A study of fractional HIV-1 infection of $CD4^+$ T–cells of immunodeficiency syndrome with the effect of antiviral drug therapy through non-singular derivative

Mostafa M.A. Khater[1,2]*, Sunil Kumar[3]†, Raghda A. M. Attia[1,4], and Abdel–Haleem Abdel–Aty[5,6]
[1] Department of Mathematics, Faculty of Science, Jiangsu University, China.
[2] Department of Mathematics, Obour Institutes, Cairo, Egypt.
[3] Department of Mathematics, National Institute of Technology, Jamshedpur 831014,India.
[4] Department of Basic Science, Higher technological institute 10^{th} of Ramadan city, Egypt.
[5] Department of Physics, College of Sciences, University of Bisha, Bisha 61922, P.O. Box 344, Saudi Arabia.
[6] Physics Department, Faculty of Science, Al–Azhar University, 71524 Assiut, Egypt.

February 23, 2020

Abstract: In this research paper, the HIV-1 infection of $CD4^+$ T–cells fractional mathematical model with the effect of antiviral drug therapy is handled by applying three new computational schemes to this biological model to investigate its analytical explicit wave solutions. This mathematical model is used to predict the evolution of the population dynamical systems involving virus particles. The modified Khater method, the extended simplest equation method, and sech–tanh method with a new fractional operator (Atangana–Baleanu derivative operator) is employing to find the analytical solutions in various distinct new formulas of the biological suggested model. Moreover, the stability of the obtained solutions is investigated by using the characterizes of the Hamiltonian system to show their applicability in making the antivirals that protect our human life. Some plots are explained under specific conditions of the contained constants to reveal the dynamical behavior of the evolution of the population dynamical systems involving virus particles. A comparison between our results and that obtained in previous work is also represented and discussed in detail to show the novelty for our solutions. The performance of the used methods shows power, practical, and ability to apply to other nonlinear partial differential equations.

Keywords: Human immunodeficiency virus (HIV)-1 infection of $CD4^+$ T–cells fractional mathematical model; Antiviral drug therapy; Modified Khater method; The extended simplest equation method; Sech–tanh method; Stability property.

AMS classification: 35Q92; 37N25; 35C07; 35C08; 35B35.

1 Introduction

Nowadays, virus study is one of the original icons in the recent survey, especially after discovering many new diseases. Some dynamical behavior of these viruses has been being formulated in nonlinear partial differential equations. These equations are classified as bio–mathematical models that are considered as a primary tool for more explanation of these viruses. These models have been formulated in mathematical structure based on the collecting data from biological experiments or statistics to allow the precise studying

*Corresponding Email: mostafa.khater2024@yahoo.com
†Corresponding Email: skumar.math@nitjsr.ac.in & skiitbhu28@gmail.com

and investigation that are usually used in the construction of these natural phenomena in isolation by using modern experimental biology. Solving these formulas gives accurate solutions that are used to improve the point of view for these models and also to control them by handling their parameters.

Based on the importance of these derived biological models, many computational and numerical schemes have been being discovered to study the analytical and approximate solutions of these models to show more physical properties of them. For more explanations of some recent analytical and numerical schemes, you can see [3, 9, 10, 11, 16, 18, 19, 22, 23, 28, 29, 30, 31, 32, 33, 35, 38]. The investigation process of the accuracy for the obtained solutions is given in the following order:

1. Solving the biological models with analytical schemes.
2. Obtaining the traveling wave solutions.
3. Calculating the initial and boundary conditions.
4. Investigating the approximate solutions of these bio–model.
5. Evaluating the absolute value of error between the exact and numerical.

