# On the Exponential Diophantine Equation $(a^n - 2)(b^n - 2) = x^2$

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October 14, 2019

**Abstract**

In this paper, we deal with the equation

$$(a^n - 2)(b^n - 2) = x^2, 2 \leq a < b, a, b, x, n \in \mathbb{N}. \tag{1}$$

We solve the equation (1) for $(a,b) \in \{(2,10),(4,100),(10,58),(3,45)\}$. Moreover, we show that $(a^n - 2)(b^n - 2) = x^2$ has no solution $n, x$ if $2|n$ and $\gcd(a,b) = 1$. We also give a conjecture which says that the equation $(2^n - 2)((2P_k)^n - 2) = x^2$ has only the solution $(n,x) = (2, Q_k)$, where $k > 3$ is odd and $P_k, Q_k$ are Pell and Pell Lucas numbers, respectively. We also conjecture that if the equation $(a^n - 2)(b^n - 2) = x^2$ has a solution $n, x$, then $n \leq 6$.

*Keywords:* Pell equation, exponential Diophantine equation, Lucas sequence
*MSC:* 11D61, 11D31, 11B39

## 1 Introduction

Over the last decades, several authors have dealt with the equation

$$(a^n - 1)(b^n - 1) = x^2, \; x, n \in \mathbb{N}, \tag{2}$$

where $a > 1$ and $b > 1$ are different fixed integers. Firstly, Szalay [14] handled the equation (2) for $(a,b) = (2,3), (2,5)$, and $(2, 2^k)$. He showed that there is no solution if $(a,b) = (2,3)$. Also he proved that there is only the solution $n = 1$ for $(a,b) = (2,5)$, and only the solution $k = 2$ and $n = 3$ for $(a,b) = (2, 2^k)$ with $k > 1$. Then, in [4], the authors determined that the equation (2) has no solutions for $(a, b) = (2, 6)$ and that has only the solutions $(a, n, k) = (2, 3, 2), (3, 1, 5)$, and $(7, 1, 4)$ for $(a, b) = (a, a^k)$ with $kn > 2$. Last result was extended by Cohn [1] to the case $a^l = b^k$. He also proved that the equation (2) has no solutions if $4|n$ except for $(a, b) = (13, 239)$, in which case $n = 4$. Later, in [7, 11, 17, 18], the authors studied the equation (2) for the different values of $a$ and $b$. Lastly, in [6], Keskin proved that the equation (2) has no solutions for $n > 4$ with $2|n$ if $a$ and $b$ have opposite parity. Keskin also proved that if $\gcd(a, b) = 1$, $2||n$, and $n > 4$, then the equation (2) has no solutions.

Motivated by the above studies, in this study, we consider the equation (1). By assuming the *abc* conjecture is true, in [8], Luca and Walsh gave the following theorem:

**Theorem 1** *Let $a, b, c, d, e$ be non-zero integers. Then the abc conjecture implies that the equation*

$$(ax^m + b)(cy^n + d) = ez^2$$

*has only finitely many solutions $(x, y, z, m, n)$ satisfying $xyz \neq 0$, $dax^m \neq bcy^n$ and* $\min(m, n) \geq 5$.

This theorem implies that the equation (1) has only finitely many solutions $n, x$ if $a$ and $b$ are different fixed positive integers. For more information on the *abc* conjecture, one can consult [13]. In this paper, we discuss on the solution of the equation (1) and solve this equation for $(a, b) \in \{(2, 10), (4, 100), (10, 58), (3, 45)\}$. Our method is elementary and use solutions of Pell equations and properties of the first and second kind Lucas sequences. In the last section, we give a conjecture which says that the equation $(2^n - 2)((2P_k)^n - 2) = x^2$ has only the solution $(n, x) = (2, Q_k)$, where $k > 3$ is odd and $P_k, Q_k$ are Pell and Pell Lucas numbers, respectively. We also conjecture that if the equation $(a^n - 2)(b^n - 2) = x^2$ has a solution $n, x$, then $n \leq 6$, where $2 \leq a < b$.

## 2 Preliminaries

In this study, while solving the equation (1), the first and second kind of Lucas sequence $(U_n(P, Q))$ and $(V_n(P, Q))$ play an essential role. So, we need to recall them.

Let $P$ and $Q$ be nonzero relatively prime integers such that $P^2 + 4Q > 0$. Define

$$U_0(P, Q) = 0,\ U_1(P, Q) = 1,\ U_{n+1}(P, Q) = PU_n(P, Q) + QU_{n-1}(P, Q) \text{ for } n \geq 1,$$

$$V_0(P, Q) = 2,\ V_1(P, Q) = P,\ V_{n+1}(P, Q) = PV_n(P, Q) + QV_{n-1}(P, Q) \text{ for } n \geq 1.$$

Sometimes, we write $U_n$ and $V_n$ instead of $U_n(P, Q)$ and $V_n(P, Q)$, respectively. For $(P, Q) = (2, 1)$, we have Pell and Pell-Lucas sequences $(P_n)$ and $(Q_n)$. The following identities concerning the sequences $(U_n)$ and $(V_n)$, which will be used in the next section, are well known (see [12, 15, 16]).

