Approximation and generic properties of McKean-Vlasov stochastic equations with continuous coefficients

Mohamed Amine Mezerdi * Khaled Bahlali† Nabil Khelfallah ‡ Brahim Mezerdi§

January 16, 2020

Abstract

We consider various approximation properties for systems driven by a Mc Kean-Vlasov stochastic differential equations (MVSDEs) with continuous coefficients, for which pathwise uniqueness holds. We prove that the solution of such equations is stable with respect to small perturbation of initial conditions, parameters and driving processes. Moreover, the unique strong solutions may be constructed by an effective approximation procedure. Finally we show that the set of bounded uniformly continuous coefficients for which the corresponding MVSDE have a unique strong solution is a set of second category in the sense of Baire.

Key words: McKean-Vlasov stochastic differential equation – Mean-field - Stability - Strong solution - Pathwise uniqueness - Wasserstein metric - Generic property - Baire space - Generic property.

2010 Mathematics Subject Classification. 60H10, 60H07, 49N90.

1 Introduction

McKean-Vlasov stochastic differential equations (MVSDE) have been investigated by McKean [17], for the first time, as the counterpart of Vlasov [22] non linear partial differential equations (PDE) arising in statistical physics. They describe the limiting bahaviour of an individual particle evolving within a large system of particles, with weak interaction, as the number of particles tends to infinity. These equations are called non linear SDEs in the sense that the coefficients depend not only on the state variable, but also on its marginal distribution and their solutions are called non linear diffusions. A pedagogical and rigorous treatment of these equations appear in the seminal Saint-Flour course by Sznitman [21].

Since the pioneering work of McKean [17], a huge literature on existence, uniqueness, numerical schemes and propagation of chaos theorems was developped. Existence and uniqueness of strong solutions were obtained under global Lipschitz coefficients in [7, 13, 21] by using the fixed point theorem on the space of continuous functions with values on the space of probability measures, equipped with Wasserstein distance. MVSDEs with non regular coefficients appear naturally in many mean-field models. The so-called mean-field FitzHugh-Nagumo model and the network of Hodgkin-Huxley neurons are typical examples (see [4]). It is clear that if the coefficients are not globally Lipschitz, the Gronwall inequality and its variants fail, so that fixed point theorems are

*Laboratory of Applied Mathematics, University of Biskra, Po. Box 145, Biskra (07000), Algeria. *(E-mail: amine.mezerdi@univ-biskra.dz)*

†Laboratoire IMATH, Université du Sud-Toulon-Var, B.P 20132, 83957 La Garde Cedex 05, France. *(E-mail: bahlali@univ-tln.fr)*

‡Laboratory of Applied Mathematics, University of Biskra, Po. Box 145, Biskra (07000), Algeria. *(E-mail: nabilkhelfallah@yahoo.fr)*

§King Fahd University of Petroleum and Minerals, Department of Mathematics and Statistics, P.O. Box 1916, Dhahran 31261, Saudi Arabia. (*E-mail: brahim.mezerdi@kfupm.edu.sa)*

no longer applicable. Let us point out that contrary to Itô's SDEs, regularity assumptions of local nature on the coefficients, such as locally Lipschitz coefficients do not lead to unique (local) strong solutions (see [19] for counterexamples).

It is a well known that if the coefficients are Lipschitz continuous, then MVSDE (2.1) has a unique strong solution $X_t(x)$, which is continuous with respect to the initial condition and coefficients. Moreover, the solution may be constructed by means of various numerical schemes (see [1]).

Our purpose in this paper, is to study strong stability properties of the solution of (2.1) under pathwise uniqueness of solutions and merely continuous coefficients. Since the coefficients are only continuous without additional regularity, one cannot expect to apply Gronwall's lemma. Instead of Gronwall's lemma, we use tightness arguments and the famous Skorokhod selection theorem to prove the desired convergence results. Of course we should not expect precise convergence speed as this last property is based on regularity of the coefficients.

The paper is organized as follows. In the second section we prove that the Euler polygonal scheme is convergent provided that there is pathwise uniqueness. This provides us with an effective way to construct strong solutions for MVSDEs. In the third section we prove that the solution is stable under small perturbation of the initial condition and coefficients. Our results generalize similar ones proved for classical Itô SDEs [2, 9, 14]. In the last section, we show that the set of bounded uniformly continuous coefficients for which strong existence and uniqueness hold is a generic property in the sense of Baire. This means that in the sense of Baire category, most of MVSDEs with bounded uniformly continuous coefficients have unique solutions. This last result extends in particular [2, 3] to MVSDEs.

2 Assumptions and preliminaries

2.1 The Wasserstein distance

Definition 2.1. Let (M, d) be a metric space, for which every probability measure on M is a Radon measure (a so-called Radon space). Denote $\mathcal{P}_p(M)$ the collection of all probability measures μ on M with finite moment of order p for some x_0 in M, $\displaystyle\int_M d(x_0, x)^p \mu(dx) < +\infty$. Then the $p-$Wasserstein distance between two probability measures μ and ν in $\mathcal{P}_p(M)$ is defined as

$$W_p(\mu,\nu) = \left(\inf \left\{ \int_{M\times M} d(x, y)^p \gamma(dx, dy);\ \gamma \in \Gamma(\mu, \nu) \right\} \right)^{1/p}$$

where $\Gamma(\mu, \nu)$ denotes the collection of all measures on $M \times M$ with marginals μ and ν on the first and second factors respectively. The set $\Gamma(\mu, \nu)$ is also called the set of all couplings of μ and ν

The Wasserstein metric may be equivalently defined by $W_p(\mu,\nu) = (\inf E\,[d(X, Y)^p])^{1/p}$ where the infimum is taken over all the joint probability distributions of the random variables X and Y with marginals μ and ν.

In the case where the metric space is replaced by the euclidian space $\mathbb{R}^d$, then the p-Wasserstein distance $W_p(\mu, \nu)$ is defined by:

$$W_p(\mu, \nu)^p = \inf_{\gamma\in\Gamma(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x - y|^p \, d\pi(x, y)^{1/p}$$

or equivalently $W_p(\mu, \nu)^p = \inf \mathbb{E}[\,|X - Y|^p]$.

In particular if X and Y are square integrable random variables, we have $W_2(P_X, P_Y) \leq \mathbb{E}[\,|X - Y|^2]^{1/2}$.

In the literature the Wasserstein metric is restricted to W_2 while W_1 is often called the Kantorovich-Rubinstein distance because of the role it plays in optimal transport.

