A new study on Riesz summability method

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Abstract. Quite recently, Bor [6] has proved a new result on weighted arithmetic mean summability factors of non-decreasing sequences, which includes some known results. In this paper, we extend his result to more general matrix summability method by using an almost increasing sequence and normal matrices in place of a positive non-decreasing sequence and weighted mean matrices, respectively.

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1 Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α the nth Cesàro mean of order α, with $\alpha > -1$, of the sequence (s_n), that is (see [7])

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \tag{1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \texttt{for} \quad n > 0. \tag{2}$$

Let $a_1 + a_2, ..., a_n$ be n arbitrary real numbers;their arithmetic mean A is defined to be

$$A = \frac{a_1 + a_2 + ... + a_n}{n}. \tag{3}$$

A series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if (see [8],[14])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty. \tag{4}$$

If we take $\alpha = 1$, then we have $|C, 1|_k$ summability. Let (p_n) be a sequence of positive numbers such that $P_n = \sum_{v=0}^{n} p_v \to \infty$ as $n \to \infty$, $(P_{-i} = p_{-i} = 0, \quad i \geq 1)$. The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{5}$$

defines the sequence (w_n) of the weighted arithmetic mean or simply the $\left(\bar{N}, p_n\right)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [9]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|w_n - w_{n-1}\right|^k < \infty. \tag{6}$$

In the special case when $p_n = 1$ for all values of n (respect. $k = 1$), then $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (respect. $|\bar{N}, p_n|$) summability.

Let $\sum a_n$ be a given series with partial sums (s_n). Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines a sequence-to-sequence transformation, mapping of the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{7}$$

A series $\sum a_n$ is said to be summable $|A, \theta_n|_k$, $k \geq 1$, if (see [15])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{8}$$

where (θ_n) is any sequence of positive constants and

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s). \tag{9}$$

(see also [13]).

In the special case, if we put $\theta_n = \frac{P_n}{p_n}$, we have $|A, p_n|_k$ summability (see [16]. When A is the matrix of weighted mean $(\bar{N}, p_n)$, and $\theta_n = \frac{P_n}{p_n}$, for all values of n), then $|A, \theta_n|_k$ reduces to summability $\left|\bar{N}, p_n\right|_k$, $k \geq 1$. Further, If $\theta_n = n$ for $n \geq 1$ and A is the matrix of Cesàro mean (C, α), then it is the same as summability $|C, \alpha|_k$ in Flett's notation. By a weighted mean matrix we state

$$a_{nv} = \begin{cases} \frac{p_v}{P_n}, & 0 \leq v \leq \text{n} \\ 0 & v > n, \end{cases}$$

where (p_n) is a sequence of positive numbers with $P_n = p_0 + p_1 + p_2 + ... + p_n \to \infty$ as $n \to \infty$.

2 The Known Results

Let K be a positive constant. If $g > 0$, then $f = O(g)$ means $|f| < K.g$ and $f = o(g)$ means $f/g \to 0$. A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (z_n) and two positive constants A and B such that $Az_n \leq b_n \leq Bz_n$ (see [1]). It is known that every increasing sequences is an almost increasing sequence but the converse need not be true. In [4] and [5], Bor obtained main theorems dealing with absolute summability. Quite recently, Bor has also proved the following theorems concerning on summability factors of the absolute weighted mean using a positive non-decreasing sequence.

Theorem 2.1 [3] Let (X_n) be a positive non-decreasing sequence and suppose that there exists sequences (β_n) and (λ_n) such that

$$|\Delta\lambda_n| \leq \beta_n, \tag{10}$$

$$\beta_n \to 0 \quad \text{as} \quad n \to \infty \tag{11}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty, \tag{12}$$

$$|\lambda_n|X_n = O(1). \tag{13}$$

If

$$\sum_{n=1}^{m} \frac{|s_n|^k}{n} = O(X_m) \quad \text{as} \quad m \to \infty, \tag{14}$$

and (p_n) is a sequence that

$$P_n = O(np_n), \tag{15}$$

$$P_n\Delta p_n = O(p_n p_{n+1}), \tag{16}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n}$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Later on, Bor has proved the following theorem using under weaker conditions.

