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ARTICLE TYPE

An analytical investigation of fractional-order Biological model using an innovative technique

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Abstract

In this paper a new so called Iterative laplace transform method is implemented to investigate the solution of certain important population models of non-integer order. The iterative procedure is combined effectively with Laplace transformation to develop the suggested methodology. The Caputo operator is applied to express non-integer derivative of fractional-order. the series form solution is obtained having components of convergent behavior toward the exact solution. For justification and verification of the present method some illustrative examples are discussed. The closed contact is observed between the obtained and exact solutions. Moreover, the suggested method has small volume of calculations and therefore it can be applied to handle the solutions of various problems with fractional-order derivatives.

KEYWORDS:
Iterative Laplace Transform method; Biological population model; Caputo operator.

1 | INTRODUCTION

Fractional calculus(FC) is the subject dealing with derivative and integration of non-integer order.The concept of FC have been started from $30th$ September 1695, when a question was asked from Leibnitz about the half-order derivative of a function. In $19th$ century Riemann and Liouville have properly defined the concept of fractional-order derivative and integration by using operator. Later on this theory was further extended by other researchers to improve the mathematical models of different physical phenomena. Generally, derivatives and integration of integer-order have important physical and geometric interpretation. However, fractional derivatives and integration doesn't have fully acceptable physical and geometrical interpretation. Due to large and up-growing involvement of FC it is accepted as the calculus of $21st$ century.

FC has attracted the researchers because of its numerous applications in various scientific fields, including, fluid mechanics, engineering, electromagnetism, viscoelasticity and other areas of science [1-6]. Fractional differential equations (FDEs) are accepted as the powerful tool to model the above mentioned physical phenomena as compare to differential equations of integer-order. Therefore, it was considered an important task by the mathematician to solve these equations. In this connection, certain important techniques have been used including fractional operational matrix method (FOMM) [7,8], fractional wavelet method (FWM) [9-12], Homotopy analysis method (HAM) [13], Homotopy perturbation method (HPM) [14], Homotopy perturbation transform method (HPTM) [15], Laplace Adomian decomposition method (LADM)[16], Fractional variational iteration method (FVIM) [17]. The above mentioned techniques have the straight forward implementations to both linear and non-linear FDEs.

In the same context, Daftardar-Gejji and Jafari in 2006 have developed the iterative technique to solve non-linear functional equations [18, 19]. Later on, Iterative technique is applied to solve non-integer differential equations (DEs) [20]. In recent time, Jafari et al. have used Laplace transform together with iterative technique for the first time which is nowadays became an effective technique named as iterative Laplace transform method (ILTM) [21] ILTM is implemented to solve partial differential equations (PDEs) and Fokker–Plank problems [22]. Recently many other FPDEs have been solved by using ILTM such as time-fractional Schrödinger equations [23], fractional Telegraph equations [24], fractional heat and wave-like equations [25] and time-fractional fisher equation [26]. The main theme of the present research work is to use ILTM for obtaining the analytical solution of non-integer Biological population model [27].

$$\frac{\partial^\rho \psi(\mu,\nu,\tau)}{\partial \tau^\rho} = \frac{\partial^2 \psi^2(\mu,\nu,\tau)}{\partial \mu^2} + \frac{\partial^2 \psi^2(\mu,\nu,\tau)}{\partial \nu^2} + g(\psi(\mu,\nu,\tau)), \quad \tau > 0, 0 < \rho \leq 1 \tag{1}$$

subject to the starting values: $\psi(\mu,\nu,0) = g_0(\mu,\nu)$.
where the population density is represented by $\psi(\mu,\nu,\tau)$ and the population rate is expressed by $g(\psi(\mu,\nu,\tau))$. For $\rho \to 1$, Various properties such like Holder estimates for its solution are discussed in [28]. The three consecutive cases for $g(u)$ are :
$g(\psi) = c$, for any constant c, reduces to Malthusian Law,
$g(\psi) = \psi(d_1 - d_2\psi)$, for positive constant d_1 and d_2 reduces to Verhulst Law,
$g(\psi) = -d\psi^k, (d \geq 0, 0 < k < 1)$, for positive d reduces to Porous Media

2 | DEFINITIONS AND PREMINALARIES

In this part of the paper some important definitions related FC and Laplace transform have briefly discussed. These preliminaries are important to continue and complete the present research work.

2.1 | Definition

The fractional derivative in terms of Caputo operator is expressed as

$$\begin{aligned} D_\tau^\rho \psi(\mu,\tau) &= \frac{1}{\Gamma(n-\rho)} \int_0^\tau (\tau-\zeta)^{n-\rho-1} \psi^{(n)}(\mu,\zeta) d\zeta, \quad n-1 < \rho \leqslant n, n \in N, \\ &= J_\tau^{n-\rho} D^n u(\mu,\tau). \end{aligned} \tag{2}$$

Here

$$D^n = \frac{d^n}{d\tau^n}$$

2.2 | Definition

The fractional integral in terms of Riemann-Liouvilee is expressed as

$$j_\tau^\rho \psi(\mu,\tau) = \frac{1}{\Gamma(\rho)} \int_0^\tau (\tau-\zeta)^{\rho-1} \psi(\mu,\zeta) d\zeta, \zeta > 0 \ \ (n-1 < \rho \leqslant n), n \in N, \tag{3}$$

j_τ^ρ represents the fractional integral operator

2.3 | Definition

The Laplace transform is describe as

$$L[g(\tau)] = G(\tau) = \int_0^\infty e^{-s\tau} g(\tau) d\tau. \tag{4}$$

2.4 | Definition

The Laplace transform of the fractional derivative $D_\tau^\rho \psi(\mu,\tau)$ is defined as

$$L[D_\tau^\rho \psi(\mu,\tau)] = s^\rho L[\psi(\mu,\tau)] - \sum_{k=0}^{n-1} \psi^{(k)}(\mu,0)s^{\rho-k-1}, \;\; n-1<\rho\leqslant n, n\in N, \tag{5}$$

2.5 | Definition

The Mittag-Leffler function is expressed as

$$E_\rho(z) = \sum_{q=0}^{\infty} \frac{z^q}{\Gamma(\rho q+1)}, \qquad (\rho\in C, Re(\rho)>0). \tag{6}$$