2.2 Assumptions

Let (B_t) a d-dimensional Brownian motion defined on a probablity space $(\Omega, \mathcal{F}, P)$, equipped with a filtration $(\mathcal{F}_t)$, satisfying the usual conditions. Throughout this paper, we consider McKean-Vlasov stochastic differential equation (MVSDE), called also mean-field stochastic differential equation of the form

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t})dt + \sigma(t, X_t, \mathbb{P}_{X_t})dB_t \\ X_0 = x \end{cases} \tag{2.1}$$

For this kind of stochastic differential equations, the drift b and diffusion coefficient σ depend not only on the state process X_t, but also on its marginal distribution $\mathbb{P}_{X_t}$.

Assume that the coefficients satisfy the following conditions.

$(\mathbf{H_1})$ Assume that

$$\begin{aligned} b &: [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d \\ \sigma &: [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d \end{aligned}$$

are Borel measurable functions and continuous in (x, μ) uniformly in $t \in [0,T]$.

$(\mathbf{H_2})$ There exist $C > 0$ such that for any $t \in [0,T], x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\begin{aligned} |b(t,x,\mu)| &\leq C\,(1 + |x| + W_2(\mu, \delta_0)), \\ |\sigma(t,x,\mu)| &\leq C\,(1 + |x| + W_2(\mu, \delta_0)), \end{aligned}$$

where W_2 is the 2-Wasserstein distance and δ_0 is the Dirac measure at 0.

The following theorem states that under global Lipschitz condition, (2.1) admits a unique solution. Its complete proof is given in [21] for a drift depending linearly on the law of X_t that is $b(t,x,\mu) = \int\limits_{\mathbb{R}^d} b'(t,x,y)\mu(dy)$ and a constant diffusion. The general case as (2.1) is treated in [?] Theorem 4.21 or [13] Proposition 1.2. The proof is based on a fixed point theorem on the space of continuous functions with values in $\mathcal{P}_2(\mathbb{R}^d)$ endowed with Wasserstein metric. Note that in [7, 13] the authors consider MVSDEs driven by general Lévy process instead of a Brownian motion.

Theorem 2.2. *Assume* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and*

$(\mathbf{H_3})$ *there exist* $L > 0$ *such that for any* $t \in [0,T], x,\ x' \in \mathbb{R}^d$ *and* $\mu,\ \mu' \in \mathcal{P}_2(\mathbb{R}^d)$,
$|b(t,x,\mu) - b(t,x',\mu')| \leq C\,(|x - x'| + W_2(\mu, \mu'))$,
$|\sigma(t,x,\mu) - \sigma(t,x',\mu')| \leq C\,(|x - x'| + W_2(\mu, \mu'))$,
then MVSDE (2.1) admits a unique solution such that $E[\sup_{t \leq T} |X_t|^2] < +\infty$.

Proof. See [13]. ■

Other versions of the MVSDEs, which are particular cases of (2.1) have been considered in the literature.

1) The following MVSDE has been treated in literature

$$\begin{cases} dX_t = b(t, X_t, \int \varphi(y)\mathbb{P}_{X_t}(dy))dt + \sigma(t, X_t, \int \psi(y)\mathbb{P}_{X_t}(dy))dW_t \\ X_0 = x, \end{cases} \tag{2.2}$$

2) MVSDEs studied in the framework of statistical physics take the form

$$\begin{cases} dX_t = \int\limits_{\mathbb{R}^d} b(t, X_t, y)\mathbb{P}_{X_t}(dy)dt + dB_t \\ X_0 = x \end{cases}$$

where $b : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a Borel measurable function such that $b(t, ., .)$ is Lipschitz. This is a particular class of MVSDEs for interacting diffusions, considered by McKean (see [21] for details), where the drift is linear on the probability distribution. It is easy to see that the drift is Lipschitz in the measure variable with respect to Wasserstein metric.

The definition of pathwise uniqueness for equation (2.1) is given by the following.

Definition 2.3. We say that pathwise uniqueness holds for equation (2.1) if X and X' are two solutions defined on the same probability space $(\Omega, \mathcal{F}, P)$ with common Brownian motion (B), with possibly different filtrations such that $P[X_0 = X'_0] = 1$, then X and X' are indistinguishable.

Let us recall Kolmogorov's tightness criteia for stochastic processes and Skorokhod selection theorem, which will be extensively used in the sequel.

Lemma 2.4. *(Skorokhod selection theorem [12] page 9) Let (S, ρ) be a complete separable metric space, $P_n, n = 1, 2, ...$ and P be probability measures on $(S, \mathcal{B}(S))$ such that $P_n \underset{n \to +\infty}{\longrightarrow} P$. Then, on a probability space $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}\right)$, we can construct S-valued random variables X_n, $n = 1, 2, ...,$ and X such that:*

(i) $P_n = \widehat{P}^{X_n}, n = 1, 2, ...,$ and $P = \widehat{P}^X$.
(ii) X_n converges to X, $\widehat{P}$ almost surely.

Lemma 2.5. *(Skorokhod limit theorem [20]) Let $(\Omega, \mathcal{F}, P)$ be a probability space, (H_n) be uniformly bounded processes and (H^n) be a sequence of Brownian motions defined on the same space such that the stochastic integral $\int_O^T H_s^n dW_s^n$ are well defined for each $n \geq 0$. Assume moreover that*

a) $\lim_{h \to 0} \sup_n \sup_{|s-t|<h} P(|H_s^n - H_t^n| > \varepsilon) = 0$
b) (H_s^n, W_s^n) converges to $\left(H_s^0, W_s^0\right)$ in probability.

Then $\int_0^T H_s^n dW_s^n$ converges in probability to $\int_0^T H_s^0 dW_s^0$

Lemma 2.6. *(Kolmogorov criterion for tightness [12] page 18) Let $(X_n(t))$, $n = 1, 2, ...,$ be a sequence of d-dimensional continuous processes satisfying the following two conditions:*

(i) There exist positive constants M and γ such that $E[|X_n(0)|^\gamma] \leq M$ for every $n = 1, 2, ...$
(ii) There exist positive constants α, β, M_k, $k = 1, 2, ...,$ such that:
$E[|X_n(t) - X_n(s)|^\alpha] \leq M_k |t - s|^{1+\beta}$ for every n and $t, s \in [0, k], (k = 1, 2, ...)$.