Let $d = (m, n)$. Then

$$(U_m, U_n) = U_d \tag{3}$$

and

$$(V_m, V_n) = \begin{cases} V_d & \text{if } m/d \text{ and } n/d \text{ are odd,} \\ 1 \text{ or } 2 & \text{otherwise.} \end{cases} \tag{4}$$

$$V_{2n} = V_n^2 - 2(-Q)^n \tag{5}$$

$$V_{3n} = V_n(V_n^2 - 3(-Q)^n) \tag{6}$$

$$V_n(P, -1) = U_{n+1}(P, -1) - U_{n-1}(P, -1). \tag{7}$$

If $P$ is even, then

$$V_n \text{ is even,}$$

$$2|U_n \text{ if and only if } 2|n, \tag{8}$$

and

$$U_n(P, -1) - U_{n-1}(P, -1) \text{ is odd.} \tag{9}$$

Let $d$ be a positive integer which is not a perfect square and $N$ be any nonzero fixed integer. Then the equation $x^2 - dy^2 = N$ is known as Pell equation. For $N = 1$, the equation $x^2 - dy^2 = 1$ is known as classical Pell equation. We use the notations $(x, y)$, and $x + y\sqrt{d}$ interchangeably to denote solutions of the equation $x^2 - dy^2 = N$. Also, if $x$ and $y$ are both positive, we say that $x + y\sqrt{d}$ is positive solution to the equation $x^2 - dy^2 = N$. The least positive integer solution $x_1 + y_1\sqrt{d}$ to the equation $x^2 - dy^2 = N$ is called the fundamental solution to this equation.

Now, consider the Pell equation

$$x^2 - dy^2 = 1. \tag{10}$$

If $x_1 + y_1\sqrt{d}$ is the fundamental solution of the equation (10), then all positive integer solutions of this equation are given by

$$x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n \tag{11}$$

with $n \geq 1$. Using the identity (10) and definitions of the sequences $U_n(P, Q)$ and $V_n(P, Q)$, the following lemma can be proved (see also Lemma 13 in [6]).

**Lemma 2** *Let $x_1 + y_1\sqrt{d}$ be the fundamental solution of the equation $x^2 - dy^2 = 1$. Then all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by*

$$x_n = \frac{V_n(2x_1, -1)}{2} \text{ and } y_n = y_1 U_n(2x_1, -1)$$

*with $n \geq 1$.*

By Theorem 110 given in [10], we can give the following lemma.

**Lemma 3** *Let $k_1+t_1\sqrt{d}$ be the fundamental solution of the equation $u^2-dv^2=2$ and let $x_1+y_1\sqrt{d}$ be the fundamental solution of the equation $x^2-dy^2=1$. Then all positive integer solutions of the equation $u^2-dv^2=2$ are given by*

$$X_n+Y_n\sqrt{d}=\left(k_1+t_1\sqrt{d}\right)\left(x_1+y_1\sqrt{d}\right)^n \tag{12}$$

*with $n\geq 1$.*

Let us denote $v_2(m)$ by the exponent of 2 in the factorization of the positive integer $m$. Then we have

**Lemma 4** (*Lemma 2.1,* [2])*If $P$ is even, then*

$$v_2(V_n(P,-1))=\begin{cases} v_2(P) & n\equiv 1(\operatorname{mod}2),\\ 1 & n\equiv 0(\operatorname{mod}2).\end{cases}$$

**Lemma 5** (*Theorem 2.2,* [15])*Let $a$ be a positive integer which is not a perfect square and let $b$ be a positive integer. Let $u_1\sqrt{a}+v_1\sqrt{b}$ be the minimal solution of the equation $ax^2-by^2=1$ and $P=4au_1^2-2$. Then all positive integer solutions of the equation $ax^2-by^2=1$ are given by $(x,y)=(u_1(U_{m+1}-U_m),v_1(U_{m+1}+U_m))$ with $n\geq 0$, where $U_m=U_m(P,-1)$.*

**Lemma 6** (*3.5.Corollary,* [16])*Let $n\in\mathbb{N}\cup\{0\}$, $m,r\in\mathbb{Z}$ and $m$ be a nonzero integer. Then*

$$U_{2mn+r}(P,-1)\equiv U_r(P,-1)(\operatorname{mod}U_m(P,-1)). \tag{13}$$

*We have*

$$3|U_n(P,-1)\Longleftrightarrow\begin{cases} 3|P \text{ and } 2|n,\\ P\equiv 1(\operatorname{mod}3) \text{ and } 3|n,\end{cases} \tag{14}$$

*and*

$$5|U_n(P,-1)\Longleftrightarrow\begin{cases} P\equiv 0(\operatorname{mod}5)\text{ and }2|n,\\ P^2\equiv 1(\operatorname{mod}5)\text{ and }3|n,\\ P^2\equiv -1(\operatorname{mod}5)\text{ and }5|n,\end{cases} \tag{15}$$

for every natural number $n$.