In this paper, we study the analytical solutions of bio–mathematical model that is used to predict the evolution of dynamical population systems involving virus particles [39]. This model includes some basic biological models, such as a four–component HIV-1 dynamical model, which provides for both latently and actively infected cells [13], the general two-component antibody-viral system [20], an HIV-1 three-component model including virions, and the simplified two–component model for HIV-1 dynamics [8]. The human immunodeficiency virus causes acquired immunodeficiency syndrome (AIDS) and also infects, damages, and reduces $CD4^+$ T–cells. The body becomes more gradually sensitive to infections and loses its safety. AIDS is one of the most important and dangerous diseases in our time. The mathematical formula of the relation between HIV viruses and uninfected $CD4^+$ cells and the effect of drug therapy on infected cells is given by the following fractional system [24, 27, 41]

$$\begin{cases} \mathcal{D}_t^{\Theta}\mathcal{U} = p_1 - p_2\,\mathcal{U} - p_3\,\mathcal{U}\,\mathcal{V}, \\ \mathcal{D}_t^{\Theta}\mathcal{S} = p_3\,\mathcal{U}\,\mathcal{V} - p_4\,\mathcal{S}, \\ \mathcal{D}_t^{\Theta}\mathcal{V} = p_5\,\mathcal{S} - p_6\,\mathcal{V}, \end{cases} \tag{1}$$

where $\left[0 < \Theta < 1\right]$ and $\left[p_i, i = 1, ..., 6\right]$ are respectively, represent the rate of creation or production of $CD4^+$ T–cells, the natural death rate, the rate of infected $CD4^+$ cells from uninfected $CD4^+$ cells, the rate of virus producing cell's death, the rate of creation of virions viruses by infected cells, and the rate of virus particle death. This system studies the nonlocal property of these models since it depends on both of historical and current states of the problem in the contract of the classical calculus which depends on the current state only. Based on this importance of this kind of calculus, many definitions have been being derived such as conformable fractional derivative, fractional Riemann–Liouville derivatives, Caputo, Caputo–Fabrizio definition, and so on [1, 4, 6, 7, 25, 43]. Applying the next definition of $\mathcal{ABR}$ fractional operator [12, 36] to Eq. (1)

Definition 1.1. It is given by

$$^{\mathcal{ABR}}\mathcal{D}_{a+}^{\Theta}\,\mathcal{F}(t) = \frac{\mathcal{B}(\Theta)}{1-\Theta}\,\frac{d}{dt}\int_a^t \mathcal{F}(x)\,\mathcal{G}_{\Theta}\left(\frac{-\Theta\,(t-\Theta)^{\Theta}}{1-\Theta}\right)dx, \tag{2}$$

where $\mathcal{G}_\Theta$ is the Mittag–Leffler function, and define by the following formula

$$\mathcal{G}_\Theta\left(\frac{-\Theta\,(t-\Theta)^\Theta}{1-\Theta}\right)=\sum_{n=0}^{\infty}\frac{\left(\frac{-\Theta}{1-\Theta}\right)^n(t-x)^{\Theta\,n}}{\Gamma(\Theta\,n+1)} \tag{3}$$

and $\mathcal{B}(\Theta)$ being a normalisation function. Thus

$$^{\mathcal{ABR}}\mathcal{D}^{\Theta}_{a+}\,\mathcal{F}(x)=\frac{\mathcal{B}(\Theta)}{1-\ \Theta}\sum_{n=0}^{\infty}\left(\frac{-\Theta}{1-\Theta}\right)^n\,{}^{RL}\mathcal{I}_a^{\Theta\,n}\,\mathcal{F}(x), \tag{4}$$

leads to

$$\begin{cases}\mathcal{U}(t)=\dfrac{c\,(1-\Theta)\,t^{-\,\Theta\,n}}{B(\Theta)\sum_{n=0}^{\infty}\left(-\frac{\Theta}{1-\Theta}\right)^n\Gamma(1-\Theta n)},\\[2ex] \mathcal{S}(t)=\dfrac{c\,(1-\Theta)\,t^{-\,\Theta\,n}}{B(\Theta)\sum_{n=0}^{\infty}\left(-\frac{\Theta}{1-\Theta}\right)^n\Gamma(1-\Theta n)},\\[2ex] \mathcal{V}(t)=\dfrac{c\,(1-\Theta)\,t^{-\,\Theta\,n}}{B(\Theta)\sum_{n=0}^{\infty}\left(-\frac{\Theta}{1-\Theta}\right)^n\Gamma(1-\Theta n)},\end{cases}$$

where c are arbitrary constants.

