Distribution of residual autocorrelations for multiplicative seasonal ARMA models with uncorrelated but non-independent error terms

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In this paper we consider portmanteau tests for testing the adequacy of multiplicative seasonal autoregressive moving-average (SARMA) models under the assumption that the errors are uncorrelated but not necessarily independent. We relax the standard independence assumption on the error term in order to extend the range of application of the SARMA models. We study the asymptotic distributions of residual and normalized residual empirical autocovariances and autocorrelations under weak assumptions on the noise. We establish the asymptotic behaviour of the proposed statistics. A set of Monte Carlo experiments and an application to monthly mean total sunspot number are presented.

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# Introduction

The multiplicative seasonal autoregressive moving average (SARMA, for short) model of order for the univariate time series $X=(X\_t)\_{t\in\mathbb Z}$, is defined by

$$a\_{\theta\_0}(L) \mathbf{a}\_{\theta\_0}(L){X}\_t=b\_{\theta\_0}(L) \mathbf{b}\_{\theta\_0}(L){\epsilon}\_t,\quad
\forall t\in \mathbb Z,$$

where and stands for the backshift operator. The nonseasonal autoregressive (AR, for short) and moving average (MA, for short) operators are defined by and , respectively, while the seasonal AR and MA operators are given by and , respectively, where denotes the length of the seasonal period. It is assumed that the model defined by is stationary, invertible and not redundant. Without loss of generality, we also assume that (by convention ).

In the standard situation ${\epsilon}=({\epsilon}\_t)\_{t\in\mathbb Z}$ is assumed to be a sequence of independent and identically distributed (iid for short) random variables with zero mean and common variance. In this standard framework, is said to be a *strong white noise* and the representation (???) is called a strong SARMA process. In contrast with this previous definition, the representation (???) is said to be a weak SARMA if the noise process is a *weak white noise* , that is, if it satisfies

* , and for all and all .

A strong white noise is obviously a weak white noise, because independence entails uncorrelatedness, but the converse is not true. It is clear from these definitions that the following inclusions hold:

After estimating the SARMA process, the next important step in the modeling consists in checking if the estimated model fits satisfactorily the data. Thus, under the null hypothesis that the model has been correctly identified, the residuals () are approximately a white noise. This adequacy checking step validates or invalidates the choice of the orders and .

Based on the residual empirical autocorrelations , where is the length of the series, (Box & Pierce, 1970) have proposed a goodness-of-fit test, the so-called portmanteau test, for strong ARMA models. A modification of their test has been proposed by (Ljung & Box, 1978). It is nowadays one of the most popular diagnostic checking tools in ARMA modeling of time series. These tests are defined by

where is a fixed integer. The statistic has the same asymptotic chi-squared distribution as and has the reputation of doing better for small or medium sized sample (see (Ljung & Box, 1978)). For weak ARMA models, (Francq, Roy, & Zakoïan, 2005) show that the asymptotic distributions of the statistics defined in are no longer chi-square distributions but a mixture of chi-squared distributions, weighted by eigenvalues of the asymptotic covariance matrix of the vector of autocorrelations. Recently, (Boubacar Maïnassara & Saussereau, 2018) proposed an alternative method based on a self-normalization approach to construct a new test statistic which is asymptotically distribution-free under the null hypothesis.

In many situations, these tests are implemented to check the lack of fit of SARMA models. However, the traditional methodology of Box and Jenkins cannot be extended to the case of SARMA models when because of the multiplicative contraints on the parameters. This standard methodology needs to be adapted to take into account the possible lack of independence of the error terms. See for instance (Duchesne, 2007) and (McLeod, 1978) who considered serial correlation testing in multiplicative seasonal univariate time series models. Duchesne (see (Duchesne, 2007)) proposed his test statistic based on a kernel-based spectral density estimator, whose weighting scheme is more adapted to autocorrelations associated to seasonal lags. The standard test procedures consist in rejecting the null hypothesis of a SARMA model if the statistics (???) are larger than a certain quantile of a chi-squared distribution with degrees of freedom. Consequently, these standard tests are not applicable for .

The works on the portmanteau statistic of SARMA models are generally performed under the assumption that the errors are independent. This independence assumption is often considered too restrictive by practitioners. It precludes conditional heteroscedasticity and/or other forms of nonlinearity (see (Francq & Zakoïan, 2005), for a review on weak univariate ARMA models). In this framework, we relax the standard independence assumption on the error term in order to be able to cover SARMA representations of general nonlinear models. For the asymptotic theory of weak SARMA, notable exception are (Boubacar Maïnassara & Ilmi Amir, 2018) where the consistency and the asymptotic normality of the quasi-maximum likelihood estimator (QMLE) for weak multivariate SARMA models are studied. They also study a particular case of the asymptotic distributions of residual autocovariances and autocorrelations at the seasonal lags .

This paper is devoted to the problem of the validation step of weak SARMA representations. We consider portmanteau test statistics based on the residual empirical autocorrelations but not necessarily at multiple lags as in (Boubacar Maïnassara & Ilmi Amir, 2018). For such models, we show that the asymptotic distributions of the statistics defined in (???) are no longer chi-square distributions but a mixture of chi-squared distributions, weighted by eigenvalues of the asymptotic covariance matrix of the vector of autocorrelations as in (Boubacar Maïnassara & Ilmi Amir, 2018). We also proposed another modified statistics based on a self-normalization approach which are asymptotically distribution-free under the null hypothesis and generalize the result of (Boubacar Maïnassara & Saussereau, 2018).

In Monte Carlo experiments, we illustrate that the proposed test statistics have reasonable finite sample performance. Under nonindependent errors, it appears that the standard test statistics are generally non reliable, overrejecting severely, while the proposed tests statistics offer satisfactory levels. Even for independent errors, they seem preferable to the standard ones, when the number of autocorrelations is small. Moreover, the error of first kind is well controlled. Contrarily to the standard tests , the proposed tests can be used safely for small (see for instance Figure ???). For all these above reasons, we think that the modified versions that we propose in this paper are preferable to the standard ones for diagnosing SARMA models under nonindependent errors. Other contribution is to improve the results concerning the statistical analysis of weak SARMA models by considering the adequacy problem.

