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Exponential stabilization of wave equation with acoustic boundary conditions and its application to memory type boundary

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Summary

In this paper, we deal with the wave equation with acoustic boundary conditions. The exponential stabilization is obtained by Lyapunov approach and Riemannian geometry method. We then apply our main theorem to the wave equations with memory type acoustic boundary conditions, which is not available in the literature and give an example in the end.

KEYWORDS:
exponential stabilization, Lyapunov approach, Riemannian geometry method, memory type acoustic boundary

1 | INTRODUCTION

Let (Ω, d) be a n-dimensional compact Riemannian manifold with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are closed and disjoint, with $\Gamma_0 \neq \emptyset$. This article is devoted to the analysis of the following wave equation

$$\begin{cases} u_{tt} - \Delta u = 0, \quad \text{in } \Omega \times (0, \infty), \\ u = 0, \quad \text{on } \Gamma_0 \times (0, \infty), \\ m(x)u_{tt} + \partial_\nu u + \gamma \partial_\nu u_t + \beta u_t = \alpha(x) y_t, \quad \text{on } \Gamma_1 \times (0, \infty), \\ u_t + p(x)y_t + q(x)y = 0, \quad \text{on } \Gamma_1 \times (0, \infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ y(x,0) = y_0(x), \quad \text{in } \Omega, \end{cases} \tag{1}$$

where Δ is the associated Laplace-Beltrami operator on manifold (Ω, d). We denote by ν the unit outward normal vector along the boundary Γ. Here the functions $p, q : \Gamma_1 \to R^+$ are essentially bounded, satisfying some conditions which will be specified later. And then we apply our main theorem to the wave equations with memory type acoustic boundary conditions as follows

$$\begin{cases} u_{tt} - \Delta u = 0, \quad \text{in } \Omega \times (0, \infty), \\ u = 0, \quad \text{on } \Gamma_0 \times (0, \infty), \\ \int_0^t g(t-s)(\partial_\nu u(s) + \gamma \partial_\nu u_t(s)) = y_t, \quad \text{on } \Gamma_1 \times (0, \infty), \\ u_t + p(x)y_t + q(x)y = 0, \quad \text{on } \Gamma_1 \times (0, \infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ y(x,0) = y_0(x), \quad \text{in } \Omega. \end{cases} \tag{2}$$

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Acoustic boundary conditions were discussed in Morse and Ingard[18] and the models of wave equations with acoustic boundary conditions were first introduced by Beale and Rosencrans[2,3,4]. In[2,3,4], they proved the global existence and regularity of solutions in a Hilbert space with semigroup methods. The asymptotic behaviour was obtained in[3] (Theorem 2.6). Recently, wave equations with acoustic boundary conditions have been studied by many authors, see[5,6,7,8,9,10,12,14,15,16,17,20,23,22,24].

Compared with previous articles on this subject, the main difference of this article is related with the equations $(1)_3$ and $(2)_3$. In most previous work, for instance,[9,14,15,16,20,22,24], the memory term of $\partial_\nu u_t$ has not been considered. In[15], they discussed the equation including the memory term for $\partial_\nu u_t$ on Γ_0, and also with the viscoelastic term Δu_t in the domain Ω. Liu and Sun[16] studied the following problem:

$$\begin{cases} u_{tt} - \Delta u + \alpha(t)\int_0^t g(t-s)\Delta u(s)ds = 0, \quad \text{in } \Omega\times(0,\infty), \\ u = 0, \quad \text{on } \Gamma_0\times(0,\infty), \\ \partial_\nu u - \alpha(t)\int_0^t g(t-s)\partial_\nu u(s)ds = y_t, \quad \text{on } \Gamma_1\times(0,\infty), \\ u_t + p(x)y_t + q(x)y = 0, \quad \text{on } \Gamma_1\times(0,\infty), \\ u(x,0) = u_0(x),\ u'(x,0) = u_1(x),\ y(x,0) = y_0(x), \quad \text{in } \Omega. \end{cases}$$

With the perturbed energy functional technique, they established a general decay result relying on the behavior of both $\alpha(t)$ and $g(t)$. Motivated by[9,15,16], we intend to study the energy decay rate of (1) with Lyapunov approach in the framework of Riemannian geometry. Then as an application, we get the exponential stabilization of the wave equations with memory type acoustic boundary conditions on Γ_1 and no damping terms in the domain Ω.

The paper is organized as follows. In section 2, we give some assumptions, notations and the main results. In section 3, we drive some important lemmas and give the proof of the theorems. Section 4 is devoted to the applications and examples.

2 | PRELIMINARIES AND MAIN RESULTS

To state the results, we begin with the conditions on the functions coefficients of the system.

Assumption (A) There are some basic assumptions about the functions in the system:

(A1) The coefficients in the equation $(1)_3$ satisfy

$$m(x) > 0,\ \ \gamma > 0,\ \ \beta > 0,\ \ m(x) - \beta\gamma > 0,\ \ \text{for } x \in \Gamma_1. \tag{3}$$

(A2) The positive functions $p, q \in C(\Gamma_1)$ are essentially bounded such that

$$p(x) \ge p_0 > \max_{x\in\Gamma_1}\left\{\frac{1}{\beta} + \frac{m(x)\alpha^2(x)}{\beta(m(x)-\beta\gamma)}\right\},\ \ q(x) \ge q_0 > 0,\ \ for\ a.e.\ x \in \Gamma_1. \tag{4}$$

For the Riemannian manifold (Ω, d), we assume that

Geometrical assumption (G) Given the triple $\{\Omega, \Gamma_0, \Gamma_1\}$, there exists a vector field H on Riemannian manifold (Ω, d) such that the following properties hold true:

(G1) $DH(\cdot,\cdot)$ is strictly positive definite on $\overline{\Omega}$, that is, there exists a $W^{1,\infty}(\overline{\Omega})$ function $h(x) \ge \rho$ such that for some $\rho > 0$ and all $x \in \overline{\Omega}$, for all $X \in M_x$(the tangent space at x):

$$DH(X,X) \equiv \langle D_X H, X\rangle = h(x)|X|^2 \ge \rho|X|^2. \tag{5}$$

(G2) The boundary Γ_1 satisfies

$$\Gamma_1 = \{x \in \Gamma \big| \langle H, \nu\rangle > 0\}, \tag{6}$$

and we have $\langle H, \nu\rangle \leq 0$ on Γ_0.