3 | THE BASIC CONCEPT OF ILTM

In this section, we will briefly discuss ILTM, to solve fractional-order nonlinear PDEs.

$$D_\tau^\rho \psi(\mu,\nu,\tau) + R\psi(\mu,\nu,\tau) + N\psi(\mu,\nu,\tau) = g(\mu,\nu,\tau), \quad n-1<\rho\leqslant n, n\in N, \tag{7}$$

$$\psi^{(k)}(\mu,\nu,0) = h_k(\mu,\nu), \quad k=0,1,2,...,n-1, \tag{8}$$

where $D_\tau^\rho \psi(\mu,\nu,\tau)$ is the fractional Caputo operator of order ρ, $n-1<\rho\leqslant n$, denoted by Eq. (3), R and N are linear and nonlinear operators. The $g(\mu,\nu,\tau)$ is source function.
Using Laplace transform of Eq. (7) we get

$$L[D_\tau^\rho \psi(\mu,\nu,\tau)] + L[R\psi(\mu,\nu,\tau) + N\psi(\mu,\nu,\tau)] = L[g(\mu,\nu,\tau)]. \tag{9}$$

Applying the property of laplace differentiation

$$L[\psi(\mu,\nu,\tau)] = \frac{1}{s^\rho}\sum_{k=0}^{m-1} s^{\rho-1-k}\psi^{(k)}(\mu,\nu,0) + \frac{1}{s^\rho}L[g(\mu,\nu,\tau)] - \frac{1}{s^\rho}L[R\psi(\mu,\nu,\tau)+N\psi(\mu,\nu,\tau)]. \tag{10}$$

By using inverse Laplace transform of Eq. (10), we obtain

$$\psi(\mu,\nu,\tau) = L^{-1}\left[\frac{1}{s^\rho}\left(\sum_{k=0}^{m-1} s^{\rho-1-k}\psi^{k}(\mu,\nu,0) + L[g(\mu,\nu,\tau)]\right)\right] - L^{-1}\left[\frac{1}{s^\rho}L[R\psi(\mu,\nu,\tau)+N\psi(\mu,\nu,\tau)]\right]. \tag{11}$$

From iterative technique,

$$\psi(\mu,\nu,\tau) = \sum_{i=0}^{\infty}\psi_i(\mu,\nu,\tau). \tag{12}$$

Since R is a linear operator

$$R\left(\sum_{i=0}^{\infty}\psi_i(\mu,\nu,\tau)\right) = \sum_{i=0}^{\infty} R\left[\psi_i(\mu,\nu,\tau)\right], \tag{13}$$

and the non-linear operator N is splitted as

$$N\left(\sum_{i=0}^{\infty}\psi_i(\mu,\nu,\tau)\right) = N[\psi_0(\mu,\nu,\tau)] + \sum_{i=1}^{\infty}\left\{N\left(\sum_{k=0}^{i}\psi_k(\mu,\nu,\tau\right) - N\left(\sum_{k=0}^{i-1}\psi_k(\mu,\nu,\tau)\right)\right\}. \tag{14}$$

Putting equations (12-14) in equation (11), we obtain

$$\begin{aligned}\sum_{i=0}^{\infty}\psi_i(\mu,\nu,\tau) = {} & L^{-1}\left[\frac{1}{s^\rho}\left(\sum_{k=0}^{m-1} s^{\rho-1-k}\psi^{k}(\mu,\nu,0) + L[g(\mu,\nu,\tau)]\right)\right] - L^{-1}\left[\frac{1}{s^\rho}L\right. \\ & \left.\left[\sum_{i=0}^{\infty}R[\psi_i(\mu,\nu,\tau)] + N[\psi_0(\mu,\nu,\tau)] + \sum_{i=1}^{\infty}\left\{N\left(\sum_{k=0}^{i}\psi_k(\mu,\nu,\tau\right) - N\left(\sum_{k=0}^{i-1}\psi_k(\mu,\nu,\tau)\right)\right\}\right]\right].\end{aligned} \tag{15}$$

Using equation (15), we defined the following iterative formula

$$\psi_0(\mu,\nu,\tau) = L^{-1}\left[\frac{1}{s^\rho}\left(\sum_{k=0}^{m-1} s^{\rho-1-k}\psi^k(\mu,\nu,0) + \frac{1}{s^\rho}L(g(\mu,\nu,\tau))\right)\right] \tag{16}$$

$$\psi_1(\mu,\nu,\tau) = -L^{-1}\left[\frac{1}{s^\rho}L[R[\psi_0(\mu,\nu,\tau)] + N[\psi_0(\mu,\nu,\tau)]]\right], \tag{17}$$

$$\psi_{m+1}(\mu,\nu,\tau) = -L^{-1}\left[\frac{1}{s^\rho}L\left[R(\psi_m(\mu,\nu,\tau)) - \left\{N\left(\sum_{k=0}^{m}\psi_k(\mu,\nu,\tau)\right) - N\left(\sum_{k=0}^{m-1}\psi_k(\mu,\nu,\tau)\right)\right\}\right]\right], \quad m \geq 1 \tag{18}$$

The approximate m-term solution of equation (7) and (8) in form of series as

$$\psi(\mu,\nu,\tau) \cong \psi_0(\mu,\nu,\tau) + \psi_1(\mu,\nu,\tau) + \psi_2(\mu,\nu,\tau) + \ldots.., +\psi_m(\mu,\nu,\tau), \quad m = 1,2,..., \tag{19}$$

4 | IMPLEMENTATION OF ILTM

In this section, ILTM is applied to determine the exact solution of some special cases of Eq. (1). It has been shown that the ILTM is an accurate and appropriate analytical technique to solve non-linear FPDEs.