Then there exist a subsequence (n_k), a probability space $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}\right)$ and d-dimensional continuous processes $\widehat{X}_{n_k}, k = 1, 2, ...,$ and $\widehat{X}$ defined on it such that
1) The laws of $\widehat{X}_{n_k}$ and X_{n_k} coincide.
2) $\widehat{X}_{n_k}(t)$ converges to $\widehat{X}(t)$ uniformly on every finite time interval $\widehat{P}$ almost surely.

3 Construction of strong solutions by approximation

It is well known for classical Itô SDEs [12] as well as for McKeanVlasov SDEs [15] that weak existence and pathwise uniqueness imply the existence and uniqueness of a strong solution. This is a corollary of the famous Yamada-Watanabe theorem (see [12]). In this section we prove that under the pathwise uniqueness, the strong solution, may be constructed by means of an approximation procedure and may be written as a measurable functional of the initial condition and the Brownian motion without appealing to the famous Yamada-Watanabe theorem.

Let (Δ^n) be a sequence of partitions of the interval $[0,T]$ where $\Delta^n : 0 = t_0^n < t_1^n < ..... < t_n^n = T$ such that

$$\lim_{n\to+\infty} \|\Delta^n\| = \lim_{n\to+\infty} \max_i \left(t_{i+1}^n - t_i^n\right) = 0$$

Define the Euler polygonal approximation for equation (2.1) by:

$$X_{\Delta^n}(x,t) = x + \int_0^t b(\phi_{\Delta^n}(s), X_{\Delta^n}, P_{\Delta^n})ds + \int_0^t \sigma(\phi_{\Delta^n}(s), X_{\Delta^n}, P_{X_{\Delta^n}})dB_s$$

where $\phi_{\Delta^n}(s) = t_i$, if $t_i^n \le s < t_{i+1}^n$ and $\|\Delta^n\| = \max_i \left(t_{i+1}^n - t_i^n\right)$ and $X_{\Delta^n} = X_{\Delta^n}(x, \phi_{\Delta^n}(s))$

Theorem 3.1. *Assume* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$*, then under pathwise uniqueness we have:*

1) $\lim_{n\to 0} E\left[\sup_{t\le T} |X_{\Delta^n}(x,t) - X(x,t)|^2\right] = 0$

2) There exists a measurable functional $F : \mathbb{R}^d \times W_0^d \longrightarrow W^d$ *which is adapted such that the unique solution* X_t *can be written* $X(.) = F(X(0), B(.))$*, where* $W^d = C\left(\mathbb{R}_+, \mathbb{R}^d\right)$ *and* $W_0^d = \left\{w \in C\left(\mathbb{R}_+, \mathbb{R}^d\right) : w(0) = 0\right\}$ *are equipped with their Borel* $\sigma-$*fields and the filtrations of coordinates.*

Proof. 1) Suppose that the conclusion of our theorem is false, then there exists a sequence(Δ_n) and $\delta \ge 0$ such that

$$\liminf_{n\to\infty} E\left[\sup_{t\le T} |X_{\Delta_n}(x,t) - X_t|^2\right] \ge \delta. \tag{3.1}$$

Let $\mathcal{C}([0,T])$ be the space of continuous functions equipped with the topology of uniform convergence and $\mathcal{P}_2\left(\mathcal{C}([0,T])\right)$ the space of probability measures equipped with the Wasserstein metric.

Using assumptions $(\mathbf{H_1})$, $(\mathbf{H_2})$ and classical arguments of stochastic calculus, it is easy to see that the sequence $(X_{\Delta_n}, X, B, P_{X_{\Delta_n}}, P_X)$ satisfies the conditions of Kolmogorov criteria, then it is tight in $\mathcal{C}([0,T])^3 \times \mathcal{P}_2\left(\mathcal{C}([0,T])\right)^2$.

Then by Skorokhod limit Theorem (Lemma 2.4), there exist a probability space $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}\right)$ and a sequence of stochastic processes $\left(\widehat{X_t^n}, \widehat{Y_t^n}, \widehat{B_t^n}, \widehat{\mu_t^n}, \widehat{\nu_t^n}\right)$ defined on it such that:

$i)$ the laws of $(X_{\Delta_n}, X, B, P_{X_{\Delta_n}}, P_X)$ and $\left(\widehat{X_t^n}, \widehat{Y_t^n}, \widehat{B_t^n}, \widehat{\mu_t^n}, \widehat{\nu_t^n}\right)$ coincide for every $n \in \mathbb{N}$.

$ii)$ there exists a subsequence also denoted by $\left(\widehat{X_t^n}, \widehat{Y_t^n}, \widehat{B_t^n}, \widehat{\mu_t^n}, \widehat{\nu_t^n}\right)$ converging to $\left(\widehat{X_t}, \widehat{Y_t}, \widehat{B_t}, \widehat{\mu_t}, \widehat{\nu_t}\right)$ uniformly on every finite time interval $\widehat{P}-$a.s..

It is clear that $\left(\widehat{B_t^n}, \widehat{\mathcal{F}_t^n}\right)$ and $\left(\widehat{B_t}, \widehat{\mathcal{F}_t}\right)$ are Brownian motions with respect the filtrations $\widehat{\mathcal{F}_t^n} = \sigma\left(\widehat{X_s^n}, \widehat{Y_s^n}, \widehat{B_s^n}; s \le t\right)$ and $\widehat{\mathcal{F}_t} = \sigma\left(\widehat{X_s}, \widehat{Y_s}, \widehat{B_s}; s \le t\right)$.

Note that the probability measures do not depend upon the random element ω, then $(P_{X^n}, P_X) = \left(\widehat{\mu_t^n}, \widehat{\nu_t^n}\right)$ and consequently $\left(\widehat{\mu_t^n}, \widehat{\nu_t^n}\right) = \left(P_{\widehat{X_t^n}}, P_{\widehat{Y_t^n}}\right)$ and $(\widehat{\mu_t}, \widehat{\nu_t}) = \left(P_{\widehat{X_t}}, P_{\widehat{Y_t}}\right)$

According to property $i)$ and the fact that X_{Δ_n} and X_t satisfy equation (2.1) and using the fact that the finite-dimensional distributions coincide, we can easily prove that $\forall n \ge 1$, $\forall t \ge 0$

$$E\left|\widehat{X_t^n} - x - \int_0^t \sigma\left(\phi_{\Delta_n}(s), \widehat{X_s^n}, P_{\widehat{X_t^n}}\right) d\widehat{B_s^n} - \int_0^t b\left(\phi_{\Delta_n}(s), \widehat{X_s^n}, P_{\widehat{X_t^n}}\right) ds\right|^2 = 0,$$

which means that

$$\widehat{X_t^n} = x + \int_0^t \sigma\left(\phi_{\Delta_n}(s), \widehat{X_s^n}, P_{\widehat{X_t^n}}\right) d\widehat{B_s^n} + \int_0^t b\left(\phi_{\Delta_n}(s), \widehat{X_s^n}, P_{\widehat{X_t^n}}\right) ds.$$