The article is organized as follows. In the next section, we briefly recall the results on the QMLE asymptotic distribution obtained by (Boubacar Maïnassara & Ilmi Amir, 2018) when satisfies mild mixing assumptions. We study the asymptotic behaviour of the residuals autocovariances and autocorrelations under weak assumptions on the noise in Subsection 3.1. It is also shown how the standard portmanteau tests must be adapted in the case of multiplicative seasonal ARMA models with non-independent innovations. In Subsection 3.2 we derive the asymptotic distribution of residuals autocovariances and autocorrelations using self-normalization approach and we establish the asymptotic behaviour of the proposed statistics. Section ??? proposes numerical illustrations and an illustrative application on real data. We provide a conclusion in Section ???. The technical proofs are relegated to the appendix.

# Estimating weak SARMA models

In this section, we recall the results on the QMLE asymptotic distribution obtained by (Boubacar Maïnassara & Ilmi Amir, 2018) when satisfies mild mixing assumptions in order to have a self-containing paper.

The unknown parameter of interest is supposed to belong to the parameter space

$$\begin{aligned}
\Theta & = \left
\{\theta=(a\_{1},...,a\_{p},b\_{1},...,b\_{q},\mathbf{a}\_{1},...,\mathbf{a}\_{P},\mathbf{b}\_{1},...,\mathbf{b}\_{Q})'
\in{\Bbb R}^{k\_0},\text{ where } k\_0=p+q+P+Q,
\right.\\&\left. a\_{\theta}(z)=1-\sum\_{i=1}^{p}a\_{i}z^{i},\, \mathbf{a}\_{\theta}(z)=1-\sum\_{j=1}^{P}\mathbf{a}\_{j}z^{sj},\, b\_{\theta}(z)=1-\sum\_{i=1}^{p}b\_{i}z^{i}\text{
and } \mathbf{b}\_{\theta}(z)=1-\sum\_{j=1}^{Q}\mathbf{b}\_{j}z^{sj} \right.\\&\left. \hspace{0.5cm} \text{ have all
their zeros outside the unit disk and have no zero in common}\right \}.\end{aligned}$$

To ensure the asymptotic theory of the QMLE, we assume that the parametrization satisfies the following smoothness conditions. Without loss of generality, we may assume that is compact.

* .

For the asymptotic normality of the QMLE, additional assumptions are required. It is necessary to assume that is not on the boundary of the parameter space .

* We have , where denotes the interior of .

To control the serial dependence of the stationary process , we introduce the strong mixing coefficients defined by

$$\alpha\_{\epsilon}\left(h\right)=\sup\_{A\in\mathcal F^t\_{-\infty},B\in\mathcal F\_{t+h}^{+\infty}}\left|\mathbb{P}\left(A\cap
B\right)-\mathbb{P}(A)\mathbb{P}(B)\right| ,$$

where $\mathcal F\_{-\infty}^t=\sigma (\epsilon\_u, u\leq t )$ and $\mathcal F\_{t+h}^{+\infty}=\sigma (\epsilon\_u, u\geq t+h )$. We use to denote the Euclidian norm of a column vector . We will make an integrability assumption on the moment of the noise and a summability condition on the strong mixing coefficients .

Assumption **(A3)** from (Francq & Zakoïan, 1998; Francq & Zakoïan, 2005) is a technical condition for proving the asymptotic theory of the QMLE. The integrability assumption on the moment of the noise is not very restrictive in this framework because the innovation process is directly observed (see (Romano & Thombs, 1996)).

For the estimation of SARMA and multivariate SARMA models, the commonly used estimation method is the quasi-maximum likelihood estimation, which can be also viewed as a nonlinear least squares estimation (LSE). Given a realization satisfying (???), the variable can be approximated, for by defined recursively by

where the unknown initial values are set to zero: . The Gaussian quasi-likelihood is given by

A QMLE of is a measurable solution of

In all the sequel, we denote by $\xrightarrow[]{\mathrm{d}}$, the convergence in distribution. The symbol $\mathrm{o}\_{\mathbb P}(1)$ is used for a sequence of random variables that converges to zero in probability. Under the above assumptions, (Boubacar Maïnassara & Ilmi Amir, 2018) showed that as and

$$\begin{aligned}
\sqrt{n}\left(\hat{\theta}\_n-\theta\_0\right)=-J ^{-1}\frac{1}{\sqrt{n}}\sum\_{t=1}^n \Upsilon\_t+\mathrm{o}\_\mathbb P(1)\xrightarrow[n\to\infty]{\mathrm{d}} {\mathcal
N}(0,\Sigma=J^{-1}IJ^{-1}),\end{aligned}$$

where

$$\begin{aligned}
J&=J(\theta\_0)= \frac{2}{\sigma\_0^2} \mathbb E \left [ \frac{\partial \epsilon\_t(\theta\_0)}{\partial\theta}\frac{\partial \epsilon\_t(\theta\_0)}{\partial\theta'} \right ],\quad
I=I(\theta\_0)=\sum\_{h=-\infty}^{+\infty}\mathrm{Cov}({\Upsilon}\_t,{\Upsilon}\_{t-h})\\ \text{ and } {\Upsilon}\_t&=\frac{2}{\sigma\_0^2}
\epsilon\_t(\theta\_0)\frac{\partial
\epsilon\_t(\theta\_0)}{\partial\theta}.\end{aligned}$$

Note that, the existence of the matrix is a consequence of **(A3)** and of Davydov’s inequality (see (Davydov, 1968)).