4.1 | Example

The Biological population model with time non-integer derivative is express as

$$\begin{aligned}\frac{\partial^\rho\psi}{\partial\tau^\rho} &= \frac{\partial^2}{\partial\mu^2}(\psi^2) + \frac{\partial^2}{\partial\nu^2}(\psi^2) + h\psi^{-1}(1-r\psi),\\ &0 < \rho \leq 1, \mu,\nu \in \mathfrak{R}, \tau > 0,\end{aligned} \tag{20}$$

with starting values

$$\psi(\mu,\nu,0) = \sqrt{\frac{hr}{4}\mu^2 + \frac{hr}{4}\nu^2 + \nu + 5}, \tag{21}$$

The Laplace transform to Eq. (20) is expressed as

$$\begin{aligned}s^\rho L[\psi(\mu,\nu,\tau)] - \sum_{k=0}^{m-1}\psi^{(k)}(\mu,\nu,0)s^{\rho-k-1} &= L(\frac{\partial^2}{\partial\mu^2}(\psi^2) + \frac{\partial^2}{\partial\nu^2}(\psi^2) + h\psi^{-1}(1-r\psi)),\\ s^\rho L[\psi(\mu,\nu,\tau)] = \psi^{(0)}(\mu,\nu,0)\frac{s^\rho}{s} &+ L(\frac{\partial^2}{\partial\mu^2}(\psi^2) + \frac{\partial^2}{\partial\nu^2}(\psi^2) + h\psi^{-1}(1-r\psi)),\end{aligned} \tag{22}$$

$$L[\psi(\mu,\nu,\tau)] = \frac{1}{s}\sqrt{\frac{hr}{4}\mu^2 + \frac{hr}{4}\nu^2 + \nu + 5} + \frac{1}{s^\rho}\left[L(\frac{\partial^2}{\partial\mu^2}(\psi^2) + \frac{\partial^2}{\partial\nu^2}(\psi^2) + h\psi^{-1}(1-r\psi))\right]. \tag{23}$$

Using inverse Laplace transform of Eq. (23),

$$\psi(\mu,\nu,\tau) = \sqrt{\frac{hr}{4}\mu^2 + \frac{hr}{4}\nu^2 + \nu + 5} + L^{-1}\left[\frac{1}{s^\rho}\left[L(\frac{\partial^2}{\partial\mu^2}(\psi^2) + \frac{\partial^2}{\partial\nu^2}(\psi^2) + h\psi^{-1}(1-r\psi))\right]\right]. \tag{24}$$

Using iterative technique describe in Eqs (12-14), we obtain the following solution components of example 4.1

$$\begin{aligned}\psi_0(\mu,\nu,\tau) &= \sqrt{\frac{hr}{4}\mu^2 + \frac{hr}{4}\nu^2 + \nu + 5}.\\ \psi_1(\mu,\nu,\tau) &= L^{-1}\left[\frac{1}{s^\rho}\left[L(\frac{\partial^2}{\partial\mu^2}(\psi_0^2) + \frac{\partial^2}{\partial\nu^2}(\psi_0^2) + h\psi_0^{-1}(1-r\psi_0))\right]\right].\end{aligned} \tag{25}$$

$$= h((\frac{hr}{4}\mu^2 + \frac{hr}{4}\nu^2 + \nu + 5)^{-\frac{1}{2}})\frac{\tau^\rho}{\Gamma(\rho+1)}, \tag{26}$$

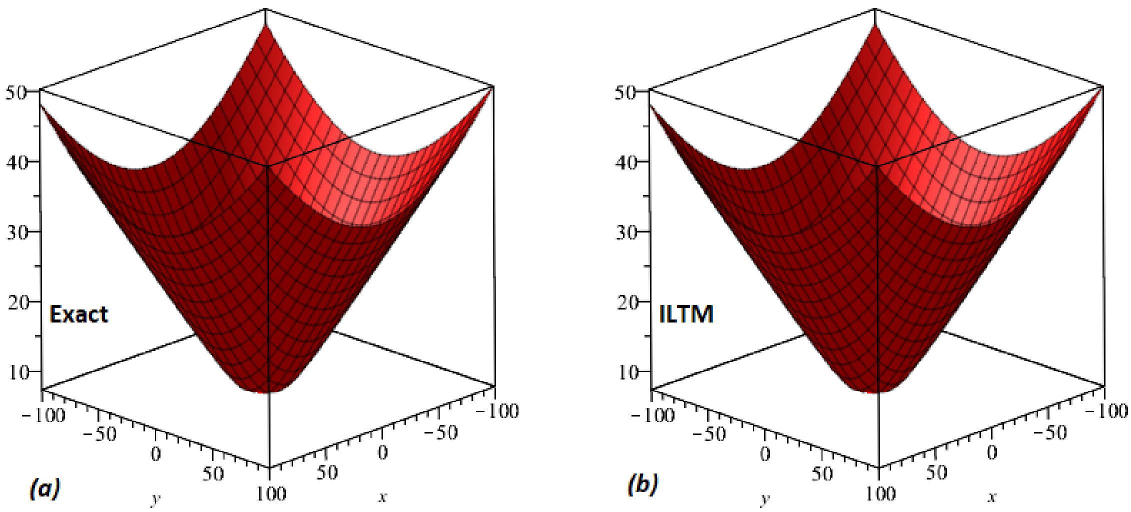

FIGURE 1 The solution plot of example **1**, (a)Exact solution and (b) ILTM solution at $\rho = 1$.

$$\psi_2(\mu,\nu,\tau) = L^{-1}\left[\frac{1}{s^\rho}\left[L(\frac{\partial^2}{\partial\mu^2}(\psi_1^2) + \frac{\partial^2}{\partial\nu^2}(\psi_1^2) + h\psi_1^{-1}(1 - r\psi_1))\right]\right]. \tag{27}$$
$$= -2h^2((\frac{hr}{4}\mu^2 + \frac{hr}{4}\nu^2 + \nu + 5)^{-\frac{3}{2}})\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}$$

$$\psi_3(\mu,\nu,\tau) = L^{-1}\left[\frac{1}{s^\rho}\left[L(\frac{\partial^2}{\partial\mu^2}(\psi_2^2) + \frac{\partial^2}{\partial\nu^2}(\psi_2^2) + h\psi_2^{-1}(1 - r\psi_2))\right]\right]. \tag{28}$$

$$= 3h^3((\frac{hr}{4}\mu^2 + \frac{hr}{4}\nu^2 + \nu + 5)^{-\frac{5}{2}})\frac{\tau^{3\rho}}{\Gamma(3\rho+1)} \tag{29}$$