Using similar arguments for $\widehat{Y_t^n}$, we obtain:

$$\widehat{Y_t^n} = x + \int_0^t \sigma\left(s, \widehat{Y_t^n}, P_{\widehat{Y_t^n}}\right) d\widehat{B_s^n} + \int_0^t b\left(s, \widehat{Y_t^n}, P_{\widehat{Y_t^n}}\right) ds$$

Now, by Skorokhod's limit Theorem (see [20] or [9] Lemma 3.1) and according to $ii)$ and the fact that $\phi_\Delta(s) \to s$, it holds that,

$$\int_0^t \sigma\left(\phi_{\Delta_n}(s), \widehat{X}_s^{n_k}, P_{\widehat{X}_s^{n_k}}\right) d\widehat{B_s^{n_k}} \xrightarrow[k\to\infty]{P} \int_0^t \sigma\left(s, \widehat{X}_s, P_{\widehat{X}_s}\right) d\widehat{B}_s,$$

$$\int_0^t b\left(\phi_{\Delta_n}(s), \widehat{X}_s^{n_k}, P_{\widehat{X}_s^{n_k}}\right) ds \xrightarrow[k\to+\infty]{P} \int_0^t b\left(s, \widehat{X}_s, P_{\widehat{X}_s}\right) ds.$$

We conclude that $\widehat{X_t}$ and $\widehat{Y}_t$ satisfy the same stochastic differential equation (2.1) on the new probability space $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}\right)$, with the same initial condition x and common Brownian motion $\widehat{B_t}$. Therefore, according to the pathwise uniqueness for (2.1) it holds that $\widehat{X} = \widehat{Y}$.

By uniform integrability, it holds that:

$$\delta \leq \liminf_{n\in\mathbf{N}} E\left[\sup_{t\leq T} |X_{\Delta_n} - X_t|^2\right] \leq \lim \widehat{E}\left[\sup_{t\leq T}\left|\widehat{X_t^{n_k}} - \widehat{Y_t^{n_k}}\right|^2\right] = \widehat{E}\left[\sup_{t\leq T}\left|\widehat{X_t} - \widehat{Y_t}\right|^2\right] = 0$$

which contradicts our hypothesis (3.1). ■

2) Let $(W, \mathcal{B}(W), P^B, B(t))$ be the standard Wiener process and $X_{\Delta_n}(x,.,w)$ be the polygonal approximation . It is clear that the functional $F_{\Delta_n} : \mathbb{R}^d \times W_0^d \longrightarrow W^d$ defined by $F_{\Delta_n}(x,w) = X_{\Delta_n}(x,.,w)$ is measurable. Moreover property 1) and Borel Cantelli lemma imply that $(F_{\Delta_n}(x,w))$ converges uniformly in W^d a.s..

Let $F(x,w) = \lim F_{\Delta_n}(x,w)$, then $F(x,w)$ is measurable and that the unique solution is written as $X(X(0),t) = F(X(0),w)$ which achieves the proof. ■

Remark 3.2. 1) Under the same assumptions and using the same proof, we can prove

$\lim_{n\to 0} \sup_{x\in K} E\left[\sup_{t\leq T} |X_{\Delta^n}(x,t) - X(x,t)|^2\right] = 0$ where K is any compact set in $\mathbb{R}^d$.

4 Stability with respect to initial conditions and coefficients

In this section we will prove that under minimal assumptions on the coefficients and pathwise uniqueness of solutions, the unique solution is continuous with respect the initial condition and coefficients.

We denote by (X_t^x) the unique solution of (2.1) corresponding to the initial condition $X_0^x = x$.

$$\begin{cases} dX_t^x = b(t, X_t^x, \mathbb{P}_{X_t^x})dt + \sigma(t, X_t^x, \mathbb{P}_{X_t^x})dB_t \\ X_0^x = x. \end{cases}$$

Theorem 4.1. *Assume that* $b(t,x,\mu)$ *and* $\sigma(t,x,\mu)$ *satisfy* $(\mathbf{H_1})$, $(\mathbf{H_2})$.*Then if the pathwise uniqueness holds for equation (2.1) then the mapping*

$$\Phi : \mathbb{R}^d \longrightarrow L^2(\Omega, \mathcal{C}([0,T], \mathbb{R}^d))$$

defined by $(\Phi(x)_t) = (X_t^x)$ is continuous.

Proof. Suppose that the conclusion of our theorem is false, then there exists a sequence (x_n) in $\mathbb{R}^d$converging to x and $\delta \geq 0$ such that

$$\liminf_{n\to\infty} E\left[\sup_{t\leq T} |X_t^n - X_t|^2\right] \geq \delta \tag{4.1}$$

where $X_t^n = X_t^{x_n}$ and $X_t = X_t^x$.

Using assumptions $(\mathbf{H_1})$, $(\mathbf{H_2})$ and classical arguments of stochastic calculus, it is easy to see that

$$E\left[|X^n(t) - X^n(s)|^4\right] \leq C(T)|t-s|^2.$$

where $C(T)$ is a constant which does not depend on n. Similar to estimate holds true also for X and the Brownian motion B. Then by Prokhorov's Theorem , the sequence $(X^n, X, P_{X^n}, P_X, B)$ satisfy i) and ii) of Lemma 1 (appendix), then this sequence is tight, which implies that it is relatively compact in the topology of weak convergence of probability measures.Therefore by Skorokhod selection Theorem (Lemma 2.4), there exists a probability space $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}\right)$ carrying a sequence of stochastic processes $\left(\widehat{X_t^n}, \widehat{Y_t^n}, \widehat{B_t^n}, \widehat{\mu_t^n}, \widehat{\nu_t^n}\right)$ defined on it such that:

i) the laws of $(X^n, X, B, P_{X^n}, P_X)$ and $\left(\widehat{X_t^n}, \widehat{Y_t^n}, \widehat{B_t^n}, \widehat{\mu_t^n}, \widehat{\nu_t^n}\right)$ coincide for every $n \in \mathbb{N}$.

ii) there exists a subsequence also denoted by $\left(\widehat{X_t^n}, \widehat{Y_t^n}, \widehat{B_t^n}, \widehat{\mu_t^n}, \widehat{\nu_t^n}\right)$ converging to $\left(\widehat{X_t}, \widehat{Y_t}, \widehat{B_t}, \widehat{\mu_t}, \widehat{\nu_t}\right)$ uniformly on every finite time interval $\widehat{P}$-a.s., where $\left(\widehat{B}_t^n, \widehat{\mathcal{F}}_t^n\right)$ and $\left(\widehat{B}_t, \widehat{\mathcal{F}}_t\right)$ are Brownian motions with respect the filtrations $\widehat{\mathcal{F}}_t^n = \sigma\left(\widehat{X_s^n}, \widehat{Y_s^n}, \widehat{B_s^n}; s \leq t\right)$ and $\widehat{\mathcal{F}}_t = \sigma\left(\widehat{X_s}, \widehat{Y_s}, \widehat{B_s}; s \leq t\right)$.