Remark 1. The equality (5) in (G1) implies that $divH = nh(x)$ *on* $\overline{\Omega}$.

Throughout this paper, we use the notation

$$V = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\},$$

which is a Hilbert space endowed with the inner product

$$(u, v)_V = \int_\Omega \langle \nabla u(x), \nabla v(x)\rangle dx\,. \tag{7}$$

We set

$$\eta(x,t) = m(x)u_t(x,t) + \gamma \partial_\nu u, \quad x \in \Gamma_1. \tag{8}$$

We consider the unknown

$$U = \left(u, v = u_t|_\Omega, \eta, y\right)^T,$$

in the state space, denoted by

$$\mathcal{H} = V \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1)\,, \tag{9}$$

with the norm defined by

$$\begin{aligned} \|U\|^2_{\mathcal{H}} &= \|(u, v, \eta, y)^T\|^2_{\mathcal{H}} \\ &= \int_\Omega \left(|\nabla u|^2 + v^2\right) dx + \frac{1}{2}\int_{\Gamma_1} \frac{1}{m(x) - \beta\gamma}\eta^2 d\Gamma + \frac{1}{2}\int_{\Gamma_1} q(x) y^2 d\Gamma. \end{aligned} \tag{10}$$

A simple computation yields

$$\begin{aligned} \eta_t &= m(x)u_{tt} + \gamma \partial_{\nu_A} u_t \\ &= -\partial_{\nu_A} u - \beta u_t + \alpha(x) y_t \\ &= -\frac{\eta - m(x)u_t}{\gamma} - \beta u_t + \alpha(x) y_t \\ &= -\frac{1}{\gamma}\eta + \left(\frac{m(x)}{\gamma} - \beta - \frac{\alpha(x)}{p(x)}\right)u_t - \frac{\alpha(x)q(x)}{p(x)} y. \end{aligned} \tag{11}$$

Thus system (1) can be rewritten in the abstract form

$$\begin{cases} U' = \mathcal{A}U, \\ U_0 = \left(u_0, u_1, \eta_0, y_0\right)^T, \end{cases} \tag{12}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \eta \\ y \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \\ -\frac{1}{\gamma}\eta + \left(\frac{m(x)}{\gamma} - \beta - \frac{\alpha(x)}{p(x)}\right)v(x,t) - \frac{\alpha(x)q(x)}{p(x)} y \\ -\frac{1}{p(x)} v - \frac{q(x)}{p(x)} y \end{pmatrix}$$

with domain

$$D(\mathcal{A}) := \left\{(u, v, \eta, y)^T \in \mathcal{H} : \Delta u \in L^2(\Omega), \eta = m(x)v|_{\Gamma_1} + \gamma \partial_\nu u\right\}. \tag{13}$$

According to the norm defined in (11) in the state space $\mathcal{H}$, we give the associated energy of system (1) by

$$E(t) = \frac{1}{2}\int_{\Omega}(u_t^2 + |\nabla u|^2)dx + \frac{1}{2}\int_{\Gamma_1}\frac{1}{m(x)-\beta\gamma}\eta^2 d\Gamma + \frac{1}{2}\int_{\Gamma_1} q(x)y^2 d\Gamma. \tag{14}$$

The wellposedness of (12) can be obtained by using linear semigroup theory in [19]. These are our main results.

Theorem 1. Assume that the assumptions (A) and (G) hold, for each initial data $(u_0, u_1, \eta_0, y_0) \in \mathcal{H}$, there exists a unique solution (u, u_t, η, y) to system (1) in the class

$$(u, u_t, \eta, y) \in C(0, \infty; \mathcal{H}),$$

furthermore, for $(u_0, u_1, \eta_0, y_0) \in V \cap H^2(\Omega) \times V \times L^2(\Gamma_1) \times L^2(\Gamma_1)$, system (1) has a strong solution in the class

$$(u, u_t, \eta, y) \in C(0, \infty; V \cap H^2(\Omega) \times V \times L^2(\Gamma_1) \times L^2(\Gamma_1)) \cap C^1(0, \infty; \mathcal{H}).$$

Now we show the decreasing of the energy functional $E(t)$ given in (14). Noticing (8), (11) and $(1)_4$, a simple computation yields

