies and modes of the system. We will use the K and M matrices gathered from the Lagrangian. The K matrix is:

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 & 0 & 0 \\ 0 & 0 & k_1 + k_2 + mg & -k_2 \\ 0 & 0 & -k_2 & k_2 + mg \end{bmatrix}$$

And the M matrix is:

$$M = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix}$$

Using the linear algebra package in scipy, we can find the eigenvalues that will give us our normal modes:

$$\omega_1^2 = 71.62 \; rad/s$$
$$\omega_2^2 = 8.37 \; rad/s$$
$$\omega_3^2 = 81.42 \; rad/s$$
$$\omega_4^2 = 18.17 \; rad/s$$

These frequencies give us our normalized normal modes:

$$A_1 = \begin{bmatrix} 0.811 \\ -0.584 \\ 0 \\ 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0.584 \\ 0.811 \\ 0 \\ 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 \\ 0 \\ 0.811 \\ -0.584 \end{bmatrix}, A_4 = \begin{bmatrix} 0 \\ 0 \\ 0.584 \\ 0.811 \end{bmatrix}$$

The first of the two normal modes correspond to the simply vertical motion of the two masses if there is no swing at all. These are the easier to find modes. The methods used to calculate these modes are found in Classical Mechanics by Taylor [2].

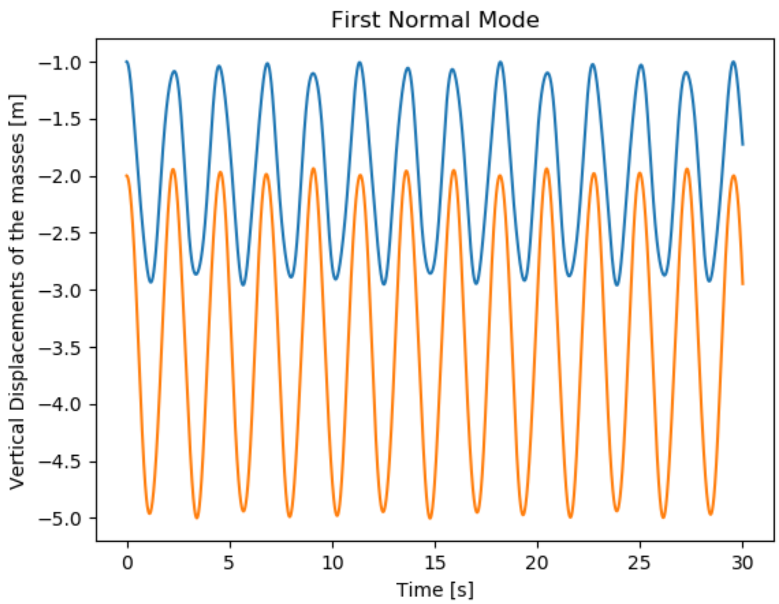

Figure 3: The First normal mode

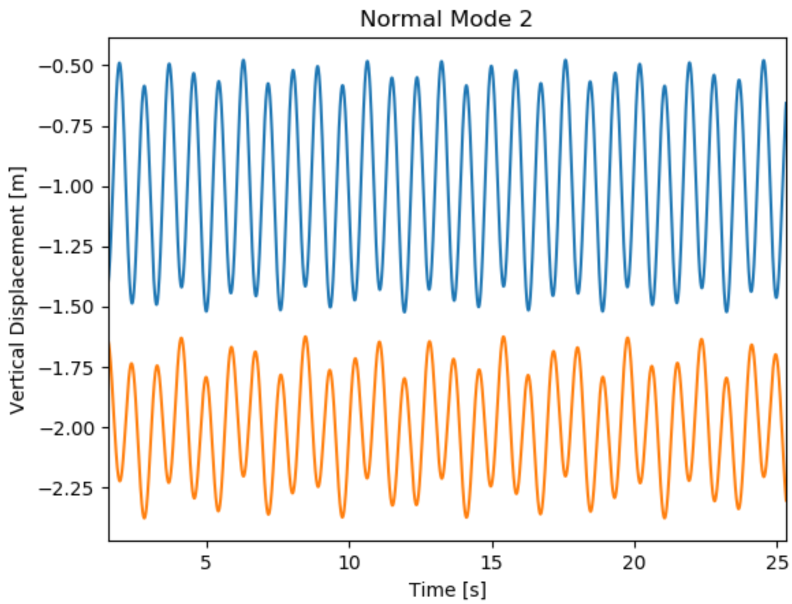

Figure 4: The Sencond Normal mode

The third and fourth normal modes are harder to find as they are the horizontal version of the first two normal modes and finding them becomes very difficult as you have to contend with swing as well as the spring oscillations.

Now we will explore the behavior of the system around its equilibrium point. The x-position of the lower mass will be moved to the right only. This will increase the initial spring stretch as well as the angle off of the vertical. For very small displacement, the motion is uniform. As we increase the amount of displacement, the motion grows more complex. I have picked a few of the most interesting ones.