**Lemma 7** *Let $P\equiv 1(\operatorname{mod}a)$ and $n\geq 1$. Then $a|U_n(P,-1)-U_{n-1}(P,-1)$ if and only if $n\equiv 2{,}5(\operatorname{mod}6)$.*

The above identities and Lemma 7 can be proved by using the identity (13).

## 3 Main Theorems

From now on, we assume that the numbers $a$ and $b$ are fixed integers such that $2 \leq a < b$ and $n > 1$.

**Theorem 8** *The equation $(a^n - 2)(b^n - 2) = x^2$ has no solution $(n, x)$ in the following cases:*

$i)$ if $a$ and $b$ have opposite parity,
$ii)$ if $a = 2^t r$ and $b = 2^l s$ with $t \neq l$ and $r, s$ odd, and $2 \mid n$,
$iii)$ if $a = 2^t r$ and $b = 2^l r$ such that $t$ and $l$ have the same parity with $t \neq l$ and $r$ is odd.

**Proof.** $i)$ Without loss of generality, assume that $a$ is even and $b$ is odd. Say $a = 2^t r$ with $r$ odd and $t \geq 1$. Then we have the equation

$$2(2^{nt-1}r^n - 1)(b^n - 2) = x^2.$$

It can be seen that the number $v_2\left(2(2^{nt-1}r^n - 1)(b^n - 2)\right)$ is odd. This contradicts the fact that the number $v_2\left(x^2\right)$ is even.

$ii)$ Let $2|n$. Assume that $a = 2^t r$ and $b = 2^l s$ such that $r$ and $s$ are odd, and $t \neq l$. Then we get

$$(2^{nt-1}r^n - 1)(2^{nl-1}s^n - 1) = (x/2)^2. \tag{16}$$

Let $d = \gcd\left(2^{nt-1}r^n - 1, 2^{nl-1}s^n - 1\right)$. Thus, from the equation (16), we obtain

$$2\left(2^{\frac{nt-2}{2}} r^{\frac{n}{2}}\right)^2 - du^2 = 1 \text{ and } 2\left(2^{\frac{nl-2}{2}} s^{\frac{n}{2}}\right)^2 - dv^2 = 1$$

for some integers $u$ and $v$ with $\gcd(u, v) = 1$. By Lemma 5, it follows that

$$2^{\frac{nt-2}{2}} r^{\frac{n}{2}} = u_1(U_{c+1} - U_c) \text{ and } 2^{\frac{nl-2}{2}} s^{\frac{n}{2}} = u_1(U_{k+1} - U_k),$$

where $u_1\sqrt{2} + v_1\sqrt{d}$ is the minimal solution of the equation $2x^2 - dy^2 = 1$. Since $U_{c+1} - U_c$ and $U_{k+1} - U_k$ are odd by the identity (9), it can be seen that

$$v_2(u_1) = v_2\left(2^{\frac{nt-2}{2}}\right) = v_2\left(2^{\frac{nl-2}{2}}\right).$$

This shows that $t = l$, which contradicts the fact that $t \neq l$.

$iii)$ If $2 \mid n$, then the proof is obvious from $ii)$. Now let $2 \nmid n$. Assume that $a = 2^t r$ and $b = 2^l r$ such that $t$ and $l$ have the same parity with $t \neq l$. Then, we get

$$(2^{nt-1}r^n - 1)(2^{nl-1}r^n - 1) = (x/2)^2.$$

Let $n = 2k + 1$. We shall discuss separately the proof according to whether both of $t$ and $l$ are even or odd.

Let $t$ and $l$ be odd. In this case, we obtain the equation

$$r\left(2^{\frac{nt-1}{2}}r^k\right)^2 - du^2 = 1 \text{ and } r\left(2^{\frac{nl-1}{2}}r^k\right)^2 - dv^2 = 1 \tag{17}$$

for some integers $u$ and $v$ with $\gcd(u,v)=1$, where $d=\gcd\left(2^{nt-1}r^n-1, 2^{nl-1}r^n-1\right)$. If $r$ is not a perfect square, then Lemma 5 and the identity (17) imply that

$$2^{\frac{nt-1}{2}}r^k = u_1(U_{c+1}-U_c) \text{ and } 2^{\frac{nl-1}{2}}r^k = u_1(U_{k+1}-U_k), \tag{18}$$

and if $r=z^2$, then Lemma 2 and (17) imply that

$$2^{\frac{nt-1}{2}}z^{2k+1} = \frac{V_{m_1}(2x_1,-1)}{2} \text{ and } 2^{\frac{nl-1}{2}}z^{2k+1} = \frac{V_{m_2}(2x_1,-1)}{2}, \tag{19}$$

where $u_1\sqrt{r}+v_1\sqrt{d}$ and $x_1+y_1\sqrt{d}$ are, respectively, the minimal solution of the equations $rx^2-dy^2=1$ and $x^2-dy^2=1$. Since $U_{c+1}-U_c$ and $U_{k+1}-U_k$ are odd by the identity (9), (18) gives us that

$$v_2(u_1) = v_2\left(2^{\frac{nt-1}{2}}\right) = v_2\left(2^{\frac{nl-1}{2}}\right).$$