# Diagnostic checking in weak SARMA models

In order to check the validity of the SARMA model, it is a common practice to examine the QMLE residuals where is given by for all . For a fixed integer , consider the vector of residual autocovariances

In the sequel, we will also need the vector of the first sample autocorrelations

The statistics are usually used to test the following null hypothesis

* $({X}\_t)\_{t\in \mathbb Z}$ satisfies a SARMA representation;

against the alternative

* $({X}\_t)\_{t\in \mathbb Z}$ does not admit a SARMA representation or $({X}\_t)\_{t\in \mathbb Z}$ satisfies a SARMA representation with or or or .

## Asymptotic distribution of the residual autocorrelations

First note that the mixing assumptions **(A3)** entail the asymptotic normality of the “empirical” autocovariances

It should be noted that is not a computable statistic because it depends on the unobserved innovations except when . Define the matrix

$$\begin{aligned}
\Xi=\left(
\begin{array}
{cc} \Sigma&
\Sigma\_{\hat\theta\_n,\gamma\_m}
\\ \Sigma'\_{\hat\theta\_n,\gamma\_m} &
\Sigma\_{\gamma\_m}
\end{array}
\right)=\sum\_{h=-\infty}^{+\infty}\mathbb{E}w\_tw'\_{t-h},\text{ where } w\_t=\left(
\begin{array}
{c}w\_{1t}\\w\_{2t}\end{array}
\right)\in\mathbb R^{k\_0+m}\end{aligned}$$

with and . Note that, the existence of the matrix is a consequence of **(A3)** and of Davydov’s inequality (see (Davydov, 1968)).

The asymptotic distribution of will be obtained from the joint asymptotic distribution of

$$\sqrt{n}\left(\hat{\theta}'\_n-\theta'\_0,\gamma'\_m\right)'=\frac{1}{\sqrt{n}}\sum\_{t=1}^n w\_t+\mathrm{o}\_{\mathbb{P}}(1)\xrightarrow[n\to\infty]{\mathrm{d}} {\mathcal
N}(0,\Xi),$$

by applying the central limit theorem for mixing processes (see (Herrndorf, 1984)).

Now, considering and as values of the same function at the points and , a Taylor expansion about gives

where is between and The last equality follows from the consistency of and the fact that is not correlated with when Then for we have

where

The following Proposition, which is a generalization of Theorem 4 obtained by (Boubacar Maïnassara & Ilmi Amir, 2018) gives the limiting distribution of the residual autocovariances and autocorrelations of SARMA models.

Proposition 1

When, , , and , under the above assumptions, we have

$$\sqrt{n}\hat\gamma\_m\xrightarrow[n\to\infty]{\mathrm{d}}\mathcal{N}\left(0,
\Sigma\_{\hat\gamma\_m}\right)\quad and \quad
\sqrt{n}\hat\rho\_m\xrightarrow[n\to\infty]{\mathrm{d}}\mathcal{N}\left(0,
\Sigma\_{\hat\rho\_m}\right),$$

The proof of this result is similar to that given by (Boubacar Maïnassara & Ilmi Amir, 2018) for Theorem 4.

The asymptotic variance matrices and depend on the unknown matrices , and the scalar . Matrix and can be estimated by its empirical counterpart, respectively

Note that the matrix is the spectral density at frequency zero of the process , thus an estimator of is given in Theorem 6 of (Boubacar Maïnassara & Ilmi Amir, 2018). Other estimators of such long-run variances are available in the literature (see for instance (Andrews, 1991), (Berk, 1974), (den Haan & Levin, 1997), (Newey & West, 1987), for general references). For the numerical illustrations presented in this paper, we used a Vector AR (VAR) spectral estimator given in Theorem 6 of (Boubacar Maïnassara & Ilmi Amir, 2018) consisting in: i) fitting VAR models for to the series , , where is obtained by replacing by in ; ii) selecting the order which minimizes an information criterion and approximating by times the spectral density at frequency zero of the estimated VAR model. Hereafter, we used the AIC model selection criterion with .

From Proposition 1 we can deduce the following result, which gives the exact limiting distribution of the standard portmanteau statistics under general assumptions on the innovation process of the fitted SARMA model.

Theorem 1

Under Assumptions in Proposition 1 and ${\bf (H0)}$, the statistics and converge in distribution, as to

where is the vector of the eigenvalues of the matrix and are independent variables.

As in (Boubacar Maïnassara & Ilmi Amir, 2018), Theorem 1 shows that for the asymptotic distribution of and , the approximation is no longer valid in the framework of weak SARMA models. The true asymptotic distribution depends on nuisance parameters involving , the matrix and the elements of . Consequently, in order to obtain the asymptotic distribution of the portmanteau statistics under weak assumptions on the noise, one needs a consistent estimator of the asymptotic covariance matrix . We let the matrix obtained by replacing by , by and by in . Denote by the vector of the eigenvalues of . At the asymptotic level , the LB (Ljung-Box) test (resp. the BP (Box-Pierce) test) consists in rejecting the adequacy of the weak SARMA model when

where is such that . We emphasize the fact that the proposed modified versions of the Box-Pierce and Ljung-Box statistics are more difficult to implement because their critical values have to be computed from the data.

## Self-normalized asymptotic distribution of the residual autocorrelations

The nonparametric kernel estimator (see (Andrews, 1991; Newey & West, 1987)), used to estimate the matrix causes serious difficulties regarding the choice of the sequence of weights. The parametric approach based on an autoregressive estimate of the spectral density of studied for instance by (Berk, 1974), (Boubacar Mainassara, 2011; Boubacar Mainassara & Francq, 2011; Boubacar Maïnassara & Ilmi Amir, 2018; den Haan & Levin, 1997) is also facing the problem of choosing the truncation parameter. So the choice of the order of truncation is often crucial and difficult. In this section, we propose as in (Boubacar Maïnassara & Saussereau, 2018) an alternative method where we do not estimate an asymptotic covariance matrix. It is based on a self-normalization based approach to construct a test-statistic which is asymptotically distribution-free under the null hypothesis (see (Boubacar Maïnassara & Saussereau, 2018), for a reference in the ARMA cases). The idea comes from (Lobato, 2001) and has been already extended by (Kuan & Lee, 2006), (Shao, 2010; Shao, 2010; Shao, 2012) to more general frameworks. See also (Shao, 2015) for a review on some recent developments on the inference of time series data using the self-normalized approach. In this case, the critical values are not computed from the data since they are tabulated. In some sense, this method is finally closer to the standard method in which the critical values are simply deduced from a -table.