Applying the above transformation to the above system (1), yields

$$\begin{cases}c\,\mathcal{U}'=p_1-p_2\,\mathcal{U}-p_3\,\mathcal{U}\,\mathcal{V},\\[1ex] c\,\mathcal{S}'=p_3\,\mathcal{U}\,\mathcal{V}-p_4\,\mathcal{S},\\[1ex] c\,\mathcal{V}'=p_5\,\mathcal{S}-p_6\,\mathcal{V},\end{cases} \tag{5}$$

Substituting third and second equation of the system (5) into the first equation of the same equation, yields.

$$\begin{aligned}&c^2\Big(c\mathcal{V}\,\mathcal{V}^{(3)}-c\mathcal{V}'\,\mathcal{V}''+p_4\Big(\mathcal{V}\,\mathcal{V}''-\mathcal{V}'^{\,2}\Big)+p_6\Big(\mathcal{V}\mathcal{V}''-\mathcal{V}'^{\,2}\Big)\Big)+p_3\mathcal{V}^{\,2}\Big(p_4\Big(c\mathcal{V}'+p_6\mathcal{V}\Big)+c\Big(c\mathcal{V}''\\&+p_6\mathcal{V}'\Big)\Big)+p_2\mathcal{V}\left(p_4\Big(c\mathcal{V}'+p_6\mathcal{V}\Big)+c\Big(c\mathcal{V}''+p_6\mathcal{V}'\Big)\right)-p_1p_3p_5\mathcal{V}^{\,2}=0.\end{aligned} \tag{6}$$

By fixing the value of death rate of virus to be equal zero, Eq. (6) transforms to be in the following formula [24, 40]

$$\begin{aligned}&p_3\mathcal{V}^{\,2}\Big(c^2\mathcal{V}''+cp_4\mathcal{V}'\Big)+p_2\mathcal{V}\left(c^2\mathcal{V}''+cp_4\mathcal{V}'\right)+c^2\Big(c\mathcal{V}\,\mathcal{V}^{(3)}-c\mathcal{V}'\,\mathcal{V}''+p_4\Big(\mathcal{V}\,\mathcal{V}''-\mathcal{V}'^{\,2}\Big)\Big)\\&-p_1p_3p_5\mathcal{V}^{\,2}=0.\end{aligned} \tag{7}$$

Determine the value of balance between the terms of Eq. (7), yield $\Big[n=1\Big]$. Using the modified Khater method, the extended simplest equation method, and sech–tanh expansion method give the general solutions of Eq. (7) in the following formula

$$\mathcal{V}=\begin{cases}a_1k^{f(t)}+a_0+b_1k^{-f(t)},\\[1ex] a_0+a_1\,f(t),\\[1ex] a_1\mathrm{sech}(t)+a_0+b_1\tanh(t).\end{cases} \tag{8}$$

With the following respective auxiliary equations:

$$f'(t) = \begin{cases} \frac{1}{\ln(K)} \left[\delta K^{f(t)} + \varrho K^{-f(t)} + \chi\right], \\ \\ \alpha + \lambda f(t) + \mu f(t)^2. \end{cases} \tag{9}$$

The rest sections of this paper is ordered as following. Section 2 applies the modified Khater method [2, 14, 15, 17, 26], the extended simplest equation method [5, 21, 42], and sech–tanh expansion method [34, 37] to the human immunodeficiency virus (HIV)-1 infection of $CD4^+$ T–cells fractional mathematical model to study the solutions of it in various explicit solutions. Moreover, some plots are represented to show more physical properties of the physical behavior of the HIV-1 infection of $CD4^+$ T–cells. Section 3 studies the stability property of the obtained solutions to their applicability in using in different studies. Section 4 shows the novelty of our obtained solutions by representing a comparison between them and that obtained in previous research papers. Section 5 explains the conclusion of the whole paper.