Thus, we get $t=l$, which contradicts the fact that $t\neq l$. On the other hand, since $V_{m_1}(2x_1,-1)/2$ and $V_{m_2}(2x_1,-1)/2$ are even by (19), it is obvious that $v_2(V_{m_1}), v_2(V_{m_2}) \geq 2$. Therefore, we see that

$$v_2(2x_1) = v_2\left(2^{\frac{nt-1}{2}}z^{2k+1}\right) = v_2\left(2^{\frac{nl-1}{2}}z^{2k+1}\right)$$

by Lemma 4. From here, we get $\frac{nt-1}{2}=\frac{nl-1}{2}$, i.e., $t=l$, which contradicts the fact that $t\neq l$.

In case both $t$ and $l$ are even, the proof is similar and we omit the proof. ∎

From Theorems 8, we can conclude the following result.

**Corollary 9** *Let $a$ be even, $b=2^k a$ with $k\geq 1$ and let $v_2(a)$ and $v_2(b)$ have the same parity. Then the equation $(a^n-2)(b^n-2)=x^2$ has no solution $(n,x)$.*

By simple congruence modulo 3 arguments, one can prove the following two corollaries.

**Corollary 10** *Let $3\nmid a$ and $3\mid b$. If $2\mid n$, then the equation $(a^n-2)(b^n-2)=x^2$ has no solution $(n,x)$.*

**Corollary 11** *If $a\equiv 1 \pmod 3$ and $3\mid b$, then the equation $(a^n-2)(b^n-2)=x^2$ has no solution $(n,x)$.*

If $a+b\sqrt{d}$ is a solution of the equation $x^2-dy^2=2$, then $(a+b\sqrt{d})^2/2 = \left(a^2+db^2\right)/2+ab\sqrt{d}$ is a solution of the equation $x^2-dy^2=1$.

The proof of the following lemma can be found in Lemma 1 [5].

**Lemma 12** *Let $d > 2$. If $k_1 + t_1\sqrt{d}$ is the fundamental solution of the equation $u^2 - dv^2 = 2$, then $\left(k_1^2 + dt_1^2\right)/2 + k_1t_1\sqrt{d}$ is the fundamental solution of the equation $x^2 - dy^2 = 1$.*

**Theorem 13** *Let $d > 2$. Let $x_1 + y_1\sqrt{d}$ and $k_1 + t_1\sqrt{d}$ be the fundamental solutions of the equations $x^2 - dy^2 = 1$ and $u^2 - dv^2 = 2$, respectively. Then $(x_1, y_1) = (k_1^2 - 1, k_1t_1)$ and all solutions of the equation $u^2 - dv^2 = 2$ are given by*

$$(X_n, Y_n) = (k_1\left(U_{n+1} - U_n\right), t_1\left(U_{n+1} + U_n\right))$$

*with $n \geq 1$, where $U_n = U_n(2x_1, -1)$.*

**Proof.** Since $k_1 + t_1\sqrt{d}$ is the fundamental solutions of the equation $u^2 - dv^2 = 2$, then $x_1 + y_1\sqrt{d} = \left(k_1^2 + dt_1^2\right)/2 + k_1t_1\sqrt{d}$ is the fundamental solution of the equation $x^2 - dy^2 = 1$ by Lemma 12. It is immediately seen that $(x_1, y_1) = (k_1^2 - 1, k_1t_1)$. By the identity (12), all positive solutions of the equation $u^2 - dv^2 = 2$ are given by $X_n + Y_n\sqrt{d} = \left(k_1 + t_1\sqrt{d}\right)\left(x_1 + y_1\sqrt{d}\right)^n$. Therefore $X_n + Y_n\sqrt{d} = \left(k_1 + t_1\sqrt{d}\right)\left(x_n + y_n\sqrt{d}\right)$. It follows that

$$\begin{aligned}
X_n + Y_n\sqrt{d} &= \left(\frac{k_1^2 + y_1\sqrt{d}}{k_1}\right)\left(x_1 + y_1\sqrt{d}\right)^n \\
&= \left(\frac{x_1 + 1 + y_1\sqrt{d}}{k_1}\right)\left(x_1 + y_1\sqrt{d}\right)^n \\
&= \frac{1}{k_1}\left[\left(x_1 + y_1\sqrt{d}\right)^{n+1} + \left(x_1 + y_1\sqrt{d}\right)^n\right].
\end{aligned}$$