We denote the matrix in $\mathbb R^{m\times(k\_0+m)}$ defined in block formed by , where is the identity matrix of order . In view of and , we deduce that

$$\begin{aligned}
\sqrt{n}\ \hat\gamma\_m & = \frac{1}{\sqrt{n}}\sum\_{t=1}^n\ \Lambda w\_t + \mathrm{o}\_{\mathbb P}(1).\end{aligned}$$

Contrarily to Subsection 3.1, we do not rely on the classical method that would consist in estimating the asymptotic covariance matrix of . We need to apply the functional central limit theorem holds for the process (see Lemma 1 in (Lobato, 2001)).

Finally, we define the normalization matrix $C\_{m}\in\mathbb R^{m\times m}$ by

To ensure the invertibility of the normalization matrix which is proved in Lemma 6 of (Boubacar Maïnassara & Saussereau, 2018), we need the following technical assumption on the distribution of .

* .

Let be a -dimensional Brownian motion starting from . For , we denote $\mathcal U\_K$ the random variable defined by

$$\mathcal U\_K={B}'\_{K}(1){V}\_{K}^{-1}{B}\_{K}(1)\text{ where }
{V}\_{K}=\int\_0^1\left ( {B}\_{K}(r)-r{B}\_{K}(1)\right)\left({B}\_{K}(r)-r{B}\_{K}(1)\right)'dr.$$

The following theorem states the asymptotic distributions of the sample autocovariances and autocorrelations.

Theorem 2

We assume that , , or . Under Assumptions of Proposition 1 , **(A4)** and under the null hypothesis **(H0)**, we have

$$\begin{aligned}
n\, \hat\gamma\_m'C\_{m}^{-1}\hat\gamma\_m \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{U}\_{m}
\text{ and }n\,\sigma\_0^4\hat{{\rho}}\_m'C\_{m}^{-1}\hat{{\rho}}\_m
\xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{U}\_{m}\end{aligned}$$

The proof of this result is postponed to Section ???.

Of course, the above theorem is useless for practical purpose, because it does not involve any observable quantities. In practice, one has to replace the matrix and the variance of the noise by their empirical or observable counterparts. The matrix can be easily estimated by his empirical counterpart

Thus we define

Finally we denote the normalization matrix $\hat C\_m\in\mathbb R^{m\times m}$ by

The following result is the applicable counterpart of Theorem 2.

Theorem 3

Assume that , , or . Under Assumptions of Theorem 2, we have

$$\begin{aligned}
n\, \hat\gamma\_m'\hat C\_m^{-1}\hat\gamma\_m \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{U}\_m\text{ and }
Q\_m^\mathrm{SN}=n\,\hat\sigma^4 \,\hat\rho\_m'\hat C\_m^{-1}\hat\rho\_m \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{U}\_m.\end{aligned}$$

The proof of this result is postponed to Section ???.

Based on the above result, we propose a modified version of the Ljung-Box statistic when one uses the statistic

where the matrix $D\_{n,m}\in\mathbb R^{m\times m}$ is diagonal with as diagonal terms.

# Numerical illustrations

In this section, by means of Monte Carlo experiments, we investigate the finite sample properties of the modified version of the portmanteau tests that we introduced in this work. The numerical illustrations of this section are made with the open source statistical software R (see R Development Core Team, 2017) or (see http://cran.r-project.org/).

## Simulated models

First of all, we introduce the models that we simulate and we indicate the conventions that we adopt in the discussion and in the tables:

* and refer to modified LB and BP tests using and in Section 3.1
* and refer to LB and BP tests using the standard statistics .
* and refer to modified tests using the self-normalized statistics in Section 3.2

To generate the strong and the weak SARMA models, we consider the following SARMA model

with and the innovation process follows a strong or weak white noise.

The generalized autoregressive conditional heteroscedastic (GARCH) models is an important example of weak white noises in the univariate case (see (Francq & Zakoïan, 2010)). So we first assume that in (???) the innovation process is the following ARCH model defined by

where is a sequence of iid standard Gaussian random variables. To generate the strong SARMA, we assume that in (???) the innovation process follows (???) with .

## Empirical size

We first simulate independent trajectories of size of models (???). The same series is partitioned as two series of sizes and . For each of these replications, we use the quasi-maximum likelihood estimation method to estimate the coefficient and we apply portmanteau tests to the residuals for different values of , where is the number of autocorrelations used in the portmanteau test statistic. For the nominal level , the empirical size over the independent replications should vary between the significant limits 3.6% and 6.4% with probability 95% and belong to with a probability 99%. When the relative rejection frequencies are outside the 95% significant limits, they are displayed in bold type and they are underlined when they are outside the 99% significant limits in Tables ??? and ???.

For the standard Box-Pierce test, the model is therefore rejected when the statistic or is larger than in a SARMA case (see (McLeod, 1978)). Consequently the empirical size is not available (n.a.) for the statistic or because they are not applicable for . For the proposed self-normalized test or , the model is rejected when the statistic or is larger than , where the critical values (for ) are tabulated in Lobato (see Table 1 in (Lobato, 2001)).

Table ??? displays the relative rejection frequencies of the null hypothesis ${\bf (H0)}$ that the data generating process (DGP for short) follows a strong SARMA model (???)–(???) with , over the independent replications. When the seasonal period is , for all tests, the percentages of rejection belong to the confident interval with probabilities 95% and 99%, except for and when . Consequently all these tests well control the error of first kind. In contrast, when , our proposed tests well also control the error of first kind (except for and when ) contrarily to the standard tests , for all sizes. We draw the conclusion that, in this strong SARMA case, the proposed modified version may be clearly preferable to the standard ones.