$$\begin{aligned}
E'(t) = {} & \int_{\Omega}(u_t u_{tt} + \nabla u \cdot \nabla u_t)dx + \int_{\Gamma_1}\frac{1}{(m(x)-\beta\gamma)}\eta\eta_t d\Gamma + \int_{\Gamma_1} q(x)yy_t d\Gamma \\
= {} & \int_{\Gamma_1} u_t\partial_\nu u d\Gamma - \int_{\Gamma_1}\frac{1}{\gamma(m(x)-\beta\gamma)}\eta^2 d\Gamma + \frac{1}{\gamma}\int_{\Gamma_1}\eta u_t d\Gamma \\
& + \int_{\Gamma_1}\frac{\alpha(x)}{(m(x)-\beta\gamma)}\eta y_t d\Gamma - \int_{\Gamma_1} p(x)y_t^2 d\Gamma - \int_{\Gamma_1} u_t y_t d\Gamma \\
= {} & \int_{\Gamma_1} u_t\partial_\nu u d\Gamma - \int_{\Gamma_1}\frac{1}{2m(x)\gamma}\eta^2 d\Gamma \\
& - \int_{\Gamma_1}\left(\frac{1}{\gamma(m(x)-\beta\gamma)} - \frac{1}{2m(x)\gamma}\right)\eta^2 d\Gamma + \frac{1}{\gamma}\int_{\Gamma_1}\eta u_t d\Gamma \\
& + \int_{\Gamma_1}\frac{\alpha(x)}{(m(x)-\beta\gamma)}\eta y_t d\Gamma - \int_{\Gamma_1} p(x)y_t^2 d\Gamma - \int_{\Gamma_1} u_t y_t d\Gamma \\
= {} & -\int_{\Gamma_1}\left(\frac{m(x)}{2\gamma}u_t^2 + \frac{\gamma}{2m(x)}\partial_\nu^2 u\right)d\Gamma \\
& - \int_{\Gamma_1}\left(\frac{1}{\gamma(m(x)-\beta\gamma)} - \frac{1}{2m(x)\gamma}\right)\eta^2 d\Gamma + \frac{1}{\gamma}\int_{\Gamma_1}\eta u_t d\Gamma \\
& + \int_{\Gamma_1}\frac{\alpha(x)}{(m(x)-\beta\gamma)}\eta y_t d\Gamma - \int_{\Gamma_1} p(x)y_t^2 d\Gamma - \int_{\Gamma_1} u_t y_t d\Gamma.
\end{aligned} \tag{15}$$

Now by Cauchy's inequality we have

$$\frac{1}{\gamma}\int_{\Gamma_1}\eta u_t d\Gamma \le \int_{\Gamma_1}\frac{1}{2\gamma(m(x)-\beta\gamma)}\eta^2 d\Gamma + \int_{\Gamma_1}\frac{m(x)-\beta\gamma}{2\gamma}u_t^2 d\Gamma, \tag{16}$$

$$\begin{aligned}
& \int_{\Gamma_1}\frac{\alpha(x)}{(m(x)-\beta\gamma)}\eta y_t d\Gamma \\
\le {} & \int_{\Gamma_1}\frac{\beta}{4m(x)(m(x)-\beta\gamma)}\eta^2 d\Gamma + \int_{\Gamma_1}\frac{m(x)\alpha^2(x)}{\beta(m(x)-\beta\gamma)}y_t^2 d\Gamma,
\end{aligned} \tag{17}$$

and

$$-\int_{\Gamma_1} u_t y_t d\Gamma \le \frac{\beta}{4}\int_{\Gamma_1} u_t^2 d\Gamma + \frac{1}{\beta}\int_{\Gamma_1} y_t^2 d\Gamma. \tag{18}$$

Substituting the inequality (16)-(18) to (15) yields

$$\begin{aligned}
E'(t) \le & -\int_{\Gamma_1} (\frac{m(x)}{2\gamma} - \frac{m(x)-\beta\gamma}{2\gamma} - \frac{\beta}{4})u_t^2 d\Gamma - \int_{\Gamma_1} \frac{\gamma}{2m(x)}\partial_\nu^2 u d\Gamma \\
& -\int_{\Gamma_1} (\frac{1}{\gamma(m(x)-\beta\gamma)} - \frac{1}{2m(x)\gamma} - \frac{1}{2\gamma(m(x)-\beta\gamma)})\eta^2 d\Gamma \\
& +\int_{\Gamma_1} \frac{\beta}{4m(x)(m(x)-\beta\gamma)}\eta^2 d\Gamma \\
& -\int_{\Gamma_1} (p(x) - \frac{m(x)\alpha^2(x)}{\beta(m(x)-\beta\gamma)} - \frac{1}{\beta})y_t^2 d\Gamma \\
= & -\int_{\Gamma_1} \frac{\beta}{4}u_t^2 d\Gamma - \int_{\Gamma_1} \frac{\gamma}{2m(x)}\partial_\nu^2 u d\Gamma - \int_{\Gamma_1} \frac{\beta}{4m(x)(m(x)-\beta\gamma)}\eta^2 d\Gamma \\
& -\int_{\Gamma_1} (p(x) - \frac{m(x)\alpha^2(x)}{\beta(m(x)-\beta\gamma)} - \frac{1}{\beta})y_t^2 d\Gamma.
\end{aligned} \tag{19}$$

Moreover we have

Theorem 2. Let the assumptions (A) and (G) hold. The energy of the system (1) defined in (14) satisfies that

$$E(t) \le Ce^{-\omega t},$$

for $t \ge 0$ and some positive constants C and ω.

3 | EXPONENTIAL STABILIZATION OF SYSTEM (1)

This section is devoted to the proof of Theorem 2.

Next, we construct a Lyapunov functional which is equivalent to E(t).

Define

$$\begin{aligned}
\Psi(t) = \ & 2\int_{\Omega} u_t H(u) dx + (n-1)\int_{\Omega} h(x) u_t u dx \\
& + \frac{\rho}{2}\int_{\Gamma_1} uy d\Gamma + \frac{\rho}{4}\int_{\Gamma_1} p(x) y^2 d\Gamma,
\end{aligned} \tag{20}$$

where $h(x)$ and ρ are given in (5).