2 Application

This section applies three different analytical schemes to the HIV-1 infection of $CD4^+$ T–cells fractional mathematical model with the effect of antiviral drug therapy. Substituting Eq. (8) along (9) into Eq. (7) and collecting all term with the same power of $\left[f^i(t),\, K^{i\,f(t)},\, i = -1, 0, 1\right]$, yield a system of algebraic equations. Solving This system by Mathematica 11.3, gives

2.1 Modified Khater method

Family I

$$\Bigg[a_0 \to \frac{\sqrt{4\delta\varrho - \chi^2}\sqrt{p_3^2 (p_4 - p_6)^2 (\chi^2 - 4\delta\varrho)^2} - i p_3 (p_4 - p_6)\chi(\chi^2 - 4\delta\varrho)}{p_3^2 (4\delta\varrho - \chi^2)^{3/2}}, a_1 \to \frac{2i\delta (p_4 - p_6)}{p_3\sqrt{4\delta\varrho - \chi^2}}, b_1 \to 0,$$

$$c \to -\frac{i (p_4 - p_6)}{\sqrt{4\delta\varrho - \chi^2}}, p_1 \to \frac{p_4 p_6 \left(3\sqrt{p_3^2 (p_4 - p_6)^2 (\chi^2 - 4\delta\varrho)^2} + p_3 (p_4 + p_6)(4\delta\varrho - \chi^2)\right)}{2 p_3^2 p_5 (4\delta\varrho - \chi^2)},$$

$$p_2 \to \frac{1}{2}\left(-\frac{\sqrt{p_3^2 (p_4 - p_6)^2 (\chi^2 - 4\delta\varrho)^2}}{p_3 (4\delta\varrho - \chi^2)} + p_4 + p_6\right), \Bigg]$$

where $\left[4\delta\varrho - \chi^2 < 0\right]$. Thus, the solutions of Eq. (1) are given by

For $\left[\chi^2 - 4\delta\varrho > 0 \,\&\, \delta \neq 0\right]$

$$\mathcal{V}_1(t) = \frac{\sqrt{4\delta\varrho - \chi^2}\sqrt{p_3^2 (p_4 - p_6)^2 (\chi^2 - 4\delta\varrho)^2} + i p_3 (p_4 - p_6)(\chi^2 - 4\delta\varrho)^{3/2} \tanh\left(\frac{1}{2}\mho\sqrt{\chi^2 - 4\delta\varrho}\right)}{p_3^2 (4\delta\varrho - \chi^2)^{3/2}}, \tag{10}$$

$$\mathcal{V}_2(t) = \frac{\sqrt{4\delta\varrho - \chi^2}\sqrt{p_3^2 (p_4 - p_6)^2 (\chi^2 - 4\delta\varrho)^2} + i p_3 (p_4 - p_6)(\chi^2 - 4\delta\varrho)^{3/2} \coth\left(\frac{1}{2}\mho\sqrt{\chi^2 - 4\delta\varrho}\right)}{p_3^2 (4\delta\varrho - \chi^2)^{3/2}}. \tag{11}$$

For $\left[\chi^2-4\delta\varrho>0\,\&\,\delta\neq 0\right]$

$$\mathcal{V}_3(t)=\frac{\sqrt{\delta^2p_3^2\left(p_4-p_6\right)^2\varrho^2}-ip_3\left(p_4-p_6\right)\sqrt{-\delta^2\varrho^2}\tanh\left(\mho\sqrt{-\delta\varrho}\right)}{\delta p_3^2\varrho}, \tag{12}$$

$$\mathcal{V}_4(t)=\frac{\sqrt{\delta^2p_3^2\left(p_4-p_6\right)^2\varrho^2}-ip_3\left(p_4-p_6\right)\sqrt{-\delta^2\varrho^2}\coth\left(\mho\sqrt{-\delta\varrho}\right)}{\delta p_3^2\varrho}. \tag{13}$$

For $\left[\chi=0\,\&\,\varrho=-\delta\right]$

$$\mathcal{V}_5(t)=\frac{-\sqrt{p_3^2\left(p_4-p_6\right)^2\varrho^4}+ip_3\left(p_4-p_6\right)\varrho\sqrt{-\varrho^2}\coth(\varrho\mho)}{p_3^2\varrho^2}. \tag{14}$$

For $\left[\chi=\delta=\kappa\,\&\,\varrho=0\right]$

$$\mathcal{V}_6(t)=-\frac{\frac{\sqrt{\kappa^4p_3^2(p_4-p_6)^2}}{\kappa^2}+\frac{i\kappa p_3(p_4-p_6)\coth\left(\frac{\kappa\mho}{2}\right)}{\sqrt{-\kappa^2}}}{p_3^2}. \tag{15}$$