Thus

$$X_n = \frac{x_{n+1} + x_n}{k_1} \text{ and } Y_n = \frac{y_{n+1} + y_n}{k_1}$$

Since $(x_n, y_n)$ is a solution of the equation $x^2 - dy^2 = 1$, we have

$$(x_n, y_n) = (V_n(2x_1, -1)/2, y_1U_n(2x_1, -1))$$

by Lemma 2. Using the identity (7), we get

$$\begin{aligned}
X_n &= \frac{1}{2k_1}\left(V_{n+1} + V_n\right) = \frac{1}{2k_1}\left(U_{n+2} - U_n + U_{n+1} - U_{n-1}\right) \\
&= \frac{1}{2k_1}\left(2x_1U_{n+1} - U_n - U_n + U_{n+1} + U_{n+1} - 2x_1U_n\right) \\
&= \frac{1}{2k_1}\left[\left(2x_1 + 2\right)U_{n+1} - \left(2x_1 + 2\right)U_n\right] \\
&= \frac{1}{2k_1}\left[2k_1^2U_{n+1} - 2k_1^2U_n\right] \\
&= k_1\left(U_{n+1} - U_n\right)
\end{aligned}$$

and

$$Y_n = \frac{1}{k_1}\left(y_1 U_{n+1} + y_1 U_n\right) = \frac{y_1}{k_1}\left(U_{n+1} + U_n\right) = t_1\left(U_{n+1} + U_n\right). \blacksquare$$

Now, we can give the following result.

**Corollary 14** *Let* $\gcd(a,b) = 1$. *Then the equation* $(a^n - 2)(b^n - 2) = x^2$ *has no solutions if* $2|n$.

**Proof.** Assume that $\gcd(a,b) = 1$ and $2|n$. Let $n = 2m$. Then

$$\left(a^m\right)^2 - du^2 = 2 \text{ and } \left(b^m\right)^2 - dv^2 = 2, \tag{20}$$

for some integers $u$ and $v$ with $\gcd(u,v) = 1$, where $d = (a^n - 2, b^n - 2)$. If $d = 1$, then we get $\left(a^m\right)^2 - u^2 = 2$, a contradiction. Therefore $d > 2$. Assume that $(x_1, y_1)$ and $(k_1, t_1)$ are the fundamental solutions of the equations $x^2 - dy^2 = 1$ and $u^2 - dv^2 = 2$, respectively. Then by Theorem 13, we get

$$a^m = k_1(U_r - U_{r-1}),\ u = t_1(U_r + U_{r-1})$$

and

$$b^m = k_1(U_s - U_{s-1}),\ v = t_1(U_s + U_{s-1}).$$

Since $\gcd(a,b) = 1$, we get $\gcd(a^m, b^m) = 1$. From the above equations it follows that $(k_1, t_1) = (1,1)$ since $\gcd(a^m, b^m) = \gcd(u,v) = 1$. This implies that $d = -1$, which is impossible. $\blacksquare$

Using Mathematica, we verified for all $2 \le a < b \le 300$ and $n$ in the range $2 \le n \le 1000$ that the equation $(a^n - 2)(b^n - 2) = x^2$ has only solutions $(a,b,n,x) = (2,10,2,14), (2,10,6,7874), (2,58,2,82), (3,45,2,119), (4,100,3,7874), (10,58,2,574), (5,235,2,1127), (4,116,3,434)$. We will solve the equation $(a^n - 2)(b^n - 2) = x^2$ for $(a,b) = (2,10), (3,45), (4,100), (10,58)$.

The proofs of the following two lemmas can be done by induction on $m$.

**Lemma 15** *Let* $m \ge 4$. *Then* $5^m > 2^{2m+1} - 3$.

**Lemma 16** *Let* $m \ge 2$. *Then* $2 \cdot 3^{4m-3} > 5^m + 1$.

From Lemma 15, we can give the following corollary.

**Corollary 17** *Let* $m$ *and* $z$ *be positive integers. If* $(z+1)(2z-1)^2 = 10^{2m}$, *then* $m = 1, z = 3$ *or* $m = 3, z = 63$.

**Theorem 18** *The equation* $(2^n - 2)(10^n - 2) = x^2$ *has only the solutions* $(n,x) = (2,14), (6,6874)$.

**Proof.** It is obvious that $(n,x) = (2,14)$ is a solution. Let $n > 2$. Firstly, assume that $n$ is even, say $n = 2m$. Then

$$\left(2^m\right)^2 - 2du^2 = 2 \text{ and } \left(10^m\right)^2 - 2dv^2 = 2$$

for some integers $u$ and $v$ with $\gcd(u,v)=1$, where $2d=\gcd(2^n-2,10^n-2)$. Since $m>1$, it can be seen that $2d>2$. Hence, by Theorem 13, it follows that

$$2^m=k_1(U_r-U_{r-1}),\ u=t_1(U_r+U_{r-1})$$

and

$$10^m=k_1(U_s-U_{s-1}),\ v=t_1(U_s+U_{s-1}),$$

where $U_t=U_t(2x_1,-1)$ and $(x_1,y_1)$ and $(k_1,t_1)$ are the fundamental solutions of the equations $x^2-2dy^2=1$ and $u^2-2dv^2=2$, respectively. Since $U_r+U_{r-1}$ is odd by the identity (9) and $\gcd(u,v)=1$, it follows that $k_1=2^m$, $r=1$, and $t_1=1$, which implies that $u=1$. Thus $(2^m)^2-2d=2$, i.e., $d=2^{n-1}-1$. By Lemma 12, $x_1+y_1\sqrt{2d}=(2^m+\sqrt{2d})^2/2=2^n-1+2^m\sqrt{2d}$. This shows that $x_1=2^n-1$ and $y_1=2^m$. On the other hand, since $(10^m)^2-2dv^2=2$, it follows that

$$\begin{aligned}(10^m+v\sqrt{2d})^2 &= 10^{2m}+2dv^2+10^m2v\sqrt{2d}\\ &= 10^n-1+10^mv\sqrt{2d}.\end{aligned}$$