Lemma 1. Assume that the assumption (A) holds. Let (u, u_t, η, y) be the solution to (1), then there exists a constant $C_1 > 0$ such that

$$|\Psi(t)| \le C_1 E(t). \tag{21}$$

Proof.

$$|\Psi(t)| \leq \; 2\int_{\Omega} |u_t||H||\nabla u|dx + (n-1)\|h\|_{L^\infty(\Omega)}\int_{\Omega} |u_t||u|dx$$
$$+\frac{\rho}{2}\int_{\Gamma_1} |u||y|d\Gamma + \frac{\rho}{4}\int_{\Gamma_1} p(x)y^2 d\Gamma.$$

Using Cauchy inequality, Poincaré inequality, Hölder inequality and trace embedding theorem in [1], we get

$$2\int_{\Omega} |u_t||H||\nabla u|dx \leq \int_{\Omega} u_t^2 dx + M^2\int_{\Omega} |\nabla u|^2 dx, \tag{22}$$

where $M = max_{\overline{\Omega}}|H|$,

$$\int_{\Omega} |u_t||u|dx \leq \frac{1}{2}\int_{\Omega} u_t^2 dx + \frac{1}{2}c_1\int_{\Omega} |\nabla u|^2 dx, \tag{23}$$

$$\begin{aligned}\int_{\Gamma_1} |u||y|d\Gamma \; &\leq \Big(\int_{\Gamma_1} \frac{q^2}{q_0^2}y^2 dx\Big)^{\frac{1}{2}}\Big(\int_{\Gamma_1} u^2 d\Gamma\Big)^{\frac{1}{2}} \\ &\leq \frac{\|q\|_{L^\infty}(\Gamma_1)}{2q_0^2}\int_{\Gamma_1} q(x)y^2 d\Gamma + c_2\int_{\Omega} |\nabla u|^2 dx,\end{aligned} \tag{24}$$

where c_1, c_2 are constants given in the Poincaré inequality and trace embedding theorem, respectively. Due to the assumptions on $p(x)$ and $q(x)$ we have

$$\frac{\rho}{4}\int_{\Gamma_1} p(x)y^2 d\Gamma \leq \frac{\rho\|p\|_{L^\infty(\Gamma_1)}}{4q_0}\int_{\Gamma_1} q(x)y^2 d\Gamma. \tag{25}$$

Combining (22)-(25) yields that

$$|\Psi(t)| \leq C_1 E(t),$$

where $C_1 = C_1(n, \rho, M, c_1, c_2, \|p\|_{L^\infty(\Gamma_1)}, q_0, \|q\|_{L^\infty(\Gamma_1)}, \|h\|_{L^\infty(\Omega))}$ is a positive constant.

□

Lemma 2. Assume that the assumptions (A) and (G) hold. Let (u, u_t, η, y) be the solution to (1), then the functional $\Psi(t)$ defined by (20) satisfies

$$\begin{aligned}\Psi'(t) \leq& -\frac{\rho}{2}\int_{\Omega} \big(|\nabla u|^2 + u_t^2\big)dx + C\int_{\Omega} u^2 dx + C\int_{\Gamma_1} \partial_\nu^2 u d\Gamma + \int_{\Gamma_1} |\nabla_\tau u|^2 d\Gamma \\ &+M\int_{\Gamma_1} u_t^2 d\Gamma + C\int_{\Gamma_1} y_t^2 d\Gamma - \frac{\rho}{2}\int_{\Gamma_1} q(x)y^2 d\Gamma,\end{aligned} \tag{26}$$

where ρ is the constant given in (5) and $M = max_{\overline{\Omega}}|H|$.

Proof. Due to the definition of Ψ in (20) we have

$$\begin{aligned}
\Psi'(t) = &\ 2\int_\Omega u_{tt}H(u)dx + 2\int_\Omega u_t H(u_t)dx + (n-1)\int_\Omega h(x)u_{tt}u dx \\
&+(n-1)\int_\Omega h(x)u_t^2 dx + \frac{\rho}{2}\int_{\Gamma_1} uy_t d\Gamma \\
&+\frac{\rho}{2}\int_{\Gamma_1} u_t y d\Gamma + \frac{\rho}{2}\int_{\Gamma_1} p(x)y_t y d\Gamma \\
= &\ \{2\int_\Omega \Delta u H(u)dx + (n-1)\int_\Omega h(x)u\Delta u dx\} \\
&+\{2\int_\Omega u_t H(u_t)dx + (n-1)\int_\Omega h(x)u_t^2 dx\} \\
&+\{\frac{\rho}{2}\int_{\Gamma_1} uy_t d\Gamma - \frac{\rho}{2}\int_{\Gamma_1} q(x)y^2 d\Gamma\} \\
\triangleq: &\ I_1(t) + I_2(t) + B(t).
\end{aligned} \tag{27}$$

Here we estimate the iterms of (27). By[25] ,we have

$$\langle \nabla u, \nabla (H(u))\rangle = DH\langle \nabla u, \nabla u\rangle + \frac{1}{2}div(|\nabla u|^2 H) - \frac{1}{2}|\nabla u|^2 divH,$$

then

$$\begin{aligned}
I_1(t) = &\ 2\int_\Omega \Delta u H(u)dx + (n-1)\int_\Omega h(x)u\Delta u dx \\
= &\ 2\int_\Omega div(\nabla u H(u))dx - 2\int_\Omega \langle \nabla u, \nabla(H(u))\rangle dx \\
&+(n-1)\int_\Omega div(h(x)u\nabla u)dx - (n-1)\int_\Omega h(x)|\nabla u|^2 dx \\
&-(n-1)\int_\Omega u\nabla h(x)\cdot\nabla u dx \\
= &\ 2\int_\Gamma \partial_\nu u H(u)d\Gamma - 2\int_\Omega DH\langle \nabla u, \nabla u\rangle dx - \int_\Omega div(|\nabla u|^2 H)dx \\
&+\int_\Omega |\nabla u|^2 divH dx + (n-1)\int_{\Gamma_1} h(x)u\partial_\nu u d\Gamma \\
&-(n-1)\int_\Omega h(x)|\nabla u|^2 dx - (n-1)\int_\Omega u\nabla h(x)\cdot\nabla u dx \\
= &\ \int_\Gamma 2\partial_\nu u H(u)d\Gamma - \int_\Gamma |\nabla u|^2\langle H, \nu\rangle d\Gamma + (n-1)\int_{\Gamma_1} h(x)u\partial_\nu u d\Gamma \\
&-\int_\Omega h(x)|\nabla u|^2 dx - (n-1)\int_\Omega u\nabla h(x)\nabla u dx.
\end{aligned} \tag{28}$$