Now, we repeat the same experiments on a weak SARMA models. As expected, Table ??? shows that the standard or test poorly performs in assessing the adequacy of this particular weak SARMA model. It can be seen that: 1) the observed relative rejection frequencies of and are definitely outside the significant limits, 2) the errors of the first kind are only globally well controlled by the proposed tests, for all when is large. We also tried the case where the ARCH model (???) have infinite fourth moments. As showing in Figure ???, the results are qualitatively similar to what we observe here.

Figure ??? displays the residual autocorrelations of a realization of size for weak SARMA model (???)-(???) with and their 5% significance limits under the strong SARMA and weak SARMA assumptions. This figure confirms clearly the conclusions drawn from Table ???. The horizontal dotted lines (blue color) correspond to the 5% significant limits obtained under the strong SARMA assumption. The solid lines (red color) and dashed lines (green color) correspond also to the 5% significant limits under the weak SARMA assumption. The full lines correspond to the asymptotic significance limits for the residual autocorrelations obtained in Proposition 1. The dashed lines (green color) correspond to the self-normalized asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.

In these Monte Carlo experiments, we illustrate that the proposed test statistics have reasonable finite sample performance. Under nonindependent errors, it appears that the standard test statistics are generally non reliable, overrejecting severely, while the proposed tests statistics offer satisfactory levels. Even for independent errors, they seem preferable to the standard ones, when the number of autocorrelations is small and when . Moreover, the error of first kind is well controlled. Contrarily to the standard tests based on or , the proposed tests can be used safely for small (see for instance Figure ???). For all these above reasons, we think that the modified versions that we propose in this paper are preferable to the standard ones for diagnosing SARMA models under nonindependent errors.

c c ccc ccc c  
& Length & Lag & &&&&&  
&& &4.3& 4.0& **6.8**& **6.8**& &  
&& &5.8& 5.4& 5.8& 5.6 & &**6.7**  
4 && &4.9& 4.8& 6.0& 5.4& & 6.3  
&&&6.3& 5.3& 5.1& 4.7& 5.8& 5.3  
&&&5.7& 5.1& 5.4& 4.3 &5.7 &4.9  
&&&5.7& 5.0& 5.2& 3.8& 5.4& 4.1  
&& &4.1& 4.1& 5.7& 5.6&&  
&& &4.2 &4.2& 4.7& 4.7& 5.9& 5.6  
4 && &4.9& 4.8& 3.9& 3.9& 4.6& 4.4  
&&&4.6& 4.6& 4.9& 4.4& 4.9& 4.5  
&&&4.0& 3.8& 4.4& 4.2& 4.6& 4.1  
&&&4.5& 4.5& 5.1& 5.0& 5.4& 5.1  
&& &4.7 & 4.5& & & &  
&& & **6.5** & 6.4 &&& &  
12 && &5.4 & 5.1&&&&  
&&&6.0& 5.6&&&&  
&&&5.1& 4.5& &&&  
&&&5.5& 4.3&&&&  
&& &4.4 & 4.3 & & &&  
&& &4.9& 4.9& 5.8& 5.7&&  
12 && &4.5& 4.4& 6.0& 5.6&&**6.6**  
&&&4.5& 4.5& 6.1& 6.0&&**6.9**  
&&&5.4& 5.3& 6.0& 5.7&**6.9**&**6.5**  
&&&4.9& 4.8& 6.3& 6.1 &&

c c ccc ccc c  
& Length & Lag & &&&&&  
&& & & & 4.5 & 4.3& &  
&& & & &**3.3**&**3.2**& &  
4 && &**3.5**&**3.2**&&& &  
&&& & & & & &  
&&&&& &&&  
&&&&&&&&  
&& &5.5 & 5.4 & 4.2 & 4.2&&  
&& &4.8 & 4.8 & 3.6 & 3.6& &  
4 && &4.5 & 4.3 & 3.8 & 3.6& &  
&&&4.8& 4.8 & 3.6 & 3.6&&  
&&&3.7 & 3.7 & **3.4** &**3.4**&&  
&&&**3.2**&&**3.5**&**3.5**&&  
&& &3.6 &**3.5** & &&&  
&& &4.0 & 3.8 & 6.4 & 6.0 & &  
12 && &4.2 &**3.5** & 5.5 & 5.2& &  
&&& & & 5.4 & 4.8& &  
&&& & & 4.3 & 4.2&&  
&&&&& 4.9 & 4.2&&  
&& &4.4& 4.4 & 4.5 & 4.5&&  
&& &4.8 & 4.7 & 4.1& 3.9& &  
12 && &3.8 & 3.8 & 4.6& 4.4&&  
&&&4.6 & 4.5 &**3.5** &**3.4**& &  
&&&3.8 & 3.7 & 3.6& **3.5**&&  
&&&**3.5** &**3.3** & 4.1 & 3.8&&

 Autocorrelation of a realization of size n=2,000 for weak SARMA(0,1)(0,1)_{12} model (???)-(???) with \alpha_1=1.3. The horizontal dotted lines (blue color) correspond to the 5% significant limits obtained under the strong SARMA assumption. The solid lines (red color) and dashed lines (green color) correspond also to the 5% significant limits under the weak SARMA assumption. The full lines correspond to the asymptotic significance limits for the residual autocorrelations obtained in Proposition 1. The dashed lines (green color) correspond to the self-normalized asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.

Autocorrelation of a realization of size for weak SARMA model (???)-(???) with . The horizontal dotted lines (blue color) correspond to the 5% significant limits obtained under the strong SARMA assumption. The solid lines (red color) and dashed lines (green color) correspond also to the 5% significant limits under the weak SARMA assumption. The full lines correspond to the asymptotic significance limits for the residual autocorrelations obtained in Proposition 1. The dashed lines (green color) correspond to the self-normalized asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.

## Empirical power

In this section we repeat the same experiments as in Section ??? to examine the power of the tests for the null hypothesis of a SARMA against the following SARMA alternative defined by

with and where the innovation process follows a strong or weak white noise introduced in Section ???. For each of these replications we fit a SARMA models and perform standard and modified tests based on , and residual autocorrelations.