For $\left[\varrho=0\&\chi\neq 0\,\&\,\delta\neq 0\right]$

$$\mathcal{V}_7(t)=-\frac{\frac{\sqrt{p_3^2(p_4-p_6)^2\chi^4}}{\chi^2}+\frac{ip_3(p_4-p_6)\chi\left(\delta e^{\chi\mho}+2\right)}{\sqrt{-\chi^2}\left(\delta e^{\chi\mho}-2\right)}}{p_3^2}. \tag{16}$$

Family II

$$\begin{aligned}&\Bigg[a_0\rightarrow\frac{\sqrt{4\delta\varrho-\chi^2}\sqrt{p_3^2\left(p_4-p_6\right)^2\left(\chi^2-4\delta\varrho\right)^2}+ip_3\left(p_4-p_6\right)\chi\left(\chi^2-4\delta\varrho\right)}{p_3^2\left(4\delta\varrho-\chi^2\right)^{3/2}},a_1\rightarrow 0,b_1\rightarrow-\frac{2i\left(p_4-p_6\right)\varrho}{p_3\sqrt{4\delta\varrho-\chi^2}},\\&c\rightarrow-\frac{i\left(p_4-p_6\right)}{\sqrt{4\delta\varrho-\chi^2}},p_1\rightarrow\frac{p_4p_6\left(3\sqrt{p_3^2\left(p_4-p_6\right)^2\left(\chi^2-4\delta\varrho\right)^2}+p_3\left(p_4+p_6\right)\left(4\delta\varrho-\chi^2\right)\right)}{2p_3^2p_5\left(4\delta\varrho-\chi^2\right)},\\&p_2\rightarrow\frac{1}{2}\left(-\frac{\sqrt{p_3^2\left(p_4-p_6\right)^2\left(\chi^2-4\delta\varrho\right)^2}}{p_3\left(4\delta\varrho-\chi^2\right)}+p_4+p_6\right),\Bigg]\end{aligned} \tag{17}$$

For $\left[\chi^2-4\delta\varrho>0\,\&\,\delta\neq 0\right]$

$$\begin{aligned}\mathcal{V}_8(t)=&\frac{1}{p_3^2\left(4\delta\varrho-\chi^2\right)^{3/2}\left(\sqrt{\chi^2-4\delta\varrho}\tanh\left(\frac{1}{2}\mho\sqrt{\chi^2-4\delta\varrho}\right)+\chi\right)}\Bigg[\sqrt{p_3^2\left(p_4-p_6\right)^2\left(\chi^2-4\delta\varrho\right)^2}\\&\times\left(\sqrt{-\left(\chi^2-4\delta\varrho\right)^2}\tanh\left(\frac{1}{2}\mho\sqrt{\chi^2-4\delta\varrho}\right)+\chi\sqrt{4\delta\varrho-\chi^2}\right)+ip_3\left(p_4-p_6\right)\left(4\delta\varrho-\chi^2\right)\\&\times\left(-\chi\sqrt{\chi^2-4\delta\varrho}\tanh\left(\frac{1}{2}\mho\sqrt{\chi^2-4\delta\varrho}\right)+4\delta\varrho-\chi^2\right)\Bigg],\end{aligned} \tag{18}$$

$$\mathcal{V}_9(t) = \frac{1}{p_3^2\left(4\delta\varrho-\chi^2\right)^{3/2}\left(\sqrt{\chi^2-4\delta\varrho}\coth\left(\frac{1}{2}\mho\sqrt{\chi^2-4\delta\varrho}\right)+\chi\right)}\left[\sqrt{p_3^2\left(p_4-p_6\right)^2\left(\chi^2-4\delta\varrho\right)^2}\right.$$
$$\times\left(\sqrt{-\left(\chi^2-4\delta\varrho\right)^2}\coth\left(\frac{1}{2}\mho\sqrt{\chi^2-4\delta\varrho}\right)+\chi\sqrt{4\delta\varrho-\chi^2}\right)+ip_3\left(p_4-p_6\right)\left(4\delta\varrho-\chi^2\right) \tag{19}$$
$$\left.\times\left(-\chi\sqrt{\chi^2-4\delta\varrho}\coth\left(\frac{1}{2}\mho\sqrt{\chi^2-4\delta\varrho}\right)+4\delta\varrho-\chi^2\right)\right],$$