The series form of analytical solution is given as

$$\begin{aligned}\psi(\mu,\nu,\tau) &= \psi_0(\mu,\nu,\tau) + \psi_1(\mu,\nu,\tau) + \psi_2(\mu,\nu,\tau) + \psi_3(\mu,\nu,\tau) +, ...,\\ &= (\frac{hr}{4}\mu^2 + \frac{hr}{4}\nu^2 + \nu + 5)^{\frac{1}{2}} + h((\frac{hr}{4}\mu^2 + \frac{hr}{4}\nu^2 + \nu + 5)^{-\frac{1}{2}})\frac{t^\rho}{\Gamma(\rho+1)}\\ &- 2h^2((\frac{hr}{4}\mu^2 + \frac{hr}{4}\nu^2 + \nu + 5)^{-\frac{3}{2}})\frac{\tau^{2\rho}}{\Gamma(2\rho+1)} + 3h^3((\frac{hr}{4}\mu^2 + \frac{hr}{4}\nu^2\\ &+ \nu + 5)^{-\frac{5}{2}})\frac{\tau^{3\rho}}{\Gamma(3\rho+1)} +, ...,\end{aligned} \tag{30}$$

$$\psi(\mu,\nu,\tau) = \psi_0 + \frac{h\tau^\rho}{\psi_0}\sum_{n=0}^{\infty}\frac{n+1}{\Gamma((n+1)\rho+1)}(\frac{-h\tau^\rho}{\psi_0^2})^n. \tag{31}$$

The exact result is given by

$$\psi(\mu,\nu,\tau) = \sqrt{\frac{hr}{4}\mu^2 + \frac{hr}{4}\nu^2 + \nu + 2h\tau + 5}. \tag{32}$$

4.2 | Example

The Biological population model with time non-integer derivative is express as

$$\frac{\partial^\rho\psi}{\partial\tau^\rho} = \frac{\partial^2}{\partial\mu^2}(\psi^2) + \frac{\partial^2}{\partial\nu^2}(\psi^2) + h\psi, \tag{33}$$

with initial condition

$$\psi(\mu,\nu,0) = \sqrt{\mu\nu}, \tag{34}$$

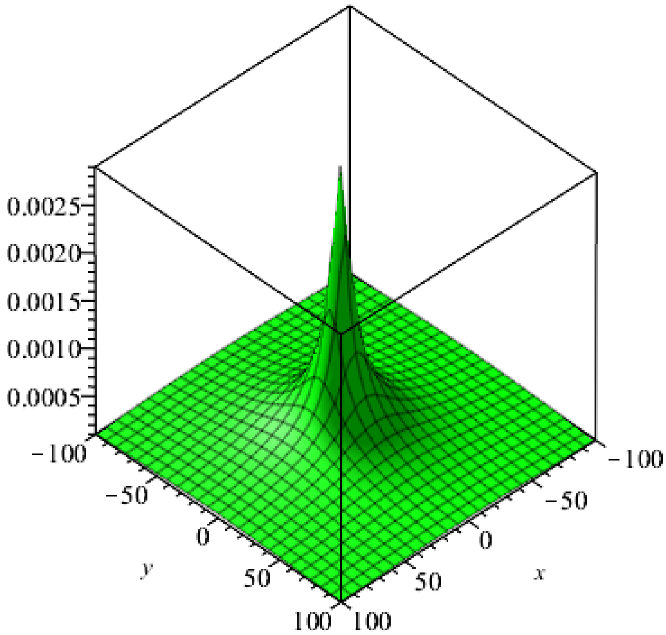

FIGURE 2 The Absolute Error for Example **1** at $\rho = 1$

The Laplace transform to Eq. (33) is expressed as

$$s^{\rho}L[\psi(\mu,\nu,\tau)] - \sum_{k=0}^{m-1}\psi^{(k)}(\mu,\nu,0)s^{\rho-k-1} = L(\frac{\partial^2}{\partial\mu^2}(\psi^2) + \frac{\partial^2}{\partial\nu^2}(\psi^2) + h\psi),$$
$$s^{\rho}L[\psi(\mu,\nu,\tau)] = \psi^{(0)}(\mu,\nu,0)\frac{s^{\rho}}{s} + L(\frac{\partial^2}{\partial\mu^2}(\psi^2) + \frac{\partial^2}{\partial\nu^2}(\psi^2) + h\psi), \tag{35}$$

$$L[\psi(\mu,\nu,\tau)] = \frac{1}{s}\sqrt{\mu\nu} + \frac{1}{s^{\rho}}\left[L(\frac{\partial^2}{\partial\mu^2}(\psi^2) + \frac{\partial^2}{\partial\nu^2}(\psi^2) + h\psi)\right]. \tag{36}$$

Using inverse Laplace transform of equation (36)

$$\psi(\mu,\nu,\tau) = \sqrt{\mu\nu} + L^{-1}\left[\frac{1}{s^{\rho}}\left[L(\frac{\partial^2}{\partial\mu^2}(\psi^2) + \frac{\partial^2}{\partial\nu^2}(\psi^2) + h\psi)\right]\right]. \tag{37}$$

Using iterative technique describe in Eqs (12-14), we obtain the following solution components of example 4.2

$$\psi_0(\mu,\nu,\tau) = \sqrt{\mu\nu},$$
$$\psi_1(\mu,\nu,\tau) = L^{-1}\left[\frac{1}{s^{\rho}}\left[L(\frac{\partial^2}{\partial\mu^2}(\psi_0^2) + \frac{\partial^2}{\partial\nu^2}(\psi_0^2) + h\psi_0))\right]\right]. \tag{38}$$

$$= h\sqrt{\mu\nu}\frac{\tau^{\rho}}{\Gamma(\rho+1)}(39)$$

$$\psi_2(\mu,\nu,\tau) = L^{-1}\left[\frac{1}{s^{\rho}}\left[L(\frac{\partial^2}{\partial\mu^2}(\psi_1^2) + \frac{\partial^2}{\partial\nu^2}(\psi_1^2) + h\psi_1)\right]\right].$$
$$= h^2\sqrt{\mu\nu}\frac{\tau^{2\rho}}{\Gamma(2\rho+1)} \tag{40}$$