Denote by

$$I_{1b}(t) = \int_\Gamma 2\partial_\nu u H(u)d\Gamma - \int_\Gamma |\nabla u|^2\langle H, \nu\rangle d\Gamma + (n-1)\int_{\Gamma_1} h(x)u\partial_\nu u d\Gamma.$$

As $u = 0$ on Γ_0, we have on Γ_0 that $\nabla u = \partial_\nu u \cdot \nu$, $|\nabla u|^2 = \partial_\nu^2 u$ and $H(u) = \partial_\nu u\langle H, \nu\rangle$. Meanwhile on Γ_1 we have $\nabla u = \partial_\nu u \cdot \nu + \nabla_\tau u$, $|\nabla u|^2 = \partial_\nu^2 u + |\nabla_\tau u|^2$ and $H(u) = \partial_\nu u\langle H, \nu\rangle + \langle H, \nabla_\tau u\rangle$. Thus we have

$$\begin{aligned}
I_{1b}(t) = & \int_{\Gamma_0} 2\partial_\nu^2 u\langle H, \nu\rangle d\Gamma + \int_{\Gamma_1} 2\partial_\nu^2 u\langle H, \nu\rangle d\Gamma + \int_{\Gamma_1} 2\partial_\nu u\langle H, \nabla_\tau u\rangle d\Gamma \\
& - \int_{\Gamma_0} \partial_\nu^2 u\langle H, \nu\rangle d\Gamma - \int_{\Gamma_1} \left(\partial_\nu^2 u + |\nabla_\tau u|^2\right)\langle H, \nu\rangle d\Gamma \\
& +(n-1)\int_{\Gamma_1} h(x)u\partial_\nu u d\Gamma \\
= & \int_{\Gamma_0} \partial_\nu^2 u\langle H, \nu\rangle d\Gamma + \int_{\Gamma_1} \partial_\nu^2 u\langle H, \nu\rangle d\Gamma - \int_{\Gamma_1} |\nabla_\tau u|^2\langle H, \nu\rangle d\Gamma \\
& + \int_{\Gamma_1} 2\partial_\nu u\langle H, \nabla_\tau u\rangle d\Gamma + (n-1)\int_{\Gamma_1} h(x)u\partial_\nu u d\Gamma \\
\leq & \; 0 + M\int_{\Gamma_1} \partial_\nu^2 u d\Gamma + 0 + M^2\int_{\Gamma_1} \partial_\nu^2 u d\Gamma + \int_{\Gamma_1} |\nabla_\tau u|^2 d\Gamma \\
& +\epsilon\int_\Omega |\nabla u|^2 dx + C_\epsilon\int_{\Gamma_1} \partial_\nu^2 u d\Gamma \\
= & \; (M + M^2 + C_\epsilon)\int_{\Gamma_1} \partial_\nu^2 u d\Gamma + \int_{\Gamma_1} |\nabla_\tau u|^2 d\Gamma + \epsilon\int_\Omega |\nabla u|^2 dx,
\end{aligned} \tag{29}$$

where we notice the assumptions (5) and (6). Here $M = max_{\overline{\Omega}}|H|$ and ϵ is a constant small enough. And it's easy to know

$$(n-1)\int_\Omega u\nabla h(x)\nabla u dx \leq \epsilon\int_\Omega |\nabla u|^2 dx + C_\epsilon\int_\Omega u^2 dx \tag{30}$$

Combining (29), (30) with (28) to get

$$\begin{aligned}
I_1(t) \leq & -\int_\Omega \left(h(x) - 2\epsilon\right)|\nabla u|^2 dx + C_\epsilon\int_{\Gamma_1} \partial_\nu^2 u d\Gamma \\
& + \int_{\Gamma_1} |\nabla_\tau u|^2 d\Gamma + C_\epsilon\int_\Omega u^2 dx.
\end{aligned} \tag{31}$$

By divergence theorem we have,

$$\begin{aligned}
I_2(t) = & \; 2\int_\Omega u_t H(u_t) dx + (n-1)\int_\Omega h(x)u_t^2 dx \\
= & \int_{\Gamma_1} u_t^2\langle H, \nu\rangle d\Gamma - \int_\Omega \left(divH - (n-1)h(x)\right)u_t^2 dx \\
\leq & \; M\int_{\Gamma_1} u_t^2 d\Gamma - \int_\Omega h(x)u_t^2 dx.
\end{aligned} \tag{32}$$

Using Hölder inequality, Young's inequality with ϵ and trace embedding theorem,

$$\begin{aligned} B(t) = {}& \frac{\rho}{2}\int_{\Gamma_1} u y_t d\Gamma - \frac{\rho}{2}\int_{\Gamma_1} q(x) y^2 d\Gamma \\ \leq {}& \frac{\rho}{2}\Big(\int_{\Gamma_1} u^2 d\Gamma\Big)^{\frac{1}{2}}\Big(\int_{\Gamma_1} y_t^2 d\Gamma\Big)^{\frac{1}{2}} - \frac{\rho}{2}\int_{\Gamma_1} q(x) y^2 d\Gamma \\ \leq {}& \epsilon \int_{\Omega} |\nabla u|^2 dx + C_\epsilon \int_{\Gamma_1} y_t^2 d\Gamma - \frac{\rho}{2}\int_{\Gamma_1} q(x) y^2 d\Gamma. \end{aligned} \tag{33}$$

Substituting (31)-(33) to (27) yields that

$$\begin{aligned} \Psi'(t) \leq {}& -\int_{\Omega} \big(h(x) - 3\epsilon\big)|\nabla u|^2 dx - \int_{\Omega} h(x) u_t^2 dx + C_\epsilon \int_{\Omega} u^2 dx \\ & + C_\epsilon \int_{\Gamma_1} \partial_\nu^2 u d\Gamma + \int_{\Gamma_1} |\nabla_\tau u|^2 d\Gamma + M \int_{\Gamma_1} u_t^2 d\Gamma \\ & + C_\epsilon \int_{\Gamma_1} y_t^2 d\Gamma - \frac{\rho}{2}\int_{\Gamma_1} q(x) y^2 d\Gamma. \end{aligned} \tag{34}$$