Tables ??? and ??? compare the empirical powers of Model - with and respectively over the independent replications. For these particular strong and weak SARMA models, we notice that the standard and and our proposed tests have very similar powers except for and when in the weak case.

c c ccc ccc c  
& Length & Lag & &&&&&  
&& &98.9& 98.9& 100.0 &100.0 &100.0 &100.0  
&& &98.1 & 98.0& 100.0& 100.0 &100.0& 100.0  
4 && &96.3 & 96.3& 100.0& 100.0 &100.0 &100.0  
&&&94.9 & 94.9 & 99.9 & 99.9& 100.0& 100.0  
&&&93.7 & 93.5 & 99.8 & 99.7& 100.0 &100.0  
&&&91.6 & 91.4 & 99.7 & 99.7 &100.0& 100.0  
&& &100.0 &100.0& 100.0& 100.0 &100.0& 100.0  
&& &100.0 &100.0& 100.0& 100.0 &100.0& 100.0  
4 && &100.0 &100.0& 100.0& 100.0 &100.0& 100.0  
&&&100.0 &100.0& 100.0& 100.0 &100.0& 100.0  
&&&100.0 &100.0& 100.0& 100.0 &100.0& 100.0  
&&&100.0 &100.0& 100.0& 100.0 &100.0& 100.0  
&& &98.1 & 98.1& 100.0& 100.0& 100.0 &100.0  
&& &97.7& 97.7 & 99.9 & 99.9& 100.0& 100.0  
12 && &96.6 & 96.6 & 99.9 & 99.9& 100.0 &100.0  
&&&95.5 & 95.4 & 99.9 & 99.9& 100.0& 100.0  
&&&92.9 & 92.7 & 99.9 & 99.9& 100.0& 100.0  
&&&90.9 & 90.5 & 99.9 & 99.9& 100.0 &100.0  
&& &100.0& 100.0& 100.0& 100.0& 100.0& 100.0  
&& &100.0& 100.0& 100.0& 100.0 &100.0& 100.0  
12 && &100.0 &100.0& 100.0& 100.0 &100.0& 100.0  
&&&100.0& 100.0& 100.0& 100.0& 100.0& 100.0  
&&&100.0& 100.0& 100.0& 100.0& 100.0& 100.0  
&&&100.0 &100.0& 100.0& 100.0 &100.0& 100.0

c c ccc ccc c  
& Length & Lag & &&&&&  
&& &89.2 & 89.1 & 98.5 & 98.5& 100.0& 100.0  
&& &83.0 & 82.9 & 96.8 & 96.7 &100.0& 100.0  
4 && &71.5 & 71.5 & 96.3 & 96.3& 100.0& 100.0  
&&&64.9& 64.7 & 95.6 & 95.6& 100.0& 100.0  
&&&56.2 & 55.7 & 95.1 & 95.0& 100.0& 100.0  
&&&50.6 & 49.4 & 94.5& 94.4& 100.0& 100.0  
&& &99.5 & 99.5& 100.0& 100.0& 100.0& 100.0  
&& &99.4 & 99.4& 100.0& 100.0& 100.0& 100.0  
4 && &99.1 & 99.1& 100.0& 100.0& 100.0& 100.0  
&&&99.2 & 99.2 & 99.9 & 99.9& 100.0& 100.0  
&&&99.5& 99.5& 100.0 &100.0& 100.0& 100.0  
&&&99.2 & 99.2 & 99.9 & 99.9& 100.0 &100.0  
&& &90.1 & 90.1 & 98.4 & 98.3& 100.0& 100.0  
&& &82.4& 82.2& 97.3 & 97.2& 100.0& 100.0  
12 && &74.3 & 74.0& 97.0& 96.9& 100.0 &100.0  
&&&68.7 & 68.3 & 96.9 & 96.5& 100.0 &100.0  
&&&56.9 & 56.0 & 96.1& 95.7& 100.0& 100.0  
&&&48.9 & 47.9 & 96.0 &95.7 &100.0 &100.0  
&& &99.6 & 99.6& 100.0& 100.0& 100.0& 100.0  
&& &99.6 & 99.6 & 99.9 & 99.9& 100.0& 100.0  
12 && &99.5 & 99.5& 100.0& 100.0& 100.0& 100.0  
&&&99.4 & 99.4& 100.0& 100.0& 100.0& 100.0  
&&&99.5 & 99.5& 100.0& 100.0& 100.0& 100.0  
&&&99.0 & 99.0& 100.0& 99.9& 100.0 &100.0

## Application to real data

We now consider an application to monthly mean total sunspot number obtained by taking a simple arithmetic mean of the daily total sunspot number over all days of each calendar month. The observations (sunspot) covered the period from January 01, 2010 to December 31, 2018 which correspond to observations. The series exhibit seasonal behavior . The data were obtain from the website of the World Data Center, Solar Influences Data Analysis Center, Royal Observatory of Belgium (http://www.sidc.be/silso/datafiles).

Let and denoting by the mean-corrected series. We adjust the particular SARMA model of the form

The quasi-maximum likelihood estimators of were obtained as

where the estimated asymptotic standard errors obtained from (respectively the -values), of the estimated parameters (first column), are given into brackets (respectively in parentheses). We apply portmanteau tests to the residuals of this model. Figure ??? displays the residual autocorrelations and their 5% significance limits under the strong SARMA and weak SARMA assumptions. In view of Figure ???, the diagnostic checking of residuals does not indicate any inadequacy. All of the sample autocorrelations should lie between the bands (at 95%) shown as dashed lines (green color), solid lines (red color) and the horizontal dotted (blue color).

 Autocorrelation of the particular SARMA(3,1)(0,1)_{12} model residuals for the mean-logarithm-corrected of monthly mean total sunspot number. The horizontal dotted lines (blue color) correspond to the 5% significant limits obtained under the strong SARMA assumption. The solid lines (red color) and dashed lines (green color) correspond also to the 5% significant limits under the weak SARMA assumption. The full lines correspond to the asymptotic significance limits for the residual autocorrelations obtained in Proposition 1. The dashed lines (green color) correspond to the self-normalized asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.