For $\left[\chi^2-4\delta\varrho>0\,\&\,\delta\neq0\right]$

$$\mathcal{V}_{10}(t) = \frac{\sqrt{\delta^2p_3^2\left(p_4-p_6\right)^2\varrho^2}-ip_3\left(p_4-p_6\right)\sqrt{-\delta^2\varrho^2}\coth\left(\mho\sqrt{-\delta\varrho}\right)}{\delta p_3^2\varrho}, \tag{20}$$

$$\mathcal{V}_{11}(t) = \frac{\sqrt{\delta^2p_3^2\left(p_4-p_6\right)^2\varrho^2}-ip_3\left(p_4-p_6\right)\sqrt{-\delta^2\varrho^2}\tanh\left(\mho\sqrt{-\delta\varrho}\right)}{\delta p_3^2\varrho}. \tag{21}$$

For $\left[\chi=0\,\&\,\varrho=-\delta\right]$

$$\mathcal{V}_{12}(t) = -\frac{\frac{\sqrt{p_3^2(p_4-p_6)^2\varrho^4}}{\varrho^2}+\frac{ip_3(p_4-p_6)\varrho\tanh(\varrho\mho)}{\sqrt{-\varrho^2}}}{p_3^2}. \tag{22}$$

For $\left[\chi=\frac{\varrho}{2}=\kappa\,\&\,\delta=0\right]$

$$\mathcal{V}_{13}(t) = -\frac{\frac{\sqrt{\kappa^4p_3^2(p_4-p_6)^2}}{\kappa^2}+\frac{i\kappa p_3(p_4-p_6)\left(e^{\kappa\mho}+2\right)}{\sqrt{-\kappa^2}\left(e^{\kappa\mho}-2\right)}}{p_3^2}. \tag{23}$$

For $\left[\delta=0\,\&\,\chi\neq0\,\&\,\varrho\neq0\right]$

$$\mathcal{V}_{14}(t) = \frac{-\frac{\sqrt{p_3^2(p_4-p_6)^2\chi^4}}{\chi^2}+\frac{ip_3(p_4-p_6)\chi\left(\chi e^{\chi\mho}+\varrho\right)}{\sqrt{-\chi^2}\left(\varrho-\chi e^{\chi\mho}\right)}}{p_3^2}. \tag{24}$$

Where $\left[\mho=\frac{i(p_4-p_6)(\Theta-1)t^{-m\Theta}}{B(\Theta)\sqrt{4\delta\varrho-\chi^2}\sum_{m=0}^{\infty}\left(-\frac{\Theta}{1-\Theta}\right)^m\Gamma(1-m\Theta)}\right]$

2.2 Extended simplest equation method

Family I

$$\left[a_1\to0,p_1\to0,p_2\to\frac{c\left(a_{-1}\lambda-2\alpha a_0\right)}{a_{-1}},p_3\to\frac{2\alpha c}{a_{-1}},p_4\to\frac{c\left(a_{-1}\lambda-2\alpha a_0\right)}{a_{-1}}.\right]$$

Family II

$$\left[a_1\to0,p_2\to\frac{c\left(a_{-1}\lambda-2\alpha a_0\right)}{a_{-1}},p_3\to\frac{2\alpha c}{a_{-1}},p_4\to\frac{c\left(a_{-1}\lambda-2\alpha a_0\right)}{a_{-1}},p_5\to0.\right]$$

Family III

$$\left[a_1\to0,\mu\to\frac{a_0\left(a_{-1}\lambda-\alpha a_0\right)}{a_{-1}^2},p_2\to0,p_3\to\frac{2\alpha c}{a_{-1}},p_4\to-\frac{c\left(a_{-1}\lambda-2\alpha a_0\right)}{a_{-1}},p_5\to0.\right]$$

Family IV

$$\left[a_1 \to 0, \mu \to \frac{a_0\left(a_{-1}\lambda - \alpha a_0\right)}{a_{-1}^2}, p_1 \to 0, p_2 \to \frac{c\left(a_{-1}\lambda - 2\alpha a_0\right)}{a_{-1}}, p_3 \to \frac{2\alpha c}{a_{-1}}, p_4 \to \frac{c\left(a_{-1}\lambda - 2\alpha a_0\right)}{a_{-1}}.\right]$$