$$\psi_3(\mu,\nu,\tau) = L^{-1}\left[\frac{1}{s^{\rho}}\left[L(\frac{\partial^2}{\partial\mu^2}(\psi_2^2) + \frac{\partial^2}{\partial\nu^2}(\psi_2^2) + h\psi_2)\right]\right]. \tag{41}$$

$$= h^3\sqrt{\mu\nu}\frac{\tau^{3\rho}}{\Gamma 3\rho+1} \tag{42}$$

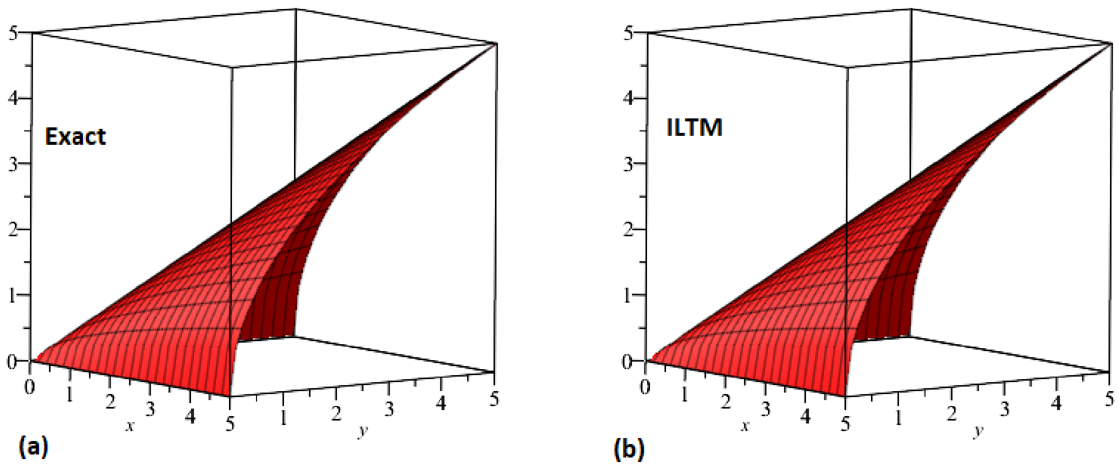

FIGURE 3 Solution graph of example **2**, (a) Exact and (b) ILTM at $\rho = 1$.

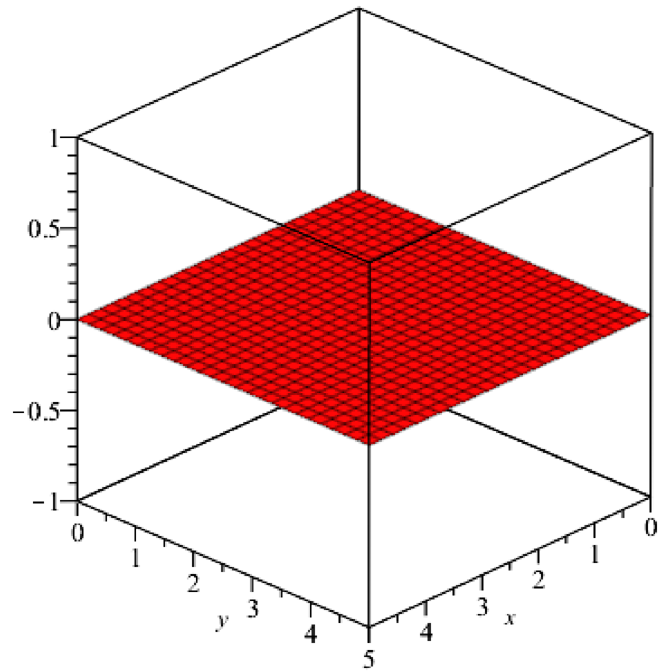

FIGURE 4 The Absolute Error for Example **2** at $\rho = 1$

The series form of analytical solution is given as

$$\begin{aligned}\psi(\mu, \nu, \tau) &= \psi_0(\mu, \nu, \tau) + \psi_1(\mu, \nu, \tau) + \psi_2(\mu, \nu, \tau) + \psi_3(\mu, \nu, \tau) +, ..., \\ &= \sqrt{\mu\nu} + h\sqrt{\mu\nu}\frac{\tau^\rho}{\Gamma(\rho+1)} + h^2\sqrt{\mu\nu}\frac{\tau^{2\rho}}{\Gamma(2\rho+1)} + h^3\sqrt{\mu\nu}\frac{\tau^{3\rho}}{\Gamma(3\rho+1)} +, ...,\end{aligned} \tag{43}$$

$$\psi(\mu, \nu, \tau) = \sqrt{\mu\nu}\sum_{k=0}^{\infty}\frac{(h\tau^\rho)^k}{\Gamma(k\rho+1)}, \tag{44}$$

The exact solution is given by

$$\psi(\mu, \nu, \tau) = \sqrt{\mu\nu}E_\rho(h\tau^\rho), \tag{45}$$

where $E_\rho(h\tau^\rho)$ is the Mittag-Leffler fuction defined as

$$E_\rho(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + 1)}. \tag{46}$$

As $\rho \to 1$ we have

$$\psi(\mu, \nu, \tau) = \sqrt{\mu\nu} \sum_{k=0}^{\infty} \frac{(h\tau)^k}{k!} = \sqrt{\mu\nu} e^{h\tau}, \tag{47}$$

4.3 | Example

The Biological population model with time non-integer derivative is express as

$$\frac{\partial^\rho \psi}{\partial \tau^\rho} = \frac{\partial^2}{\partial \mu^2}(\psi^2) + \frac{\partial^2}{\partial \nu^2}(\psi^2) + \psi, \tag{48}$$

with initial condition

$$\psi(\mu, \nu, 0) = \sqrt{\sin\mu \sinh\nu}, \tag{49}$$

The Laplace transform to Eq. (48) is expressed as

$$\begin{aligned} s^\rho L[\psi(\mu, \nu, \tau)] - \sum_{k=0}^{m-1} \psi^{(k)}(\mu, \nu, 0) s^{\rho-k-1} &= L(\frac{\partial^2}{\partial \mu^2}(\psi^2) + \frac{\partial^2}{\partial \nu^2}(\psi^2) + \psi), \\ s^\rho L[\psi(\mu, \nu, \tau)] = \psi^{(0)}(\mu, \nu, 0)\frac{s^\rho}{s} &+ L(\frac{\partial^2}{\partial \mu^2}(\psi^2) + \frac{\partial^2}{\partial \nu^2}(\psi^2) + \psi), \end{aligned} \tag{50}$$

$$L[\psi(\mu, \nu, \tau)] = \frac{1}{s}\sqrt{\mu\nu} + \frac{1}{s^\rho}\left[L(\frac{\partial^2}{\partial \mu^2}(\psi^2) + \frac{\partial^2}{\partial \nu^2}(\psi^2) + \psi)\right]. \tag{51}$$