Autocorrelation of the particular SARMA model residuals for the mean-logarithm-corrected of monthly mean total sunspot number. The horizontal dotted lines (blue color) correspond to the 5% significant limits obtained under the strong SARMA assumption. The solid lines (red color) and dashed lines (green color) correspond also to the 5% significant limits under the weak SARMA assumption. The full lines correspond to the asymptotic significance limits for the residual autocorrelations obtained in Proposition 1. The dashed lines (green color) correspond to the self-normalized asymptotic significance limits for the residual autocorrelations obtained in Theorem 3.

# Conclusion

From these simulation experiments and from the asymptotic theory, we draw the conclusion that the standard methodology, based on the QMLE, allows to fit SARMA representations of a wide class of nonlinear time series. But it is often restrictive to consider that the innovation process is directly observed. In future works, we intent to study how the existing estimation (see (Francq & Zakoïan, 2004; Francq & Zakoïan, 2010)) and diagnostic checking (see (Zhu, 2013)) procedures should be adapted in the situation where the GARCH process used in these simulation experiments is not directly observed, but constitutes the innovation of an observed SARMA-(seasonal)GARCH process which will be able to extend considerably the range of applications.

# Proofs

The proofs of Theorems 2 and 3 follow the same lines as in (Boubacar Maïnassara & Saussereau, 2018) and are similar. To have its own autonomy, the proofs will be rewrite and adapt.

**Proof of Theorem 2**

We recall that the Skorokhod space $\mathbb D^k [0{,}1]$ is the set of $\mathbb R^k-$valued functions defined on which are right continuous and have left limits. It is endowed with the Skorokhod topology and the weak convergence on $\mathbb D^k [0{,}1]$ is mentioned by $\xrightarrow[]{\mathbb{D}^k}$. We finally denote by the integer part of the real .

To prove the result we need to recall some following results of (Boubacar Maïnassara, 2014), (Francq, Roy, & Zakoïan, 2005; Francq & Zakoïan, 1998; Francq & Zakoïan, 2000; McLeod, 1978).

We denote by , , and the coefficients defined by

Following (Francq, Roy, & Zakoïan, 2005) and (McLeod, 1978) (see also (Boubacar Maïnassara, 2014)), the noise derivatives involving in the expression of and can be represented as

with when .

For any $\theta\in\Theta\subset \mathbb R^{k\_0}$ and any , under the above Assumptions, there exists absolutely summable and deterministic sequences , and such that, almost surely,

with . A useful property of the above three sequences that they are asymptotically exponentially small. Indeed there exists and a positive constant such that, for all , we have

See Lemmas A.1. and A.2. of (Francq & Zakoïan, 2000) for a more detailed treatment.

Now, in view of it is clear that the asymptotic behaviour of is related to the limit distribution of . First, we prove that converges on the Skorokhod space to a Brownian motion. More precesily, we have to show that

$$\frac{1}{\sqrt{n}}\sum\_{j=1}^{\lfloor nr\rfloor }\Lambda w\_j
\xrightarrow[n\to\infty]{\mathbb D^m}
\left(\Psi\Psi'\right)^{1/2} B\_{m}(r)$$

where is a -dimensional standard Brownian motion.

Using , the process can be rewritten as

and thus the non-correlation between implies that has zero expectation with values in $\mathbb R^{k\_0+m}$. In order to apply the functional central limit theorem for strongly mixing process, we need to identify the asymptotic covariance matrix in the classical central limit theorem for the sequence . It is proved in Subsection 3.1 that

$$\begin{aligned}
\frac{1}{\sqrt{n}}\sum\_{t=1}^{n}w\_t \xrightarrow[n\to\infty]{\mathrm{d}}\mathcal{N}(0,\Xi:=2\pi f\_w(0))\end{aligned}$$

where is the spectral density of the stationary process $(w\_t)\_{t\in\mathbb Z}$ evaluated at frequency 0. The main issue is to prove the existence of the matrix which is a consequence **(A3)** and Davydov’s inequality (Davydov, 1968). For that sake, one has to introduce for any integer , the random variables

Since depends on a finite number of values of the noise-process , it also satisfies a mixing property (see Theorem 14.1 in (Davidson, 1994), p. 210). Based on the Davydov inequality (see (Davydov, 1968)), the arguments developed in the Lemma A.1 in (Francq, Roy, & Zakoïan, 2005) (see also (Francq & Zakoïan, 1998)) imply that

$$\begin{aligned}
\frac{1}{\sqrt{n}}\sum\_{t=1}^{n}w^k\_t \xrightarrow[n\to\infty]{\mathrm{d}}\mathcal{N}(0,\Xi\_k)\end{aligned}$$

where

and thus holds. Moreover we have that .

Since the matrix is positive definite, it can be factorized as where the lower triangular matrix has nonnegative diagonal entries. Therefore, we have

$$\frac{1}{\sqrt{n}}\sum\_{t=1}^{n}\Lambda w\_t \xrightarrow[n\to\infty]{\mathrm{d}}\mathcal{N}(0,\Lambda\Xi\Lambda'),$$

and the new variance matrix can also been factorized as , where . Thus, ${n}^{-1/2}\sum\_{t=1}^{n}\left(\Psi\Psi'\right)^{-1/2}\Lambda w\_t \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}\left(0,I\_{m}\right)$ where is the identity matrix of order . The above arguments also apply to matrix with some matrix which is defined analogously as . Consequently,

$$\frac{1}{\sqrt{n}}\sum\_{t=1}^{n}\Lambda w\_t^k \xrightarrow[n\to\infty]{\mathrm{d}}\mathcal{N}(0,\Lambda\Xi\_k\Lambda'),$$

and we also have ${n}^{-1/2}\sum\_{t=1}^{n}\left(\Psi\_k\Psi\_k'\right)^{-1/2}\Lambda w^k\_t \xrightarrow[n\to\infty]{\mathrm{d}} \mathcal{N}\left(0,I\_{m}\right)$