Then taking $\epsilon = \frac{\rho}{6}$ in (34) completes the proof. □

Obviously we obtain

Lemma 3. Assume that the assumptions (A) and (G) hold. Let (u, u_t, η, y) be the solution to (1), we have

$$\begin{aligned} \rho E(t) + \Psi'(t) \leq {}& \frac{\rho}{2}\int_{\Gamma_1} \frac{1}{(m(x) - \beta\gamma)} \eta^2 d\Gamma + M \int_{\Gamma_1} u_t^2 d\Gamma + C \int_{\Gamma_1} y_t^2 d\Gamma \\ & + C \int_{\Gamma_1} \partial_\nu^2 u d\Gamma + \int_{\Gamma_1} |\nabla_\tau u|^2 d\Gamma + C \int_{\Omega} u^2 dx. \end{aligned} \tag{35}$$

Lemma 4. [11] Lemma 7.2. Let $\varepsilon > 0$ be given small. Let u solves the problem (1). Then

$$\begin{aligned} & \int_{\varepsilon}^{T-\varepsilon} \int_{\Gamma_1} |\nabla_\tau u|^2 d\Gamma dt \\ \leq {}& C_{T,\varepsilon}\Big\{ \int_0^T \int_{\Gamma_1} \big(\partial_\nu^2 u + u_t^2\big) d\Gamma dt + \|u\|_{H^{\frac{1}{2}+\varepsilon}(\Omega\times(0,T))} \Big\}. \end{aligned} \tag{36}$$

According to Lemma 3 and Lemma 4, we obtain the following observability inequality of the system (1).

Lemma 5. Assume that the assumptions (A) and (G) hold. Let (u, u_t, η, y) be the solution to (1), then for any $\varepsilon > 0$, there exists $T_0 > 0$ and a positive constant $C_{T,\varepsilon,\rho}$ such that, for all $T > T_0$,

$$E(0) \leq C_{T,\varepsilon,\rho}\Big\{ \int_0^T \int_{\Gamma_1} (u_t^2 + \partial_\nu^2 u + \eta^2 + y_t^2) d\Gamma dt + \|u\|_{H^{\frac{1}{2}+\varepsilon}(\Omega\times(0,T))} \Big\}. \tag{37}$$

Proof. For any ε small enough, integrating the inequality (35) on the interval $(\varepsilon, T-\varepsilon)$ yields

$$\begin{aligned}
&\rho \int_{\varepsilon}^{T-\varepsilon} E(t)dt + \Psi(T-\varepsilon) - \Psi(\varepsilon) \\
&\leq \frac{\rho}{2} \int_{\varepsilon}^{T-\varepsilon} \int_{\Gamma_1} \frac{1}{m(x)-\beta\gamma} \eta^2 d\Gamma + C \int_{\varepsilon}^{T-\varepsilon} \int_{\Gamma_1} (u_t^2 + \partial_\nu^2 u + y_t^2) d\Gamma \\
&\quad + \int_{\varepsilon}^{T-\varepsilon} \int_{\Gamma_1} |\nabla_\tau u|^2 d\Gamma + C \int_{\Omega} u^2 dx. \\
&\leq C_{T,\varepsilon,\rho} \Big\{ \int_0^T \int_{\Gamma_1} (u_t^2 + \partial_\nu^2 u + \eta^2 + y_t^2) d\Gamma + \|u\|_{H^{1/2+\epsilon}(\Omega\times(0,T))} \Big\} dx,
\end{aligned}$$

which yields

$$\begin{aligned}
\int_{\varepsilon}^{T-\varepsilon} E(t)dt \leq C \Big\{ \int_0^T \int_{\Gamma_1} (u_t^2 + \partial_\nu^2 u + \eta^2 + y_t^2) d\Gamma + \|u\|_{H^{1/2+\epsilon}(\Omega\times(0,T))} \Big\} dx \\
+ C_1 \big(E(T-\varepsilon) + E(\varepsilon) \big),
\end{aligned} \tag{38}$$

where we notice the inequality (21) and the constant C_1 is given there. On the other hand, from inequalities (15)-(18), we have

$$-E'(t) \leq C_2 \int_{\Gamma_1} (u_t^2 + \partial_\nu^2 u + \eta^2 + y_t^2) d\Gamma. \tag{39}$$

Using the above inequality (39) to obtain

$$\begin{aligned}
&E(0) + C_1 \big(E(T-\varepsilon) + E(\varepsilon) \big) \\
&= \int_{\varepsilon}^{2C_1+\varepsilon+1} E(t)dt + \int_{\varepsilon}^{2C_1+\varepsilon+1} \big(E(0) - E(t) \big) dt \\
&\quad + C_1 \big(E(\varepsilon) - E(0) \big) + C_1 \big(E(T-\varepsilon) - E(0) \big) \\
&= \int_{\varepsilon}^{2C_1+\varepsilon+1} E(t)dt - \int_{\varepsilon}^{2C_1+\varepsilon+1} \Big(\int_0^t E'(\tau) d\tau \Big) dt \\
&\quad + C_1 \int_0^{\varepsilon} E'(\tau) d\tau + C_1 \int_0^{T-\varepsilon} E'(\tau) d\tau \\
&\leq (2C_1+1) C_2 \int_0^{\max\{T-\varepsilon, 2C_1+\varepsilon+1\}} \int_{\Gamma_1} \big(u_t^2 + \partial_\nu^2 u + \eta^2 + y_t^2 \big) d\Gamma dt \\
&\quad + \int_{\varepsilon}^{2C_1+\varepsilon+1} E(t)dt \\
&\leq (2C_1+1) C_2 \int_0^{T-\varepsilon} \int_{\Gamma_1} \big(u_t^2 + \partial_\nu^2 u + \eta^2 + y_t^2 \big) d\Gamma dt \\
&\quad + \int_{\varepsilon}^{T-\varepsilon} E(t)dt,
\end{aligned} \tag{40}$$

where in the last step we take $T_0 = 2C_1 + 2\varepsilon + 1$ to guarantee that $T - \varepsilon > 2C_1 + \varepsilon + 1$, for all $T > T_0$. □

In the following, we use the compactness uniqueness argument to absorb the lower order term in (37). We list the lemma and omit the proof, which is similar to[21].