Theorem 2.2 [6] Let (X_n) be a positive non-decreasing sequence. If the sequences (X_n) , (β_n), (λ_n), and (p_n) satisfy the conditions (10)-(13), (15)-(16), and

$$\sum_{n=1}^{m} \frac{|s_n|^k}{nX_n^{k-1}} = O(X_m) \quad \text{as} \quad m \to \infty, \tag{17}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n}$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

3 The Main Results

The aim of this paper is to generalize Theorem 2.2 for more general matrix summability method by using almost increasing sequences in place of positive non-decreasing sequence. So, we have generalized Theorem

2.2 under weaker hypothesis by using normal matrices.

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \quad \bar{\Delta} a_{nv} = a_{nv} - a_{n-1,v}, \quad a_{-1,0} = 0 \tag{18}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{\Delta} \bar{a}_{nv}, \quad n = 1, 2, ... \tag{19}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{20}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{21}$$

With this notation we have the following theorem.

Theorem 3.1 Let $A = (a_{nv})$ be a positive normal matrix such that

$$\bar{a}_{n0} = 1, \ n = 0, 1, ..., \tag{22}$$

$$a_{n-1,v} \geq a_{nv}, \ \text{for} \ \ n \geq v + 1, \tag{23}$$

$$a_{nn} = O(\frac{p_n}{P_n}), \tag{24}$$

$$\sum_{v=1}^{n-1} a_{vv} \hat{a}_{n,v+1} = O(a_{nn}). \tag{25}$$

Let (X_n) be an almost increasing sequence and $(\theta_n a_{nn})$ be a non-increasing sequence. If the sequences (X_n), (β_n), (λ_n), and (p_n) satisfy the conditions (10)-(13) and (15)-(16) of Theorem 2.2, and the condition

$$\sum_{n=1}^{m} (\theta_n a_{nn})^{k-1} \frac{|s_n|^k}{n X_n^{k-1}} = O(X_m) \quad \text{as} \quad m \to \infty, \tag{26}$$

is satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is summable $|A, \theta_n|_k$, $k \geq 1$.

We need the following lemmas for the proof of Theorem 3.1.

Lemma 3.1 [10] Under the conditions on (X_n), (β_n), and (λ_n) as expressed in the statement of Theorem 2.1, we have the following:

$$n X_n \beta_n = O(1), \tag{27}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{28}$$

Lemma 3.2 [12] If the conditions (15) and (16) of Theorem 2.1 are satisfied, then $\Delta\left(\frac{P_n}{np_n}\right)=O\left(\frac{1}{n}\right)$.

Remark Under the conditions on the sequence (λ_n) of Theorem 2.1, we have that (λ_n) is bounded and $\Delta\lambda_n=O(1/n)$ (see [3]).

4 Proof of Theorem 3.1

Let (V_n) denotes the A-transform of the series $\sum a_n\frac{P_n\lambda_n}{np_n}$. Then, by the definition, we have that

$$\bar{\Delta}V_n=\sum_{v=1}^{n}\hat{a}_{nv}a_v\frac{P_v\lambda_v}{vp_v}.$$

Applying Abel's transformation to this sum, we have that

$$\bar{\Delta}V_n=\sum_{v=1}^{n-1}\Delta_v\left(\frac{\hat{a}_{nv}P_v\lambda_v}{vp_v}\right)\sum_{r=1}^{v}a_r+\frac{\hat{a}_{nn}P_n\lambda_n}{np_n}\sum_{r=1}^{n}a_r$$
$$\bar{\Delta}V_n=\sum_{v=1}^{n-1}\Delta_v\left(\frac{\hat{a}_{nv}P_v\lambda_v}{vp_v}\right)s_v+\frac{\hat{a}_{nn}P_n\lambda_n}{np_n}s_n,$$