Using inverse Laplace transform of Eq. (51),we obtain

$$\psi(\mu, \nu, \tau) = \sqrt{\mu\nu} + L^{-1}\left[\frac{1}{s^\rho}\left[L(\frac{\partial^2}{\partial \mu^2}(\psi^2) + \frac{\partial^2}{\partial \nu^2}(\psi^2) + \psi)\right]\right]. \tag{52}$$

Using iterative technique describe in Eqs (12-14), we obtain the following solution components of example 4.3

$$\begin{aligned} \psi_0(\mu, \nu, \tau) &= \sqrt{\sin\mu \sinh\nu}, \\ \psi_1(\mu, \nu, \tau) &= L^{-1}\left[\frac{1}{s^\rho}\left[L(\frac{\partial^2}{\partial \mu^2}(\psi_0^2) + \frac{\partial^2}{\partial \nu^2}(\psi_0^2) + \psi_0))\right]\right]. \end{aligned} \tag{53}$$

$$= \sqrt{\sin\mu \sinh\nu}\frac{\tau^\rho}{\Gamma(\rho + 1)} \tag{54}$$

$$\begin{aligned} \psi_2(\mu, \nu, \tau) &= L^{-1}\left[\frac{1}{s^\rho}\left[L(\frac{\partial^2}{\partial \mu^2}(\psi_1^2) + \frac{\partial^2}{\partial \nu^2}(\psi_1^2) + \psi_1)\right]\right]. \\ &= \sqrt{\sin\mu \sinh\nu}\frac{\tau^{2\rho}}{\Gamma(2\rho + 1)} \end{aligned} \tag{55}$$

$$\psi_3(\mu, \nu, \tau) = L^{-1}\left[\frac{1}{s^\rho}\left[L(\frac{\partial^2}{\partial \mu^2}(\psi_2^2) + \frac{\partial^2}{\partial \nu^2}(\psi_2^2) + \psi_2)\right]\right]. \tag{56}$$

$$= \sqrt{\sin\mu \sinh\nu}\frac{\tau^{3\rho}}{\Gamma(3\rho + 1)} \tag{57}$$

The series form of analytical solution is given as

$$\begin{aligned} &\psi(\mu, \nu, \tau) = \psi_0(\mu, \nu, \tau) + \psi_1(\mu, \nu, \tau) + \psi_2(\mu, \nu, \tau) + \psi_3(\mu, \nu, \tau) +, ..., \\ &= \sqrt{\sin\mu \sinh\nu} + \sqrt{\sin\mu \sinh\nu}\frac{\tau^\rho}{\Gamma\rho + 1} + \sqrt{\sin\mu \sinh\nu}\frac{\tau^{2\rho}}{\Gamma 2\rho + 1} + \sqrt{\sin\mu \sinh\nu} \\ &\frac{\tau^{3\rho}}{\Gamma 3\rho + 1} +, ..., \end{aligned} \tag{58}$$

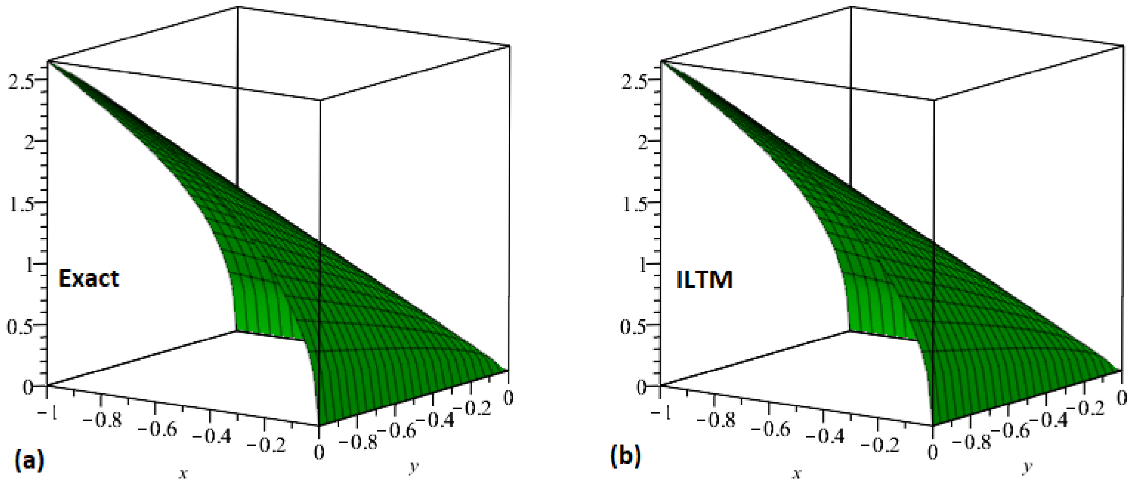

FIGURE 5 The solution plot of example **3**, (a)Exact solution and (b) ILTM solution at $\rho = 1$.