Lemma 6. Let (u, u_t, η, y) be the solution to (1), then for $T > T_0$ large enough, there exists a positive constant $C > 0$ such that

$$\|u\|_{H^{1/2+\varepsilon}(\Omega\times(0,T))} \le C\Big\{ \int_0^T \int_{\Gamma_1} (u_t^2 + \partial_\nu^2 u + \eta^2 + y_t^2) d\Gamma dt \Big\}. \tag{41}$$

Combining Lemma 5 and Lemma 6 yields

Lemma 7. Assume that the assumptions (A) and (G) hold. Let (u, u_t, η, y) be the solution to (1), then for any $T > T_0$, there exists a positive constant C depending on T, ε, ρ such that

$$E(0) \le C \int_0^T \int_{\Gamma_1} (u_t^2 + \partial_\nu^2 u + \eta^2 + y_t^2) d\Gamma dt. \tag{42}$$

Proof of Theorem 2. From (19) we know that

$$\begin{aligned} E'(t) \le & -\int_{\Gamma_1} \frac{\beta}{4} u_t^2 d\Gamma - \int_{\Gamma_1} \frac{\gamma}{2m(x)} \partial_\nu^2 u d\Gamma - \int_{\Gamma_1} \frac{\beta}{4m(x)(m(x)-\beta\gamma)} \eta^2 d\Gamma \\ & - \int_{\Gamma_1} (p(x) - \frac{m(x)\alpha^2}{\beta(m(x)-\beta\gamma)} - \frac{1}{\beta}) y_t^2 d\Gamma \\ \le & -c_3 \int_{\Gamma_1} (u_t^2 + \partial_\nu^2 u + \eta^2 + y_t^2) d\Gamma, \end{aligned} \tag{43}$$

where

$$c_3 = \min_{x\in\Gamma_1} \Big\{ \frac{\beta}{4}, \frac{\gamma}{2m(x)}, \frac{\beta}{4m(x)(m(x)-\beta\gamma)}, p_0 - \frac{1}{\beta} - \frac{m(x)\alpha^2(x)}{\beta(m(x)-\beta\gamma)} \Big\} > 0.$$

Substituting (43) to (42) yields that for all $T > T_0$,

$$\begin{aligned} E(0) \le \;& C \int_0^T \int_{\Gamma_1} (u_t^2 + \partial_\nu^2 u + \eta^2 + y_t^2) d\Gamma dt \\ \le \;& -\frac{C}{c_3} \int_0^T E'(t) = -\frac{C}{c_3}(E(T) - E(0)), \end{aligned}$$

which yields

$$E(T) \le \frac{C - c_3}{C} E(0).$$

The exponential decay result follows from the above inequality. □

4 | APPLICATIONS AND THE EXAMPLE

In this section we give an application of Theorem 2 to the wave equation with memory type acoustic boundary conditions, see system (2).

First, let's rewrite the equation $(2)_3$. Denote by

$$(g * v)(t) = \int_0^t g(t-s)v(s)ds.$$

Differentiating $(2)_3$ yields the following Volterra equation

$$\partial_\nu u + \gamma \partial_\nu u_t = \frac{1}{g(0)} y_{tt} - \frac{g'}{g(0)} * (\partial_\nu u + \gamma \partial_\nu u_t).$$

Then from Theorem 3.5 in [13], we get

$$\partial_\nu u + \gamma \partial_\nu u_t = \frac{1}{g(0)}(y_{tt} + R * y_{tt}), \tag{44}$$

where the resolvent kernel $R(t)$ is the solution of

$$R(t) = -\frac{g'}{g(0)} - \frac{g'}{g(0)} * R. \tag{45}$$

Now we differentiate $(2)_4$ to get

$$y_{tt} = -\frac{1}{p(x)} u_{tt} - \frac{q(x)}{p(x)} y_t, \tag{46}$$

where we notice the positivity of function $p(x)$. Moreover, we have

$$\begin{aligned}
R * y_{tt} &= \int_0^t R(t-s) y_{tt}(s) ds \\
&= R(t-s) y_t(s) \mid_0^t + \int_0^t R'(t-s) y_t(s) ds \\
&= R(0) y_t(t) - R(t) y_t(0) + R'(t-s) y(s) \mid_0^t + \int_0^t R''(t-s) y(s) ds \\
&= R(0) y_t(t) + R'(0) y(t) + R'' * y - R'(t) y(0) - R(t) y_t(0).
\end{aligned} \tag{47}$$

Substituting (46), (47) to (44) and combining with $(2)_4$ yield

$$\begin{aligned}
&\frac{1}{g(0)p(x)} u_{tt} + \partial_\nu u + \gamma \partial_\nu u_t \\
= \;& \frac{1}{g(0)} \Big\{ \big(-\frac{q(x)}{p(x)} + R(0) - \beta g(0) p(x) \big) y_t + \beta g(0) p(x) y_t \Big\} \\
&+ \frac{1}{g(0)} \Big\{ R'(0) y + R'' * y - \big(R'(t) y(0) + R(t) y_t(0) \big) \Big\} \\
= \;& \frac{1}{g(0)} \Big\{ \big(-\frac{q(x)}{p(x)} + R(0) - \beta g(0) p(x) \big) y_t - \beta g(0) u_t \Big\} \\
&+ \frac{1}{g(0)} \Big\{ \big(R'(0) - \beta g(0) q(x) \big) y + R'' * y - \big(R'(t) y(0) + R(t) y_t(0) \big) \Big\},
\end{aligned}$$