by the formula for the difference of products of sequences (see [9]) we have

$$\bar{\Delta}V_n=\frac{a_{nn}P_n\lambda_n}{np_n}s_n+\sum_{v=1}^{n-1}\frac{P_v\lambda_v}{vp_v}\Delta_v(\hat{a}_{nv})s_v+\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\lambda_v\Delta\left(\frac{P_v}{vp_v}\right)s_v+\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\frac{P_{v+1}}{(v+1)p_{v+1}}\Delta\lambda_v s_v$$
$$\bar{\Delta}V_n=V_{n,1}+V_{n,2}+V_{n,3}+V_{n,4}.$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\theta_n^{k-1}\mid V_{n,r}\mid^k<\infty,\quad\text{for}\quad r=1,2,3,4. \tag{29}$$

Firstly, by applying Abel's transformation and in view of the hypotheses of Theorem 3.1 we have

$$\begin{aligned}
&\sum_{n=1}^{m}\theta_n^{k-1}\mid V_{n,1}\mid^k\leq\sum_{n=1}^{m}\theta_n^{k-1}a_{nn}^{k-1}a_{nn}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\frac{P_n}{p_n}\right)|\lambda_n|^k\frac{|s_n|^k}{n^k}\\
&=O(1)\sum_{n=1}^{m}(\theta_n a_{nn})^{k-1}|\lambda_n|^k\frac{|s_n|^k}{n}=O(1)\sum_{n=1}^{m}(\theta_n a_{nn})^{k-1}|\lambda_n|^{k-1}|\lambda_n|\frac{|s_n|^k}{n}\\
&=O(1)\sum_{n=1}^{m}(\theta_n a_{nn})^{k-1}\frac{1}{n}\frac{1}{X_n^{k-1}}|\lambda_n||s_n|^k=O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}(\theta_v a_{vv})^{k-1}\frac{|s_v|^k}{vX_v^{k-1}}\\
&+O(1)|\lambda_m|\sum_{n=1}^{m}(\theta_n a_{nn})^{k-1}\frac{|s_n|^k}{nX_n^{k-1}}\\
&=O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n+O(1)|\lambda_m|X_m=O(1)\sum_{n=1}^{m-1}\beta_n X_n+O(1)|\lambda_m|X_m=O(1)\quad\text{as}\quad m\to\infty.
\end{aligned}$$

By applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$ and as in $V_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}\mid V_{n,2}\mid^k &= \sum_{n=2}^{m+1}\theta_n^{k-1}\left|\sum_{v=1}^{n-1}\frac{P_v\lambda_v}{vp_v}\Delta_v(\hat{a}_{nv})s_v\right|^k\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k\right\}\times\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}(\theta_n a_{nn})^{k-1}\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||s_v|^k\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k\sum_{n=v+1}^{m+1}(\theta_n a_{nn})^{k-1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}\frac{1}{X_v^{k-1}}|s_v|^k|\lambda_v|\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}a_{vv}\frac{1}{X_v^{k-1}}|s_v|^k|\lambda_v|\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^k\\
&= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}\frac{1}{X_v^{k-1}}|s_v|^k|\lambda_v|\frac{1}{v^k}\left(\frac{P_v}{p_v}\right)^{k-1}\\
&= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}\frac{1}{X_v^{k-1}}|s_v|^k|\lambda_v|\frac{1}{v^k}v^{k-1}\\
&= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}\frac{1}{X_v^{k-1}}|s_v|^k|\lambda_v|\frac{1}{v} = O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1. Also, since $\Delta\left(\frac{P_v}{vp_v}\right)=O\left(\frac{1}{v}\right)$, by Lemma 3.2, we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}\mid V_{n,3}\mid^k &= \sum_{n=2}^{m+1}\theta_n^{k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\Delta\left(\frac{P_v}{vp_v}\right)\lambda_v s_v\right|^k\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}a_{vv}^{1-k}\hat{a}_{n,v+1}|\lambda_v|^k|s_v|^k\frac{1}{v^k}\right\}\times\left\{\sum_{v=1}^{n-1}a_{vv}\hat{a}_{n,v+1}\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}(\theta_n a_{nn})^{k-1}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^{k-1}\hat{a}_{n,v+1}|\lambda_v|^k|s_v|^k\frac{1}{v^k}\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||s_v|^k\frac{1}{v}\sum_{n=v+1}^{m+1}(\theta_n a_{nn})^{k-1}\hat{a}_{n,v+1}\\
&= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}|\lambda_v|^{k-1}|\lambda_v||s_v|^k\frac{1}{v}\sum_{n=v+1}^{m+1}\hat{a}_{n,v+1}\\
&= O(1)\sum_{v=1}^{m}(\theta_v a_{vv})^{k-1}\frac{1}{X_v^{k-1}}|\lambda_v||s_v|^k\frac{1}{v} = O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1. Finally, by virtue of the hypotheses of Theorem 3.1, by Lemma 3.1, we have $v\beta_v = O(\frac{1}{X_v})$, then