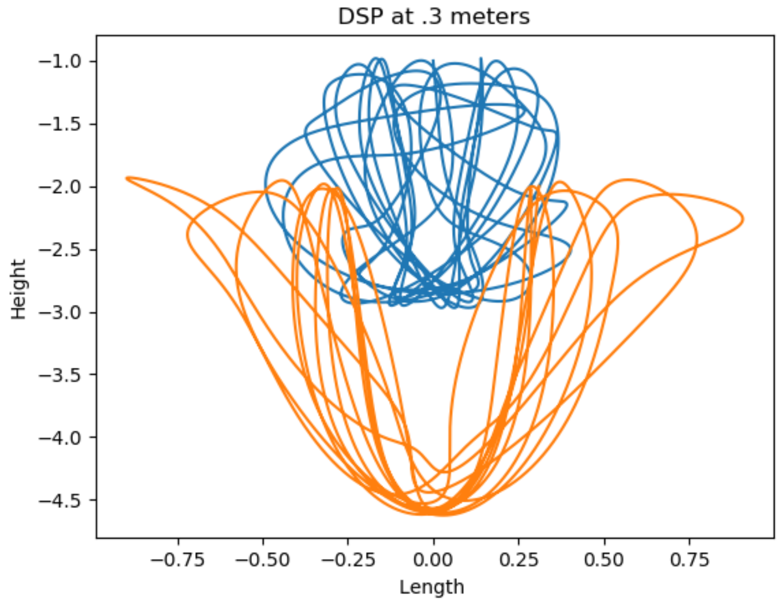

Figure 5: This is a caption

You can see at .3 meters the bottom pendulum is doing a sort of figure eight motion. This shows that there is still some periodicity to the system at this point.

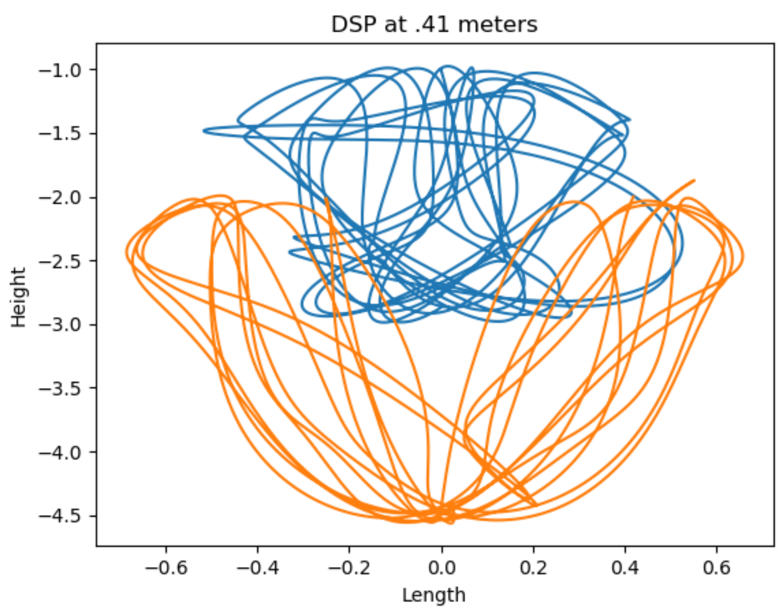

Figure 6: This is a caption

These graphs show that there is some structured oscillations in between the chaotic parts of the system. As the system continues to grow beyond this it only becomes more chaotic except for one intial condition that creates an unexpected order to the graph. When you start the intial mass at .9961 meters this is what you get:

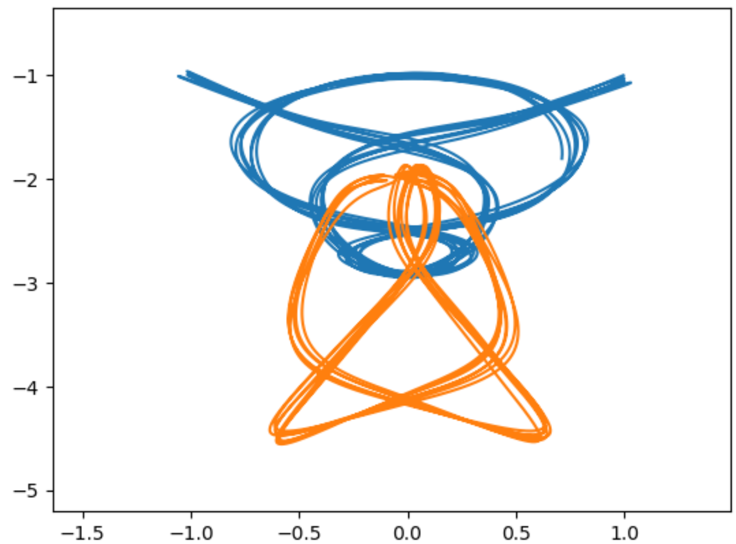

Figure 7: DSP at m1 at .9961 meters

Conclusion

The Spring Double Pendulum is a chaotic system that has some very interesting properties when you get into it. The most interesting pieces are near equilibrium when the system is still nonchaotic and we can see the divergence begin to develop.

References

[1] M. K. Ali, Quantization of the double pendulum, Canadian Journal of Physics, 1996, 74(5-6): 255-262, https://doi.org/10.1139/p96-040

[2] Taylor, John R. *Classical Mechanics*. University Science Books, 2002. Ch. 11 Section 14