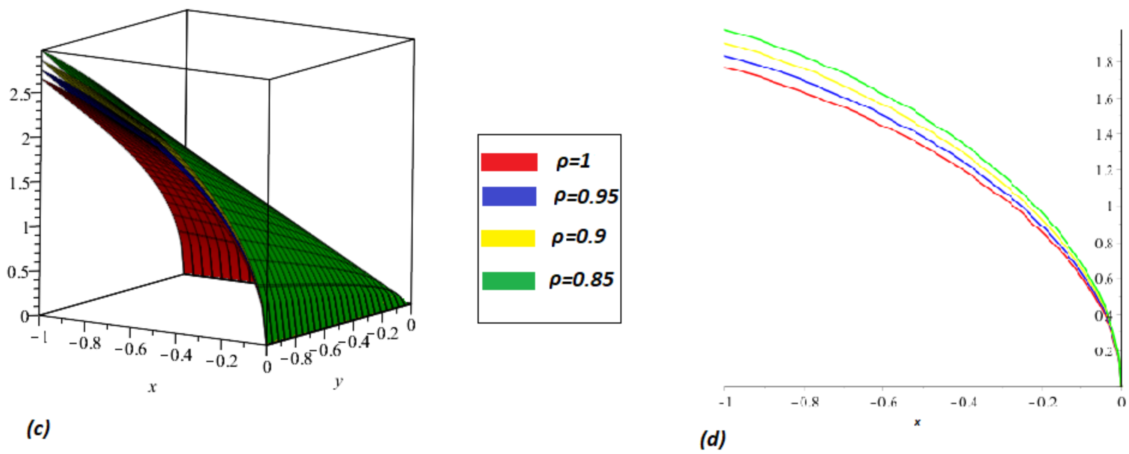

FIGURE 6 Solution graph at various fractional order ρ for example **4.3**

$$\psi(\mu, \nu, \tau) = \sqrt{\sin\mu \sinh\nu} \sum_{k=0}^{\infty} \frac{\tau^{k\rho}}{\Gamma(\rho k + 1)}, \tag{59}$$

The exact result is given by

$$\psi(\mu, \nu, \tau) = \sqrt{\sin\mu \sinh\nu} E_{\rho}(h\tau^{\rho}), \tag{60}$$

As $\rho \to 1$ we have

$$\psi(\mu, \nu, \tau) = \sqrt{\sin\mu \sinh\nu} \sum_{k=0}^{\infty} \frac{(\tau)^k}{k!} = \sqrt{\sin\mu \sinh\nu} e^{\tau}, \tag{61}$$

4.4 | Example

The Biological population model with time non-integer derivative is express as

$$\frac{\partial^{\rho}\psi}{\partial\tau^{\rho}} = \frac{\partial^2}{\partial\mu^2}(\psi^2) + \frac{\partial^2}{\partial\nu^2}(\psi^2) + \psi(1 - r\psi), \tag{62}$$

with initial condition

$$\psi(\mu,\nu,0)=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\nu)}, \tag{63}$$

The Laplace transform to Eq. (62) is expressed as

$$\begin{aligned}
s^{\rho}L[\psi(\mu,\nu,\tau)]-\sum_{k=0}^{m-1}\psi^{(k)}(\mu,\nu,0)s^{\rho-k-1}&=L(\frac{\partial^2}{\partial\mu^2}(\psi^2)+\frac{\partial^2}{\partial\nu^2}(\psi^2)+\psi(1-r\psi)),\\
s^{\rho}L[\psi(\mu,\nu,\tau)]=\psi^{(0)}(\mu,\nu,0)\frac{s^{\rho}}{s}&+L(\frac{\partial^2}{\partial\mu^2}(\psi^2)+\frac{\partial^2}{\partial\nu^2}(\psi^2)+\psi(1-r\psi)),
\end{aligned} \tag{64}$$

$$L[\psi(\mu,\nu,\tau)]=\frac{1}{s}\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\nu)}+\frac{1}{s^{\rho}}\left[L(\frac{\partial^2}{\partial\mu^2}(\psi^2)+\frac{\partial^2}{\partial\nu^2}(\psi^2)+\psi(1-r\psi))\right]. \tag{65}$$

Using inverse Laplace transform of Eq. (65),

$$\psi(\mu,\nu,\tau)=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\nu)}+L^{-1}\left[\frac{1}{s^{\rho}}\left[L(\frac{\partial^2}{\partial\mu^2}(\psi^2)+\frac{\partial^2}{\partial\nu^2}(\psi^2)+\psi(1-r\psi))\right]\right]. \tag{66}$$

Using iterative technique describe in Eqs (12-14), we obtain the following solution components of example 4.4

$$\begin{aligned}
\psi_0(\mu,\nu,\tau)&=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\nu)},\\
\psi_1(\mu,\nu,\tau)&=L^{-1}\left[\frac{1}{s^{\rho}}\left[L(\frac{\partial^2}{\partial\mu^2}(\psi_0^2)+\frac{\partial^2}{\partial\nu^2}(\psi_0^2)+\psi_0(1-r\psi_0)))\right]\right].
\end{aligned} \tag{67}$$

$$=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\nu)}\frac{\tau^{\rho}}{\Gamma(\rho+1)} \tag{68}$$

$$\begin{aligned}
\psi_2(\mu,\nu,\tau)&=L^{-1}\left[\frac{1}{s^{\rho}}\left[L(\frac{\partial^2}{\partial\mu^2}(\psi_1^2)+\frac{\partial^2}{\partial\nu^2}(\psi_1^2)+\psi_1(1-r\psi_1))\right]\right].\\
&=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\nu)}\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}
\end{aligned} \tag{69}$$

$$\psi_3(\mu,\nu,\tau)=L^{-1}\left[\frac{1}{s^{\rho}}\left[L(\frac{\partial^2}{\partial\mu^2}(\psi_2^2)+\frac{\partial^2}{\partial\nu^2}(\psi_2^2)+\psi_2(1-r\psi_2))\right]\right]. \tag{70}$$

$$=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\nu)}\frac{\tau^{3\rho}}{\Gamma(3\rho+1)} \tag{71}$$

The series form of analytical solution is given as

$$\begin{aligned}
\psi(\mu,\nu,\tau)&=\psi_0(\mu,\nu,\tau)+\psi_1(\mu,\nu,\tau)+\psi_2(\mu,\nu,\tau)+\psi_3(\mu,\nu,\tau)+,...,\\
&=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\nu)}+\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\nu)}\frac{t^{\rho}}{\Gamma(\rho+1)}+\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\nu)}\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}+\\
&\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\nu)}\frac{\tau^{3\rho}}{\Gamma(3\rho+1)}+,...,
\end{aligned} \tag{72}$$

$$\psi(\mu,\nu,\tau)=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\nu)}\sum_{k=0}^{\infty}\frac{\tau^{k\rho}}{\Gamma(\rho k+1)}, \tag{73}$$