Thus, the solutions of all these families are formulated in the following formulas
Case one $\lambda = 0$ For $\alpha\mu > 0$

$$\mathcal{V}_{15}(t) = \frac{a_{-1}\sqrt{\alpha\mu}\cot\left(\sqrt{\alpha\mu}\left(\Theta - \frac{c(\Theta-1)t^{-m\Theta}}{B(\Theta)\sum_{m=0}^{\infty}\left(-\frac{\Theta}{1-\Theta}\right)^m\Gamma(1-m\Theta)}\right)\right)}{\alpha} + a_0, \tag{25}$$

$$\mathcal{V}_{16}(t) = \frac{a_{-1}\sqrt{\alpha\mu}\tan\left(\sqrt{\alpha\mu}\left(\Theta - \frac{c(\Theta-1)t^{-m\Theta}}{B(\Theta)\sum_{m=0}^{\infty}\left(-\frac{\Theta}{1-\Theta}\right)^m\Gamma(1-m\Theta)}\right)\right)}{\alpha} + a_0, \tag{26}$$

For $\alpha\mu < 0$

$$\mathcal{V}_{17}(t) = a_0 - \frac{a_{-1}\sqrt{-\alpha\mu}\coth\left(-\frac{c(\Theta-1)\sqrt{-\alpha\mu}t^{-m\Theta}}{B(\Theta)\sum_{m=0}^{\infty}\left(-\frac{\Theta}{1-\Theta}\right)^m\Gamma(1-m\Theta)} \mp \frac{\log(\Theta)}{2}\right)}{\alpha}, \tag{27}$$

$$\mathcal{V}_{18}(t) = a_0 - \frac{a_{-1}\sqrt{-\alpha\mu}\tanh\left(-\frac{c(\Theta-1)\sqrt{-\alpha\mu}t^{-m\Theta}}{B(\Theta)\sum_{m=0}^{\infty}\left(-\frac{\Theta}{1-\Theta}\right)^m\Gamma(1-m\Theta)} \mp \frac{\log(\Theta)}{2}\right)}{\alpha}, \tag{28}$$

Case two $4\alpha\mu > \lambda^2$

$$\mathcal{V}_{19}(t) = a_0 - \frac{2a_{-1}\mu}{\lambda - \sqrt{4\alpha\mu - \lambda^2}\tan\left(\frac{1}{2}\sqrt{4\alpha\mu - \lambda^2}\left(\Theta - \frac{c(\Theta-1)t^{-m\Theta}}{B(\Theta)\sum_{m=0}^{\infty}\left(-\frac{\Theta}{1-\Theta}\right)^m\Gamma(1-m\Theta)}\right)\right)}, \tag{29}$$

$$\mathcal{V}_{20}(t) = a_0 - \frac{2a_{-1}\mu}{\lambda - \sqrt{4\alpha\mu - \lambda^2}\cot\left(\frac{1}{2}\sqrt{4\alpha\mu - \lambda^2}\left(\Theta - \frac{c(\Theta-1)t^{-m\Theta}}{B(\Theta)\sum_{m=0}^{\infty}\left(-\frac{\Theta}{1-\Theta}\right)^m\Gamma(1-m\Theta)}\right)\right)}, \tag{30}$$

2.3 Sech–tanh expansion method

Family I

$$\left[a_1 \to 0, p_1 \to 0, p_2 \to -\frac{2a_0c}{b_1}, p_3 \to \frac{2c}{b_1}, p_4 \to -\frac{2a_0c}{b_1}.\right]$$

Family II

$$\left[a_1 \to 0, p_2 \to -\frac{2a_0c}{b_1}, p_3 \to \frac{2c}{b_1}, p_4 \to -\frac{2a_0c}{b_1}, p_5 \to 0.\right]$$

Thus, the solutions of the above two families are formulated in the following formulas

$$\mathcal{V}_{21}(t) = a_0 - b_1\tanh\left(\frac{(\Theta-1)ct^{-\Theta}}{B(\Theta)\sum_{n=0}^{\infty}\left(-\frac{\Theta}{1-\Theta}\right)^n\Gamma(1-\Theta n)}\right). \tag{31}$$