Note that the probability measures do not depend upon the random element ω, then $(P_{X^n}, P_X) = \left(\widehat{\mu_t^n}, \widehat{\nu_t^n}\right)$ and consequently $\left(\widehat{\mu_t^n}, \widehat{\nu_t^n}\right) = \left(P_{\widehat{X_t^n}}, P_{\widehat{Y_t^n}}\right)$ and $(\widehat{\mu}_t, \widehat{\nu}_t) = \left(P_{\widehat{X_t}}, P_{\widehat{Y_t}}\right)$

According to property *i)* and the fact that X_t^n and X_t satisfy equation (2.1) with initial data x_n and x, and using the fact that the finite-dimensional distributions coincide, we can easily prove that $\forall n \geq 1$, $\forall t \geq 0$

$$E\left|\widehat{X_t^n} - x_n - \int_0^t \sigma\left(s, \widehat{X_s^n}, P_{\widehat{X_t^n}}\right) d\widehat{B_s^n} - \int_0^t b\left(s, \widehat{X_s^n}, P_{\widehat{X_t^n}}\right) ds\right|^2 = 0.$$

In other words,

$$\widehat{X_t^n} = x_n + \int_0^t \sigma\left(s, \widehat{X_s^n}, P_{\widehat{X_t^n}}\right) d\widehat{B_s^n} + \int_0^t b\left(s, \widehat{X_s^n}, P_{\widehat{X_t^n}}\right) ds$$

Using similar arguments for $\widehat{Y_t^n}$, we obtain:

$$\widehat{Y_t^n} = x + \int_0^t \sigma\left(s, \widehat{Y_t^n}, P_{\widehat{Y_t^n}}\right) d\widehat{B_s^n} + \int_0^t b\left(s, \widehat{Y_t^n}, P_{\widehat{Y_t^n}}\right) ds$$

Now, by Skorokhod's limit theorem (Lemma 2.5) and according to *ii)* it holds that,

$$\int_0^t \sigma\left(s, \widehat{X}_s^{n_k}, P_{\widehat{X}_s^{n_k}}\right) d\widehat{B_s^{n_k}} \xrightarrow[k\to\infty]{P} \int_0^t \sigma\left(s, \widehat{X}_s, P_{\widehat{X}_s}\right) d\widehat{B}_s,$$

$$\int_0^t b\left(s, \widehat{X}_s^{n_k}, P_{\widehat{X}_s^{n_k}}\right) ds \xrightarrow[k\to+\infty]{P} \int_0^t b\left(s, \widehat{X}_s, P_{\widehat{X}_s}\right) ds.$$

We conclude that $\widehat{X_t}$ and $\widehat{Y_t}$ satisfy the same stochastic differential equation (2.1) on the new probability space $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P}\right)$, with the same initial condition x and common Brownian motion $\widehat{B_t}$

$$\widehat{X_t} = x + \int_0^t \sigma\left(s, \widehat{X_s}, P_{\widehat{X_t}}\right) d\widehat{B_s} + \int_0^t b\left(s, \widehat{X_s}, P_{\widehat{X_t}}\right) ds$$

and

$$\widehat{Y_t} = x + \int_0^t \sigma\left(s, \widehat{Y_t}, P_{\widehat{Y_t}}\right) d\widehat{B_s} + \int_0^t b\left(s, \widehat{Y_t}, P_{\widehat{Y_t}}\right) ds.$$

According to the pathwise uniqueness for (2.1) it holds that $\widehat{X} = \widehat{Y}$.

By uniform integrability, it holds that:

$\delta \leq \liminf_{n \in \mathbf{N}} E\left[\sup_{t \leq T} |X_t^n - X_t|^2\right] \leq \lim \widehat{E}\left[\sup_{t \leq T} \left|\widehat{X_t^{n_k}} - \widehat{Y_t^{n_k}}\right|^2\right] = \widehat{E}\left[\sup_{t \leq T} \left|\widehat{X_t} - \widehat{Y_t}\right|^2\right] = 0$

which contradicts our hypothesis (4.1). ■

Using the same techniques we can prove the continuity of the solution of MVSDE with respect to a parameter. In particular the solution is continunous with respect to the coefficients. Let us consider a sequence of functions and consider the MVSDE.

$$\begin{cases} dX_t^n = \sigma_n\left(t, X_t^n, P_{X_t^n}\right) dB_t + b_n\left(t, X_t^n, P_{X_t^n}\right) dt \\ X^n(0) = x_n. \end{cases} \tag{4.2}$$

Theorem 4.2. *Suppose that $\sigma_n(t, x, \mu)$ and $b_n(t, x, \mu)$ are continuous functions . Further suppose that for each $T > 0$, and each compact set K there exists $L > 0$ such that*

i) $\sup_{t \leq T} \left(|\sigma_n(t, x, \mu)| + |b_n(t, x, \mu)|\right) \leq L\ (1 + |x|)$ uniformly in n,

ii) $\lim_{n \to +\infty} \sup_{x \in K} \sup_{t \leq T} \left(|\sigma_n(t, x, \mu) - \sigma(t, x, \mu)| + |b_n(t, x, \mu) - b(t, x, \mu)|\right) = 0$,

iii) $\lim_{n \to +\infty} x_n = x$.

If the pathwise uniqueness holds for equation (2.1)*, then:*

$$\sup_{x \in K} E\left[\sup_{t \leq T} |X_t^n - X_t|^2\right] = 0 \text{ , for every } T \geq 0 \text{ .}$$

Proof. Similar to the proof of Theorem 4.1. ■

Remark 4.3. It is clear that if we suppose that the coefficients b and σ are globally Lipschitz or continuous and satisfy any assumption ensuring pathwise uniqueness, then the conclusion of the last theorem remains true. In particular, if the coefficients satisfy Osgood condition or are monotone (see [1, ?]) then the solution depends continuously on the initial data and the coefficients b and σ.