The exact result is given by

$$\psi(\mu,\nu,\tau)=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\nu)}E_{\rho}(h\tau^{\rho}), \tag{74}$$

As $\rho\to 1$ we have

$$\psi(\mu,\nu,\tau)=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\nu)}\sum_{k=0}^{\infty}\frac{(\tau)^k}{k!} \tag{75}$$

$$=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\mu+\nu)+\tau}, \tag{76}$$

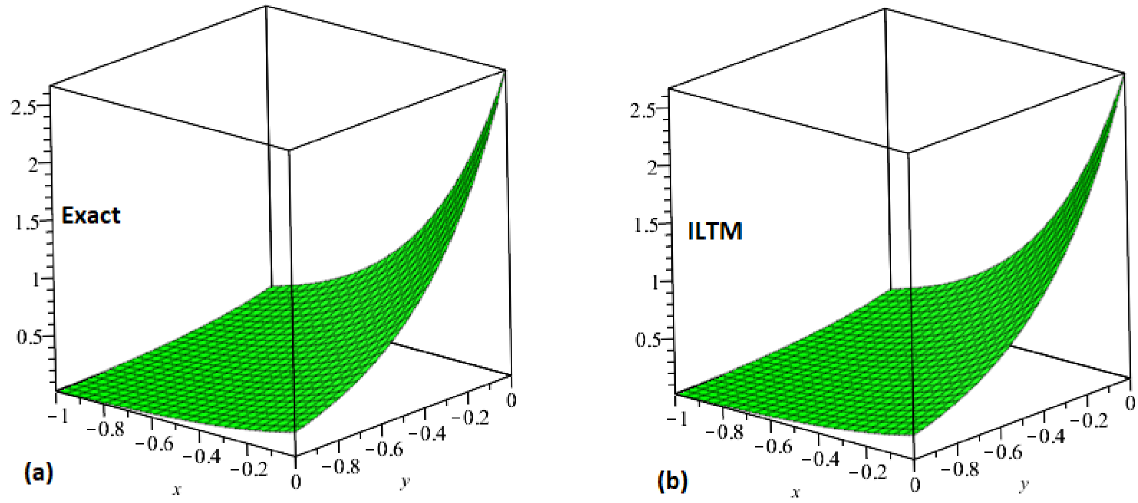

FIGURE 7 The solution plot of example **4**, (a)Exact solution and (b) ILTM solution at $\rho = 1$.

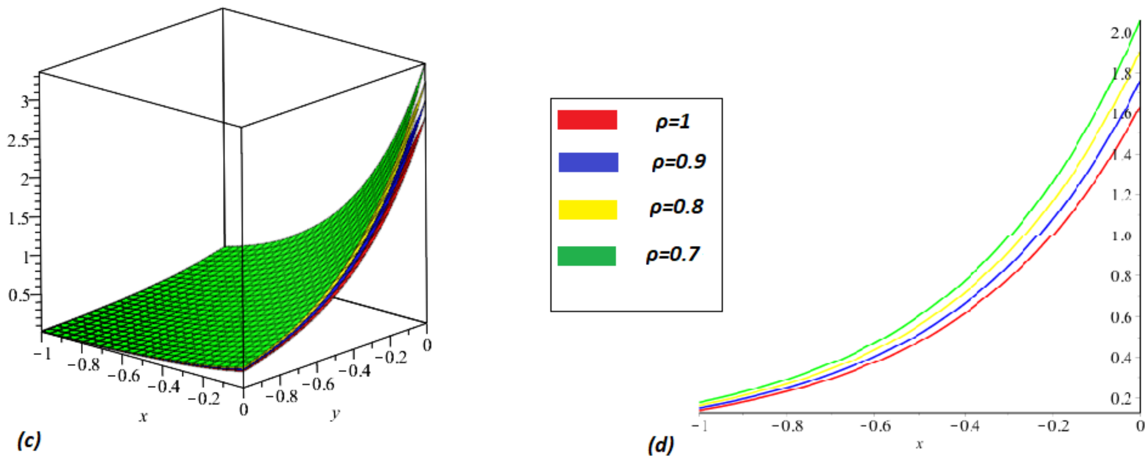

FIGURE 8 Solution graph at various fractional order ρ for example **4.4**

5 | RESULTS AND DISCUSSION

In this research work, ILTM is implemented to solve some important Biological models of non-integer order. The results obtained by the proposed method are explain with the help of its graphical representation. Figure 1 has shown the solution-graphs of exact and ILTM for example 4.1 at $\rho = 1$. It is verified that ILTM solution is closely related with the exact solution. In Figure 2, the error analysis of ILTM for example 4.1 is discussed. It is observed that the proposed method has sufficient degree of accuracy. Similarly, in figure 3, the solution-plot of exact and ILTM solution is displayed for example 4.2. These solution-graphs are very closed to each other and confirmed the validity of the suggested method. Moreover, the higher degree of accuracy is achieved as represented by Figure 4. In Figure 5, the exact and ILTM solutions for example 4.3 are compared. The solution-graphs for both exact and ILTM are identical and support the reliability of the suggested method. In Figure 6, the solution for example 4.3 at different fractional-orders are calculated. It is investigated that the solutions at different fractional-orders are converges to the solution of integer-order solution as fractional-orders approaches to an integer-order. In Figure7 the same graphical representation have been made for the exact and ILTM solution of example 4.4. Figure 7 provide the graphical layout of the solution of example 4 at different fractional-order. The convergence phenomena of the solutions at different fractional-order can be seen in Figure 8.

6 | CONCLUSION

The present research article is related to the solution of fractional-order Biological population models by using a sophisticated analytical technique. The present method is implemented for both fractional and integer-order models. The solution graphs for ILTM and exact solutions of the problems are plotted. it is investigated that the ILTM solutions are in closed contact with the exact solution of the problems even by taking two or three components of the proposed method. The ILTM solutions of the problems at different Fractional-orders are also shown with the help of their graphical representation. The convergence phenomena of the solutions of fractional-order problems toward integer-order solution is observed. This behavior of the obtained solution has confirmed the efficiency of the suggested scheme. Due to an effective and straight forward implementation, the suggested method can be modified for the solution of other FPDEs arises in applied sciences.

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