Family III

$$\left[a_1 \to -ib_1, p_1 \to 0, p_2 \to -\frac{a_0c}{b_1}, p_3 \to \frac{c}{b_1}, p_4 \to -\frac{a_0c}{b_1}.\right]$$

Family IV

$$\left[a_1 \to -ib_1, p_2 \to -\frac{a_0 c}{b_1}, p_3 \to \frac{c}{b_1}, p_4 \to -\frac{a_0 c}{b_1}, p_5 \to 0.\right]$$

Thus, the solutions of the above two families are formulated in the following formulas

$$\begin{aligned}\mathcal{V}_{22}(t) =& a_0 + b_1\Bigg(- \tanh\left(\frac{(\Theta-1)ct^{-\Theta}}{B(\Theta)\sum_{n=0}^{\infty}\left(-\frac{\Theta}{1-\Theta}\right)^n\Gamma(1-\Theta n)}\right)\\ &- i\,\mathrm{sech}\left(\frac{(\Theta-1)ct^{-\Theta}}{B(\Theta)\sum_{n=0}^{\infty}\left(-\frac{\Theta}{1-\Theta}\right)^n\Gamma(1-\Theta n)}\right)\Bigg).\end{aligned} \tag{32}$$

3 Stability

This section of our research paper investigates one of the basic properties of any model. It examines the stability property for the HIV-1 infection of CD4$^+$ T–cells fractional mathematical model with the effect of antiviral drug therapy by using a Hamiltonian system. The momentum in the Hamiltonian system given by the following formula:

$$\mathcal{M} = \frac{1}{2}\int_{-\ni}^{\ni} \mathcal{V}^2(t)\,d\,t, \tag{33}$$

where $\ni$ is arbitrary constant. Thus, the condition for stability is given in the next condition:

$$\left.\frac{\partial \mathcal{M}}{\partial c}\right|_{c=\flat} > 0. \tag{34}$$

where c, $\flat$ are arbitrary constants.
For an example of studying the stability of the solution of Eq. (1) by using (31) with the following values of the constants $\left[a_0 = 3,\ b_1 = 4\right]$, yields:

$$\mathcal{M} = \frac{1}{c}\left[32\left(\sqrt{\pi}-1\right)\tanh\left(\frac{5c}{2-2\sqrt{\pi}}\right)\right] + 125 \tag{35}$$

Thus, we obtain

$$\left.\frac{\partial \mathcal{M}}{\partial c}\right|_{c=5} = 0.98874092915848 > 0. \tag{36}$$

This means, this solution is stable, and by applying the same steps to other obtained solutions, the stability property of each one of them can be determined.

4 Results and discussion

this section, we show and explain our obtained solutions and the novelty of our paper in the following steps:

- Solving the biological model (the HIV-1 infection of CD4$^+$ T–cells fractional mathematical model with the effect of antiviral drug therapy) give more explanations of the dynamical behavior of viruses.
- Solving the bio–mathematical model by applying three analytical schemes gives more distinct types of solutions.
- All obtained solutions are new and different from that obtained in previous research work specially all previous work is numerical study of this biological model but this paper gives a computational investigation of this model for the first time.

5 Conclusion

In this paper, we applied three analytical schemes (the modified Khater method, the extended simplest equation method, and the sech-tanh expansion method) to the HIV-1 infection of $CD4^+$ T–cells fractional mathematical model with the effect of antiviral drug therapy. We obtained many novels and different solitary wave solutions that captured by using other previous techniques and have not yet been found in other literature. These obtained solutions show the ability of the used techniques to apply on many various forms of nonlinear partial differential equations. The performance of the suggested techniques reveals that these methods are fruitful and convenient for applying on many different formulas of nonlinear partial differential equations.

Acknowledgment

Mostafa Khater would like to dedicate this paper to my mother and the soul of my father, who was there for the beginning of this degree and did not make it to the end. His love, support, and constant care will never be forgotten. He is very much missed. Also, he want to dedicate this paper to my son (Adam) and i promise him to do more and more to make him proud of me.

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