$$\sum_{n=2}^{m+1} \theta_n^{k-1} \mid V_{n,4} \mid^k = \sum_{n=2}^{m+1} \theta_n^{k-1} \left| \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \frac{P_{v+1}}{(v+1)p_{v+1}} \Delta\lambda_v s_v \right|^k$$
$$\leq \sum_{n=2}^{m+1} \theta_n^{k-1} \left\{ \sum_{v=1}^{n-1} a_{vv}^{1-k} \hat{a}_{n,v+1} |\Delta\lambda_v|^k |s_v|^k \right\} \times \left\{ \sum_{v=1}^{n-1} a_{vv} \hat{a}_{n,v+1} \right\}^{k-1}$$
$$= O(1) \sum_{n=2}^{m+1} (\theta_n a_{nn})^{k-1} \sum_{v=1}^{n-1} a_{vv}^{1-k} \hat{a}_{n,v+1} |\Delta\lambda_v|^k |s_v|^k$$
$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^{k-1} |s_v|^k |\Delta\lambda_v|^k \sum_{n=v+1}^{m+1} (\theta_n a_{nn})^{k-1} \hat{a}_{n,v+1}$$
$$= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} |s_v|^k (v\beta_v)^{k-1} \beta_v \sum_{n=v+1}^{m+1} \hat{a}_{n,v+1} = O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} v\beta_v |s_v|^k \frac{1}{vX_v^{k-1}}$$
$$= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} (\theta_r a_{rr})^{k-1} \frac{|s_r|^k}{rX_r^{k-1}} + O(1) m\beta_m \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} \frac{|s_v|^k}{vX_v^{k-1}}$$
$$= O(1) \sum_{v=1}^{m-1} |\Delta(v\beta_v)| X_v + O(1) m\beta_m X_m = O(1) \sum_{v=1}^{m-1} |(v+1)\Delta\beta_v - \beta_v| X_v + O(1) m\beta_m X_m$$
$$= O(1) \sum_{v=1}^{m-1} v|\Delta\beta_v| X_v + O(1) \sum_{v=1}^{m-1} X_v \beta_v + O(1) m\beta_m X_m = O(1) \quad \text{as} \quad m \to \infty,$$

This completes the proof of Theorem 3.1 .

5 Conclusions

1. If we take $\theta_n = \frac{P_n}{p_n}$, then we have a result concerning the $|A, p_n|_k$ summability factors (see [17]).
2. If we take $a_{nv} = \frac{p_v}{P_n}$, then we have another result dealing with $\left|\bar{N}, p_n, \theta_n\right|_k$ summability.
3. If we put $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we obtain a result concerning $|C, 1, \theta_n|_k$ summability.
4. If we take (X_n) as a positive non-decreasing sequence,$\theta_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3.1, then we obtain Theorem 2.2 and if we put $k = 1$ in Theorem 2.2, we have a known result of Mishra and Srivastava dealing with $\left|\bar{N}, p_n\right|$ summability factors of infinite series (see [12]).
5. If we take $\theta_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n in Theorem 3.1, then we obtain a known result of Mishra and Srivastava concerning the $|C, 1|_k$ summability factors of infinite series (see [11]).

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