5 Existence and uniqueness is a generic property

We know that under globally Lipschitz coefficients equation (2.1) has a unique strong solution (see [7, 13, 21]). A huge literature has been produced to improve the conditions under which pathwise uniquenes holds. Moreover the continuity of the coefficients is not sufficient for the uniqueness. The objective to identify completely the set of coefficients, under which there is a unique strong solution seems to be out of reach, even for ordinary differential equations. In this section we are interested in qualitative properties of the set of coefficients for which existence and uniqueness of solutions hold. In fact we prove that "most" of the MVSDEs with bounded uniformly continuous coefficients enjoy the property of existence and uniqueness. The expression "most" should be understood in the sense of topology and is similar to the measure theoritic concept of a set whose complement is a negligible set. More precisely, we prove that in the sense of Baire, the set of coefficients (b, σ) for which existence and uniqueness of a strong solution is a residual set in the Baire space of all bounded uniformly continuous functions.

Prevalence properties for ordinary differential equations were first considered by Orlicz [18] and Lasota-Yorke [16]. These properties have been extended to Itô stochastic differential equations in [2, 3, 11].

Let us recall some facts about Baire spaces.

Definition 5.1. A Baire space X is a topological space in which the union of every countable collection of closed sets with empty interior has empty interior.

This definition is equivalent to each of the following conditions.
a) Every intersection of countably many dense open sets is dense.
b) The interior of every union of countably many closed nowhere dense sets is empty.

Remark 5.2. By the Baire category theorem, we know that a complete metric space is a Baire space.

Definition 5.3. 1) A subset of a topological space X is called nowhere dense in X, if the interior of its closure is empty

2) A subset is of first category in the sense of Baire (or meager in X), if it is a union of countably many nowhere dense subsets.

3) A subset is of second category or nonmeager in X, if it is not of first category in X.

Remark 5.4. 1) The definition for a Baire space can then be stated as follows: a topological space X is a Baire space if every non-empty open set is of second category in X.

2) In the literature, a subset of second category is also called a residual subset.

Definition 5.5. A property P is generic in the Baire space $\mathcal{X}$ if P holds is satisfied for each element in $\mathcal{X} - \mathcal{N}$, where $\mathcal{N}$ is a set of first category in the Baire space $\mathcal{X}$.

Let us introduce the appropriate Baire space.

Let $\mathcal{C}_1$be the set of bounded uniformly continuous functions $b : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2\left(\mathbb{R}^d\right) \longrightarrow \mathbb{R}^d$. Define the metric ρ_1 on $\mathcal{C}_1$ as follows:

$$\rho_1\left(b_1, b_2\right) = \sup_{(t,x,\mu)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathcal{P}_2\left(\mathbb{R}^d\right)} \left|b_1(t, x, \mu) - b_2(t, x, \mu)\right|$$

Note that the metric ρ_1 is compatible with the topology of uniform convergence on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2\left(\mathbb{R}^d\right)$.

Let $\mathcal{C}_2$ be the set of bounded unifomly continuous functions $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2\left(\mathbb{R}^d\right) \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ endowed with the corresponding metric ρ_2 :

$$\rho_2\left(\sigma_1, \sigma_2\right) = \sup_{(t,x,\mu)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathcal{P}_2\left(\mathbb{R}^d\right)} \left|\sigma_1(t, x, \mu) - \sigma_2(t, x, \mu)\right|$$

It is clear that since $\mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2\left(\mathbb{R}^d\right)$ is a complete metric space, then $\mathfrak{R} = \mathcal{C}_1 \times \mathcal{C}_2$ endowed with the metric λ is a complete metric space also, where $\lambda(\left(b_1, \sigma_1\right), \left(b_2, \sigma_2\right)) = \rho\left(b_1, b_2\right) + \rho\left(\sigma_1, \sigma_2\right)$.

Remark 5.6. Note that for ordinary or Itô stochastic differential equations, the suitable Baire space is the space of bounded continous functions. The space of continuous functions contains a dense subset formed of all locally Lipschitz functions for which there is uniqueness of solutions for Itô SDEs. This property is no more valid for MVSDEs as the uniqueness of solutions may fail for locally Lipschitz coefficients (see [19]). Instead of bounded continuous functions we consider bounded uniformly continuous functions. These functions are approximated by globally Lipschitz functions for which we have existence and uniqness. The fact that the coefficients depend on the marginal distribution of the unknown process is not suitable for localization techniques.

For (b,σ) in $\mathfrak{R}$, let $E(x,b,\sigma)$ stands for MVSDE (2.1) corresponding to coefficients b,σ and initial data x.

$$\mathbf{M}^2 = \left\{ \xi : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}^d,\ F_t^B - \text{adapted, continuous with } E\left[\sup_{t\leq T} |\xi_t|^2\right] < +\infty \right\}$$

Define a metric on $\mathbf{M}^2$ by:

$$d\left(\xi_1,\xi_2\right) = \left(E \sup_{0\leq t\leq T} \left|\xi_t^1 - \xi_t^2\right|^2 \right)^{\frac{1}{2}}$$

By using Borel-Cantelli lemma, it is easy to see that $\left(\mathbf{M}^2, d\right)$ is a complete metric space.

It is clear that a strong solution (ξ_t) of equations (2.1) is an element of the metrix space $(\mathbf{M}^2, d)$.

Let $\mathcal{L}$ be the subset of $\mathfrak{R}$ consisting of functions $h(t,x,\mu)$ which are Lipschitz in their arguments, that is:

$$\begin{aligned} |b(t,x,\mu) - b(t,x',\mu')| &\leq C\left(|x-x'| + W_2(\mu,\mu')\right), \\ |\sigma(t,x,\mu) - \sigma(t,x',\mu')| &\leq C\left(|x-x'| + W_2(\mu,\mu')\right) \end{aligned}$$

Proposition 5.7. *Every bounded uniformly continuous function in a metric space is a uniform limit of a sequence of globally Lipschitz functions.*

Proof. See [10] Theorem 6.8 ∎

The last proposition states that the subset $\mathcal{L}$ of globally Lipschitz functions is dense in the Baire space $\mathcal{R}$.

5.1 The oscillation function

Let us define the oscillation function, which was first introduced by Lasota-Yorke [16] in the case of ordinary differential equations and partial differential equations and then used by [3, 11] for Itô SDEs.