Now we are able to apply the functional central limit theorem for strongly mixing process of (Herrndorf, 1984). We have for any ,

$$\begin{aligned}
\frac{1}{\sqrt{n}}\sum\_{j=1}^{\lfloor nr\rfloor }\left(\Psi\_k\Psi\_k'\right)^{-1/2}\Lambda w\_j^k
\xrightarrow[n\to\infty]{\mathbb D^m}
B\_{m}(r) .\end{aligned}$$

For all , we write

and we obtain that

$$\begin{aligned}
\frac{1}{\sqrt{n}}\sum\_{j=1}^{\lfloor nr\rfloor }\left(\Psi\Psi'\right)^{-1/2}\Lambda w\_j^k
\xrightarrow[n\to\infty]{\mathbb D^m}
B\_{m}(r) .\end{aligned}$$

In order to conclude , it remains to observe that, uniformly with respect to ,

$$\begin{aligned}
Z^k\_n(r):=\frac{1}{\sqrt{n}}\sum\_{j=1}^{\lfloor nr\rfloor }\left(\Psi\Psi'\right)^{-1/2}\Lambda Y\_j^k
\xrightarrow[k\to\infty]{\mathbb D^m}
0 ,\end{aligned}$$

where

By Lemma 4 in (Francq & Zakoïan, 1998), we have

$$\sup\_{n} \mathrm{Var} \left ( \frac{1}{\sqrt{n}} \sum\_{j=1}^n Y\_j^k \right ) \xrightarrow[k\to\infty]{} 0$$

and since ,

$$\sup\_{0\le r\le 1}\sup\_n \left ( \| Z\_n^k(r) \| \right ) \xrightarrow[k\to\infty]{} 0 .$$

Thus is true and the proof of the first step is achieved.

The previous step ensures us that Assumption 1 in (Lobato, 2001) is satisfied for the sequence . We follow the arguments developed in Sections 2 and 3 in (Lobato, 2001), the second step is to show that

$$\begin{aligned}
C\_m=\frac{1}{n^2}\sum\_{t=1}^{n}S\_{t}S'\_{t} \xrightarrow[n\to\infty]{\mathrm{d}} \left(\Psi\Psi'\right)^{1/2}{V}\_{m}\left(\Psi\Psi'\right)^{1/2},\end{aligned}$$

by applying the continuous mapping theorem on the Skorokhod space and where the random variable is defined in . The main issue is to obtain that

$$\begin{aligned}
\nonumber
\frac{1}{\sqrt{n}}{S}\_{[nr]} &=\frac{1}{\sqrt{n}}\sum\_{j=1}^{[nr]}\left({\Lambda}{w}\_j-\Lambda\bar{w}\right)
=\frac{1}{\sqrt{n}}\sum\_{j=1}^{[nr]}{\Lambda}{w}\_j-\frac{[nr]}{n}\left(\frac{1}{\sqrt{n}}\sum\_{j=1}^{n}\Lambda {w}\_j\right)\\
&\xrightarrow[n\to\infty]{\mathbb D^m}
\left(\Psi\Psi'\right)^{1/2} B\_{m}(r)-r\left(\Psi\Psi'\right)^{1/2} B\_{m}(1), \end{aligned}$$

by using and the fact that as . In view of , it follows that by continuous mapping theorem

$$\begin{aligned}
C\_m&=\frac{1}{n^2}\sum\_{t=1}^{n}S\_{t}S'\_{t} =\frac{1}{n}\sum\_{t=1}^{n}\left(\frac{1}{\sqrt{n}}{S}\_{t}\right)\left(\frac{1}{\sqrt{n}}{S}\_{t}\right)'
\\& \xrightarrow[n\to\infty]{\mathrm{d}} \left(\Psi\Psi'\right)^{1/2}\left(\int\_0^1\left ( {B}\_{m}(r)-r{B}\_{m}(1)\right)\left({B}\_{m}(r)-r{B}\_{m}(1)\right)'dr\right)\left(\Psi\Psi'\right)^{1/2}\\&= \left(\Psi\Psi'\right)^{1/2}{V}\_{m}\left(\Psi\Psi'\right)^{1/2},\end{aligned}$$

which prove . Since , using (???) and (???) we obtain

$$\begin{aligned}
n\hat\gamma\_m'C\_m^{-1}\hat\gamma\_m & = \left(\frac{1}{\sqrt{n}}\sum\_{t=1}^{n}\Lambda w\_t+\mathrm{o}\_\mathbb{P}(1) \right)'C\_{m}^{-1}\left(\frac{1}{\sqrt{n}}\sum\_{t=1}^{n}\Lambda w\_t+\mathrm{o}\_\mathbb{P}(1) \right) \\
&\xrightarrow[n\to\infty]{\mathrm{d}} \left(\left[\Psi\Psi'\right]^{1/2}{B}\_{m}(1)\right)'\left(\left[\Psi\Psi'\right]^{1/2}{V}\_{m}\left\{\left[\Psi\Psi'\right]^{1/2}\right\}'\right)^{-1}\left(\left[\Psi\Psi'\right]^{1/2}{B}\_{m}(1)\right)
\\&\qquad={B}'\_{m}(1){V}\_{m}^{-1}{B}\_{m}(1)=:\mathcal U\_m.\end{aligned}$$

The proof of Theorem 2 is then complete.

**Proof of Theorem 3:**

We write where . There are three kinds of entries in the matrix . The first one is a sum composed of

for . Using and the consistency of , we have almost surely. The two last kinds of entries of come from the following quantities for and

and they also satisfy almost surely by using and . Consequently, almost surely as goes to infinity. Thus one may find a matrix , that tends to the null matrix almost surely, such that

Thanks to the arguments developed in the proof of Theorem 2, converges in distribution. So tends to zero in distribution, hence in probability. Then and have the same limit in distribution and the result is proved.

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