It can be easily seen that $10^n-1+10^mv\sqrt{2d}$ is a solution of the equation $x^2-2dy^2=1$. Thus we have

$$10^n-1=V_k(2x_1,-1)/2 \text{ and } 10^mv=y_1U_k(2x_1,-1)=2^mU_k$$

for some positive integer $k$ by Lemma 2. This implies that $5|U_k$. Now assume that $4|n$. Then $U_2=P=2x_1=2^{n+1}-2\equiv 0(\bmod 5)$ and this implies that $k$ is even since $5|U_k$. Taking $k=2c$, we get $2\cdot 10^n-2=V_{2c}=V_c^2-2$ by the identity (5), i.e., $2\cdot 10^n=V_c^2$. This is impossible. Hence $2||n$. Then $P\equiv 1(\bmod 5)$. Since $U_k=U_{6q+r}\equiv U_r(\bmod U_3)$ by the identity (13) and $5|U_3$, it follows that $k=3t$. This implies that $2\cdot 10^n-2=V_{3t}=V_t^3-3V_t$ by the identity (6). Taking $V_t=2z$, then, from the last equality, we get $(z+1)(2z-1)^2=10^{2m}$. By Corollary 17, it follows that $m=1$ or $m=3$. Since $n>2$, we get $n=2m=6$.

Secondly, assume that $n$ is odd. If $n=4k+3$, then $x^2=(2^n-2)(10^n-2)\equiv -2(\bmod 5)$, a contradiction. If $n=4k+1$, then we have the equation $(2^{4k}-1)(2^{4k}5^{4k+1}-1)=\left(\frac{x}{2}\right)^2$. This implies that

$$\left(2^k\right)^4-du^2=1 \text{ and } 5\left(10^k\right)^4-dv^2=1$$

for some integers $u$ and $v$ with $\gcd(u,v)=1$, where $d=\gcd(2^{4k}-1,2^{4k}5^{4k+1}-1)$. From the equation $\left(2^k\right)^4-du^2=1$, we can write that $2^{2k}-1=ra^2, 2^{2k}+1=sb^2$ for some integers $a,b,r,s$, where $rs=d$. The equation $5\left(10^k\right)^4-dv^2=1$ implies that $5\left(10^{2k}\right)^2\equiv 1(\bmod r)$ and $5\left(10^{2k}\right)^2\equiv 1(\bmod s)$. Thus $\left(\frac{r}{5}\right)=\left(\frac{s}{5}\right)=1$. If $k$ is even, then $sb^2\equiv 2(\bmod 5)$, which implies that $\left(\frac{s}{5}\right)=-1$. This is a contradiction. If $k$ is odd, then $ra^2\equiv 3(\bmod 5)$. This shows that $\left(\frac{r}{5}\right)=\left(\frac{3}{5}\right)=-1$, which is a contradiction. This completes the proof. ■

From Theorem 18, we immediately deduce the following corollary.

**Corollary 19** *The equation* $(4^n - 2)(100^n - 2) = x^2$ *has only the solution* $(n, x) = (3, 7874)$.

**Theorem 20** *The equation* $(10^n - 2)(58^n - 2) = x^2$ *has only the solution* $(n, x) = (2, 574)$.

**Proof.** Clearly, $(n, x) = (2, 574)$ is a solution. If $4|n$ or $n \equiv 1(\bmod 4)$, then it can be seen that $x^2 \equiv 2$ or $3(\bmod 5)$, which is impossible. Assume that $n \equiv 3(\bmod 4)$. Then $n$ is form of $12q+3$ or $12q+7$, or $12q+11$. If $n = 12q+3$, then we get $x^2 \equiv 3(\bmod 7)$, a contradiction. If $n = 12q + 11$, then it is seen that $x^2 \equiv 6(\bmod 7)$, which is impossible. Let $n = 12q + 7$. Then we get $n \equiv 7, 19, 31, 43,$ or $55(\bmod 60)$. If $n \equiv 31(\bmod 60)$, then we get $x^2 \equiv 8(\bmod 11)$, a contradiction. Let $n \equiv 7, 19, 43,$ or $55(\bmod 60)$. Similarly, when we investigate the equation $(10^n - 2)(58^n - 2) = x^2$ according to modulo 31, we can see that it has no solutions.