that is,

$$\begin{aligned}
&m(x) u_{tt} + \beta u_t + \partial_\nu u + \gamma \partial_\nu u_t - \alpha(x) y_t \\
= \;& \frac{1}{g(0)} \Big\{ \big(-\frac{q(x)}{p(x)} - \beta g(0) p(x) \big) y_t \Big\} \\
&+ \frac{1}{g(0)} \Big\{ \big(R'(0) - \beta g(0) q(x) \big) y + R'' * y - \big(R'(t) y(0) + R(t) y_t(0) \big) \Big\},
\end{aligned} \tag{48}$$

where we denote by

$$m(x) := \frac{1}{g(0)p(x)}, \quad \alpha(x) := \frac{R(0)}{g(0)}, \tag{49}$$

and β is a positive number which could be determined as needed.

Assumption (R) We assume the functions p, q, g and coefficients α, β satisfy that

$$\begin{aligned}&\Big\{-\frac{q(x)}{p(x)} - \beta g(0)p(x)\Big\}y_t \\ &+\big(R'(0) - \beta g(0)q(x)\big)y + R'' * y - \big(R'(t)y(0) + R(t)y_t(0)\big) = 0.\end{aligned} \tag{50}$$

Thus we get

$$m(x)u_{tt} + \partial_\nu u + \gamma\partial_\nu u_t + \beta u_t = \alpha(x)y_t, \tag{51}$$

and we know that (50) always has a solution. Due to Theorem 2, we have

Theorem 3. Let the assumptions (A), (G) and (R) hold. The energy of the system (2) defined in (14) satisfies that

$$E(t) \le Ce^{-\omega t},$$

for $t \ge 0$ and some positive constants C and ω.

Let's give an example to end this section.

Example 4.1. Let $(\Omega, d) = (\Omega, \delta)$. δ is the standard metric on R^n. Given $g(t) = e^{-kt}$ $(\frac{\sqrt{3}(1-\gamma)}{4\sqrt{\gamma}} < k < \frac{(1-\gamma)}{2\sqrt{\gamma}})$, $0 < \gamma < 1$. The functions $q(x) > 0$, $\frac{1+\gamma-\sqrt{(1-\gamma)^2-4\gamma k^2}}{2\gamma\beta} < p(x) < \frac{1+\gamma+\sqrt{(1-\gamma)^2-4\gamma k^2}}{2\gamma\beta}$. In this case, the system (2) has the exponentially stabilization.

Proof. Now we verify the assumptions of Theorem 3. In (Ω, δ), Δ is the Laplacian Operator and the geometrical assumption (G) holds with the vector field $H = x - x_0$. Given $g(t) = e^{-kt}$, (45) becomes

$$R(t) = ke^{-kt} + ke^{-kt} * R(t),$$

and by the laplace transform and its inverse transform, we have

$$R(t) = k,$$

and

$$R'(t) = 0, \quad R(0) = k, \quad R''(t) = 0,$$

therefore, from (49) we get $m(x) = \frac{1}{p(x)}$ and $\alpha(x) = k$.

Assumption (R) becomes

$$\Big(\frac{q(x)}{p(x)} + \beta p(x)\Big)y_t + \beta q(x)y + ky_t(0) = 0, \tag{52}$$

which is a one order linear ODE with the solution

$$y(t) = -\frac{\beta p^2(x) + q(x)}{\beta p(x)q(x)}y_t(0)e^{-\frac{\beta p(x)q(x)}{\beta p^2(x)+q(x)}t} - \frac{ky_t(0)}{\beta q(x)}, \tag{53}$$

where we supposed that $y_t(0) \neq 0$.

It's easy to see that $p(x) < \frac{1+\gamma+\sqrt{(1-\gamma)^2-4\gamma k^2}}{2\gamma\beta} < \frac{1+\gamma+(1-\gamma)}{2\gamma\beta} = \frac{1}{\beta\gamma}$, which corresponds to $m(x) = \frac{1}{p(x)} > \gamma\beta$ in (A1). The condition $\frac{1+\gamma-\sqrt{(1-\gamma)^2-4\gamma k^2}}{2\gamma\beta} < p(x) < \frac{1+\gamma+\sqrt{(1-\gamma)^2-4\gamma k^2}}{2\gamma\beta}$ guarantees that

$$\gamma\beta^2 p^2(x) - \beta(1+\gamma)p + 1 + k^2 < 0, \tag{54}$$

which is

$$\beta p(x)(1-\beta\gamma p(x)) > (1-\beta\gamma p(x)) + k^2,$$

which verifies (A2) when we notice $m(x) = \frac{1}{p(x)}$ and $\alpha(x) = k$. Thus the stabilization property holds true with applying Theorem 2. □

Remark 2. As an application we can also consider (2) with the fourth equation $(2)_4$ changed as

$$u_t + l(x)y_{tt} + p(x)y_t + q(x)y = 0, \qquad \text{on } \Gamma_1 \times (0,\infty), \tag{55}$$

with $l(x) > 0$ and an adding initial value $y_t(0,x) = y_1(x)$ to $(1)_5$.

Here the corresponding energy functional can be defined as

$$\begin{aligned} E_l(t) = \ & \frac{1}{2}\int_\Omega (u_t^2 + |\nabla u|^2)dx + \int_{\Gamma_1} \frac{1}{2(m(x)-\beta\gamma)}\eta^2 d\Gamma \\ & + \frac{1}{2}\int_{\Gamma_1} q(x)y^2 d\Gamma + \frac{1}{2}\int_{\Gamma_1} l(x)y_t^2 d\Gamma. \end{aligned} \tag{56}$$

Similarly we can get the exponential decay of the energy E_l. This kind of boundary condition as in (55) has been recently considered in the reference [17], where they imposed different boundary condition on y_t and got the polynomial decay.

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