Let $x \in \mathbb{R}^d$ and $(b,\sigma) \in \mathfrak{R}$, let $\xi(x,b,\sigma)$ the solution of equation $E(x,b,\sigma)$.

Define the oscillation function as follows

$$D_1(x,b,\sigma) : \mathbb{R}^d \times \mathcal{R} \longrightarrow \mathbb{R}_+$$

$$D_1(x,b,\sigma) = \lim_{\delta\to 0} \sup\left\{ d(\xi(x,b_1,\sigma_1),\xi(x,b_2,\sigma_2);\ \ (b_i,\sigma_i) \in \mathcal{L} \text{ and } \lambda((b,\sigma),(b_i,\sigma_i)) < \delta,\ i=1,2 \right\}$$

Proposition 5.8. *Let $x \in \mathbb{R}^d$ and (b,σ) are Lipschitz coefficients, that is $(b,\sigma) \in \mathcal{L}$, then* $D_1(x,b,\sigma) = 0$.

Proof.

We know that if $(b,\sigma)\in\mathcal{L}$ then equation (2.1) has a unique strong solution.

For each $i=1,2$, let $\left(X_t^i\right)$ be a solution of (2.1) coerresponding to (b_i,σ_i), then

$$\begin{aligned}
|X_t^1-X_t^2|^2 &\leq 3(\int_0^t |b_1(s,X_s^1,\mathbb{P}_{X_s^1})-b_1(s,X_s^2,\mathbb{P}_{X_s^2})|ds)^2\\
&+3(\int_0^t |b_1(s,X_s^2,\mathbb{P}_{X_s^2})-b_2(s,X_s^2,\mathbb{P}_{X_s^2}))|ds)^2\\
&+3\left|\int_0^t \left(\sigma_1(s,X_s^1,\mathbb{P}_{X_s^1})-\sigma_1(s,X_s^2,\mathbb{P}_{X_s^2})\right)dB_s\right|^2\\
&+3\left|\int_0^t \left(\sigma_1(s,X_s^2,\mathbb{P}_{X_s^2})-\sigma_2(s,X_s^2,\mathbb{P}_{X_s^2})\right)dB_s\right|^2.
\end{aligned}$$

By using the Lipschitz continuity and Burkholder Davis Gundy inequality, it holds that

$$\begin{aligned}
E\left[\sup_{t\leq T}|X_t^1-X_t^2|^2\right] &\leq 3(T+C_2)L^2\int_0^t\left(E\left[\sup_{s\leq t}|X_s^1-X_s^2|^2\right]+W_2(\mathbb{P}_{X_s^1},\mathbb{P}_{X_s^2})^2\right)ds\\
&+6(T+C_2)E[\int_0^t |b_1(s,X_s^2,\mathbb{P}_{X_s^2})-b(s,X_s^2,\mathbb{P}_{X_s^2}|^2ds]\\
&+6(T+C_2)E[\int_0^t\int_0^t |\sigma_1(s,X_s^2,\mathbb{P}_{X_s^2})-\sigma(s,X_s^2,\mathbb{P}_{X_s^2}|^2ds]\\
&\leq 6(T+C_2)\int_0^t E\left[\sup_{s\leq t}|X_s^1-X_s^2|^2\right]ds+K,
\end{aligned}$$

such that

$$K=3(T+C_2)E[\int_0^T |b_1-b|^2(s,X_s^2,\mathbb{P}_{X_s^2})+|\sigma_1-\sigma|^2(s,X_s^2,\mathbb{P}_{X_s^2})ds].$$

An application of Gronwall lemma allows us to get

$$E\left[\sup_{t\leq T}|X_t^1-X_t^2|^2\right]\leq C\delta^2.$$

where C is some constantwhcih implies that $D_1(x,b,\sigma)=0$. ■

Proposition 5.9. *The oscillation function D is upper semicontinuous function at each point of the set $\mathbb{R}^d\times\mathcal{L}$.*

Proof.

Let (x_n,b_n,σ_n) be a sequence in $\mathbb{R}^d\times\mathcal{R}$ converging to a limit $(x,b,\sigma)\in\mathbb{R}^d\times\mathcal{L}$. D is upper semicontinuous if $\lim_{n\to+\infty}D_1(x_n,b_n,\sigma_n)=0$. Suppose that the last statement is false. Then according to the definition of the function D, there exists $\varepsilon>0$ and a subsequence still denoted by $\{n\}$ (to avoid heavy notations) and functions $\left(b_n^i,\sigma_n^i\right)$ in $\mathcal{L}$ such that :

(i) $\lambda((b_n,\sigma_n),\left(b_n^i,\sigma_n^i\right))<1/2^n$

(ii) $d(\xi(x_n,b_n^1,\sigma_n^1),\xi(x_n,b_n^2,\sigma_n^2))>\varepsilon/2$

Thus according to Theorem 4.1 on the continuous dependence with respect to initial condition and coefficients, and property (i) it holds that:

$$\lim_{n\to+\infty}d(\xi(x_n,b_n^1,\sigma_n^1),\xi(x_n,b_n^2,\sigma_n^2))=0.$$

But this contradicts the property (ii), then D is upper semicontinuous. ■

Proposition 5.10. *Let (x, b, σ) be in $\mathbb{R}^d \times \mathcal{R}$ such that $D_1(x, b, \sigma) = 0$, then there exists at least one strong solution to MVSDE (2.1).*

Proof. Similar to [3] Prop. 1.4 or Proposition 5 in [11]. ■

5.2 Existence and uniqueness of solutions is a generic propery

The main result of this section is the following.

Theorem 5.11. *The subset $\mathcal{U}$ consisting of those (σ, b) for which existence and uniqueness of a strong solution holds for equation (2.1) contains a set of second category in the Baire space $\mathfrak{R}$.*

Proof. It is clear from Proposition 5.10, that if for some (σ, b) in $\mathfrak{R}$, equation (2.1) has at least one strong solution then $D_1(x, b, \sigma) = 0$. Then the set of couples (σ, b) in $\mathfrak{R}$, for which existence of strong solution holds, contains the set

$$\mathcal{A} = \{(\sigma, b) \in \mathfrak{R}; D_1(x, b, \sigma) = 0\}.$$

If we denote

$$\mathcal{A}_n = \{(\sigma, b) \in \mathfrak{R}; D_1(x, b, \sigma) < 1/n\}$$

then $\mathcal{A} = \bigcap\limits_{n=1}^{+\infty} \mathcal{A}_n$.