Now assume that $n \equiv 2(\bmod 4)$. Say $n = 2m$ with $m$ odd. Then we get $(2^{2m-1}5^{2m} - 1)(2^{2m-1}29^{2m} - 1) = (x/2)^2$. Thus

$$2\left(2^{m-1}5^m\right)^2 - du^2 = 1 \text{ and } 2\left(2^{m-1}29^m\right)^2 - dv^2 = 1$$

for some integers $u$ and $v$ with $\gcd(u, v) = 1$, where $d = \gcd(2^{2m-1}5^{2m} - 1, 2^{2m-1}29^{2m} - 1)$. By Lemma 5, we obtain

$$2^{m-1}5^m = u_1(U_r - U_{r-1}),\ u = v_1(U_r + U_{r-1})$$

and

$$2^{m-1}29^m = u_1(U_s - U_{s-1}),\ v = v_1(U_s + U_{s-1}),$$

where $U_c = U_c(P, -1)$ with $P = 8u_1^2 - 2$, and $(u_1, v_1)$ is the fundamental solution of the equation $2x^2 - dy^2 = 1$. Since $U_r + U_{r-1}$ and $U_s + U_{s-1}$ are odd by the identity (9) and $\gcd(u, v) = 1$, it follows that $u_1 = 2^{m-1}$, $v_1 = 1, U_r - U_{r-1} = 5^m$ and $U_s - U_{s-1} = 29^m$. Thus $2\left(2^{m-1}\right)^2 - d = 1$, i.e., $d = 2^{n-1} - 1$. Since $2^{m-1}\sqrt{2} + \sqrt{d}$ is the fundamental solution of the equation $2x^2 - dy^2 = 1$, $\left(2^{m-1}\sqrt{2} + \sqrt{d}\right)^2 = 2^n - 1 + 2^m\sqrt{2d}$ is the fundamental solution of the equation $x^2 - 2dy^2 = 1$. On the other hand, it is seen that $\left(2^{m-1}5^m\sqrt{2} + u\sqrt{d}\right)^2 = 10^n - 1 + 10^m u\sqrt{2d}$, and this is a solution of the equation $x^2 - 2dy^2 = 1$. Hence, by Lemma 2, we get $10^n - 1 = V_k(2x_1, -1)/2$ and $10^m u = y_1 U_k(2x_1, -1)$ for some natural number $k$, where $x_1 + y_1\sqrt{2d} = 2^n - 1 + 2^m\sqrt{d}$. From this, it is clear that $V_k(2x_1, -1) = 2 \cdot 10^n - 2$ and $U_k(2x_1, -1) = 5^m u$. Therefore $5|U_k$. Since $2x_1 = 2^{n+1} - 2 \equiv 1(\bmod 5)$, it follows that $3|k$ by the identity (15) . Let $k = 3t$. Then we get $2 \cdot 10^n - 2 = V_{3t} = V_t^3 - 3V_t$ by the identity (6). Taking $V_t = 2z$, then from the last equality we get $(z+1)(2z-1)^2 = 10^{2m}$. By Corollary 17, it follows that $m = 1$ or $m = 3$. Therefore $n = 2$ or $n = 6$. But $n = 6$ is impossible and so $n = 2$. This completes the proof. ■

**Theorem 21** *The equation* $(3^n - 2)(45^n - 2) = x^2$ *has only the solution* $(n, x) = (2, 119)$.

**Proof.** Clearly, $(n, x) = (2, 574)$ is a solution. If $4|n$ or $n \equiv 1 \pmod 4$, then it can be seen that $x^2 \equiv 2$ or $3 \pmod 5$, which is impossible. Assume that $n \equiv 3 \pmod 4$. Then $n \equiv 3, 7, 11 \pmod{12}$. In these cases, using modulo 13, it can be seen that the equation $(3^n - 2)(45^n - 2) = x^2$ is impossible.

Now assume that $n \equiv 2 \pmod 4$. Say $n = 2m$ with $m$ odd. Then we get

$$(3^m)^2 - du^2 = 2 \text{ and } (45^m)^2 - dv^2 = 2$$

for some integers $u$ and $v$ with $\gcd(u, v) = 1$, where $d = \gcd(3^n - 2, 45^n - 2)$. By Theorem 13, we obtain