Let $(b, \sigma) \in \mathcal{L}$, then according to Proposition 5.8 we have $D_1(x, b, \sigma) = 0$. Therefore $\mathcal{L} \subset \mathcal{A}_n$ and then by Proposition 5.7, $\mathcal{A}_n$ contains a dense open subset of the Baire space $(\mathfrak{R}, \lambda)$. Therefore $\mathcal{A}$ contains an intersection of open dense subsets, then $\mathcal{A}$ is a residual subset in $\mathfrak{R}$.

If $(b, \sigma) \in \mathcal{A}$, equation MVSDE (2.1) enjoys the property of existence of a solution. To obtain the property of uniqueness let us introduce the finction $D_2 : \mathcal{A} \longrightarrow [0, +\infty[$ defined by

$$D_2((b, \sigma)) = \sup\left\{d(\xi_1, \xi_2); \xi_i \in \mathbf{S}^2 \text{ and } \xi_i \text{ is a strong solution of } E(x, b, \sigma)\right\}$$

and

$$\mathcal{B}_n = \{(\sigma, b) \in \mathcal{A}; D_2(x, b, \sigma) < 1/n\}$$

Let $\mathcal{B} = \bigcap\limits_{n=1}^{+\infty} \mathcal{B}_n$

Let us note that if (b, σ) are Lipschitz functions then equation (2.1) admits a unique strong solution. Therefore if $(b, \sigma) \in \mathcal{L}$, then $D_2(x, b, \sigma) = 0$. This implies in particular that $\mathcal{B}_n$ contains the intersection of $\mathcal{A}$ and a dense open subset in $\mathcal{R}$, namely $\mathcal{L}$. Therefore $\mathcal{B}$ contains an intersection of open dense subsets in the Baire space $(\mathfrak{R}, \lambda)$. This means that $\mathcal{B}$ is a residual subset in $(\mathfrak{R}, \lambda)$. ■

Remark 5.12. By using similar techniques it is not difficult to prove that the set of coefficients (b, σ) for which the Euler polygonal scheme and the Picard scheme for MVSDE converge, is a residual set in the Baire space of all bounded uniformly continuous functions $(\mathfrak{R}, \lambda)$.

References

[1] Bahlali, K, Mezerdi, M. A., Mezerdi, B., Stability of McKean-Vlasov stochastic differential equations and applications. *Stochastics and Dynamics, Vol. 2019, online version, https://doi.org/10.1142/S0219493720500070.*

[2] Bahlali, K., Mezerdi, B., Ouknine, Y., Pathwise uniqueness and approximation of stochastic differential equations. *Sém. de Probabilités, Vol. XXXII (1998), Edit. J. Azema, M.Yor, P.A Meyer, Lect. Notes in Math.1651, Springer Verlag.*

[3] Bahlali, K. , Mezerdi, B., Ouknine, Y., Some generic properties of stochastic differential equations. *Stochastics and Stoch. Reports, Vol. 57 (1996), pp. 235-245.*

[4] Bossy, M., Faugeras, O., Talay, D., Clarification and complement to "mean-field descriptionand propagation of chaos in networks of Hodgkin–Huxley and FitzHugh–Nagumo neurons", *The Journal of Mathematical Neuroscience (JMN), 5 (2015), p. 19.*

[5] Chiang, T.S., McKean-Vlasov equations with discontinuous coeffcients. *Soochow J. Math., 20(4):507{526, 1994. Dedicated to the memory of Professor Tsing-Houa Teng.*

[6] Dieudonné, J., Choix d'oeuvres mathématiques. *Hermann, Paris 1987.*

[7] Graham, C., McKean-Vlasov Itô-Skorohod equations, and nonlinear diffusions with discrete jump sets. *Stoch. Proc. Appl., 40ss(1):69–82, 1992.*

[8] Gyongy, I., The stability of stochastic partial differential equations and applications. *Stochastics and Stoch. Reports, Vol. 27 (1986), pp. 129-150.*

[9] Gyongy, I., Krylov, N.V., Existence of strong solutions for Itô's stochastic equations via approximations. *Probab. Theory Relat. Fields 105 (1996), 143-158.*

[10] Heinonen, J., Lectures on analysis on metric spaces. *Universitext, Springer Science + Business Media, New York, 2001.*

[11] Heunis, A .J., On the prevalence of stochastic differential equations with unique solutions. *The Annals of Probability, (2) (1986), pp. 653-662.*

[12] Ikeda, N., Watanabe, S., Stochastic differential equations and diffusions processes, *2nd Edition (1989), North- Holland Publishing Company, Japan.*

[13] Jourdain, B., Méléard, S., Woyczynski, W., Nonlinear SDEs driven by Lévy processes and related PDEs. *Alea 4 (2008), 1–29.*

[14] Kaneko, H., Nakao, S., A note on approximation for stochastic differential equations. *Séminaire de Probab. de Strasbourg, Vol. 22 (1988), Lect. Notes in Math. 1321, p. 155-162*

[15] Kurtz, T., Weak and strong solutions of general stochastic models. *Electron. Comm.. Probab. Volume 19 (2014), paper no. 58, 16 pp..*

[16] Lasota, A., Yorke, J. A., The generic property of existence of solutions of differential equations in Banach space. *J. Diff. Equat. 13 (1973), pp. 1-12.*

[17] McKean, H.P., A class of Markov processes associated with nonlinear parabolic equations. *Proc. Nat. Acad. Sci. U.S.A., 56:1907-1911, 1966.*

[18] Orlicz, Zur theorie der differentialgleichung $y' = f(x, y)$. *Bull. Acad. Polon. Sci. Ser. A (1932), pp. 221-228.*

[19] Scheutzow, M., Uniqueness and non-uniqueness of solutions of Vlasov-McKean equations. *J. of the Austr. Math. Soc. (Series A), 43:246–256.*

[20] Skorokhod, A. V., Studies in the theory of random processes, *Translated from the Russian by Scripta Technica, Inc. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1965.*

[21] Sznitman, A.S., Topics in propagation of chaos. *In Ecole de Probabilités de Saint Flour, XIX-1989. Lect. Notes in Math. 1464, pp. 165–251. Springer, Berlin (1989).*

[22] Vlasov, A.A., The vibrational properties of an electron gas. *Physics-Uspekhi, 10(6):721-733, 1968.*

[23] Zvonkin, A.K., Krylov, N.V., On strong solutions of stochastic differential equations. *Sel. Math. Sov. 1, 19-61 (1981).*