$$3^m = k_1(U_r - U_{r-1}),\ u = t_1(U_r + U_{r-1}) \tag{21}$$

and

$$45^m = k_1(U_s - U_{s-1}),\ v = t_1(U_s + U_{s-1}), \tag{22}$$

where $U_c = U_c(P, -1)$ with $P = 2x_1 = 2k_1^2 - 2$, $y_1 = k_1t_1$, $(x_1, y_1)$ is the fundamental solution of the equation $x^2 - dy^2 = 1$, and $(k_1, t_1)$ is the fundamental solution of the equation $x^2 - dy^2 = 2$. So $k_1 > 1$ and this implies that $3|k_1$ by the identity (21). Then $P = 2k_1^2 - 2 \equiv 1 \pmod 3$. Assume that $3|U_r - U_{r-1}$. Then it is seen that $r \equiv 2, 5 \pmod 6$ by Lemma 7. Since $5 \nmid k_1$ by the identity (21), it follows that $P = 2k_1^2 - 2 \equiv 0, 1 \pmod 5$. Assume that $P \equiv 1 \pmod 5$. Then $5|U_r - U_{r-1}$ by Lemma 7 since $r \equiv 2, 5 \pmod 6$. This is impossible by (21). Let $P \equiv 0 \pmod 5$. Then it can be seen that $5 \nmid U_s - U_{s-1}$ by the identity (15). This is impossible by the identity (22) since $5 \nmid k_1$. We conclude that $3 \nmid U_r - U_{r-1}$ and therefore $k_1 = 3^m$ and $r = 0$ by (21). Thus $x_1 = k_1^2 - 1 = 3^n - 1$. Besides, it is clear that $u = t_1 = 1$ since $\gcd(u, v) = 1$ and $r = 0$. This implies that $y_1 = k_1t_1 = 3^m$ and $d = 3^n - 2$. On the other hand, since $45^m + v\sqrt{d}$ is a solution of the equation $x^2 - dy^2 = 2$, then $\left(45^m + v\sqrt{d}\right)^2/2 = 45^n - 1 + 45^m v\sqrt{d}$ is a solution of the equation $x^2 - dy^2 = 1$. In this case, $45^n - 1 = V_k(P, -1)/2$ and $45^m v = y_1U_k(P, -1) = 3^mU_k(P, -1)$ by Lemma 2. From this, it is clear that $V_k(P, -1) = 2 \cdot 45^{2m} - 2$ and $U_k(P, -1) = 3^m5^mv$. Therefore $3|U_k$. Since $P \equiv 1 \pmod 3$, it follows that $3|k$ by the identity (14). Let $k = 3t$. Then we get $2 \cdot 45^{2m} - 2 = V_{3t} = V_t^3 - 3V_t$ by the identity (6). Taking $V_t = 2z$, from the last equality, we get $(z + 1)(2z - 1)^2 = w^2$ with $w = 45^m$. Then $z + 1 = (\frac{w}{2z-1})^2$. Let $a = \frac{w}{2z-1}$. Then $z + 1 = a^2$ and $w = a(2z - 1) = a(2a^2 - 3)$. It can be seen that $\gcd(a, 2a^2 - 3) = 3$ since $3|a(2a^2 - 3)$. Then $a = 3b$ and $2a^2 - 3 = 3c$ for some integers $b$ and $c$ with $\gcd(b, c) = 1$. Then $9bc = 45^m = 3^{2m}5^m$ and therefore $bc = 3^{2m-2}5^m$. Since $a = 3b$ and $2a^2 - 3 = 3c$, we get $c = 6b^2 - 1$ and therefore $b(6b^2 - 1) = 3^{2m-2}5^m$. Then $b = 3^{2m-2}$ and $6b^2 - 1 = 5^m$. Thus $2 \cdot 3^{4m-3} = 5^m + 1$. This is only possible for $m = 1$ by Lemma 16. Consequently, $n = 2$ and $x = 119$. ∎

# 4 Concluding Remark

Using a program with Mathematica, we verified for all $2 \leq a < b \leq 300$ and $n$ in the range $2 \leq n \leq 1000$ that the equation $(a^n - 2)(b^n - 2) = x^2$ has solutions only when $n \leq 6$. Also, this program have showed in the range $2 \leq n \leq 1000$ that for the pairs $(a,b) = (5,235),(4,116),(2,58)$, the equation $(a^n-2)(b^n-2) = x^2$ has only solutions $(n,x) = (2,1127),(3,434),(2,82)$, respectively. We have not handled to find all integer solutions of the equations $(5^n-2)(235^n-2) = x^2$ and $(4^n-2)(116^n-2) = x^2$ yet. But we think that the equation $(5^n-2)(235^n-2) = x^2$ has only the solution $(n,x) = (2,1127)$ and the equation $(4^n-2)(116^n-2) = x^2$ has only the solution $(n,x) = (3,434)$. Besides, we have handled to determine all integer solutions of the equation $(2^n-2)(58^n-2) = x^2$ but we were not able to solve this problem. So, we think the following conjectures are true.

**Conjecture 22** *The only solutions of the equation* $(2^n-2)(58^n-2) = x^2$ *is* $(n,x) = (2,82)$.

**Conjecture 23** *Let* $2 \leq a < b$. *If the equation* $(a^n-2)(b^n-2) = x^2$ *has a solution* $(n,x)$, *then* $n \leq 6$.

A computer search with Mathematica showed that in the ranges $2 \leq n \leq 1000$ and $3 < k \leq 100$, the equation $(2^n-2)((2P_k)^n-2) = x^2$ has no solutions. This situation enables us to give the following conjectures.

**Conjecture 24** *Let* $k > 3$ *be odd. Then the equation* $(2^n-2)((2P_k)^n-2) = x^2$ *has only the solution* $(n,x) = (2,Q_k)$.

When $k = 3$, $P_k = P_3 = 5$, the above equation becomes $(2^n-2)((10^n-2) = x^2$, which has only the solutions $(n,x) = (2,14),(6,7874)$ by Theorem 18.

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