ZERO-HOPF BIFURCATION OF A QUARTIC SYSTEM OBTAINED FROM A SCALAR THIRD ORDER ORDINARY DIFFERENTIAL EQUATION

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Abstract. In this article, we study the zero-Hopf bifurcation of a quartic system in the three-dimensional space which can be obtained from a scalar the third order ordinary differential equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx + h(x) = 0.$$

where a, b, c are parameters and the dot indicates derivative with respect to the time t. For doing this, some adequate change in parameters must be done in order that the computations become easier.

1. Introduction and statement of the main results

In [2] the authors studied the existence of zero-Hopf bifurcations of the third-order ordinary differential equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx - x^2 = 0, \tag{1}$$

which is commonly known as the Genesio equation.

Here, we study the existence of zero-Hopf bifurcations of the differential equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx + h(x) = 0, \tag{2}$$

where $h(x) = -x^2 + x^4$ et $a, b, c \in \mathbb{R}$.

By defining of the variables $y = \dot{x}$, $z = \dot{y}$ and $\dot{z} = w$, differential equation (2) can be written as the system of nonlinear differential equations

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -cx - by - az + x^2 - x^4 \end{cases} \tag{3}$$

The objective is to study the existence of zero Hopf equilibria and of zero Hopf bifurcations in the system (3). We recall that a zero Hopf

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equilibrium of a three dimensional autonomous differential system is an isolated equilibrium point of system whose linear part at the equilibrium has a zero eigenvalue and a pair of purely imaginary eigenvalues.

Usually the main tool for studying a zero Hopf bifurcation is to pass the system to the normal form of a zero-Hopf bifurcation. However, our analysis of the zero Hopf bifurcation occurring in the system (3) will use the averaging theory, a summary of the results of this theory that we need here is given in section 2. The averaging theory has already been used to study Hopf and zero-Hopf bifurcations in some others differential systems, see for instance [[1], [3], [4], [5]].

Proposition 1. Differential system (3) has a unique zero Hopf equilibrium localized at the origin of coordinates when $a = c = 0$ and $b > 0$.

Theorem 2. Consider the system (3) with the parameters $a = \varepsilon\alpha, b = \omega^2 + \varepsilon\beta$ and $c = \varepsilon\gamma$, with $\omega > 0$ and ε a su ciently small parameter. Then this system exhibits a zero-Hopf bifurcation at the equilibrium point located at the origin of coordinates when $\varepsilon = 0$ if $\gamma - \alpha^2\omega^4 > 0$. Moreover, the periodic orbit $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ bifurcating from this equilibrium point satis es that $(x(0,\varepsilon), y(0,\varepsilon), z(0,\varepsilon))$ is

$$\varepsilon\left(\frac{(\gamma-\alpha\omega^2)}{2} - \frac{\sqrt{-2\,\omega^4\alpha^2+2\,\gamma^2}}{2}, 0, \frac{\omega^2\,(\gamma-\alpha\omega^2)}{2}\right) + O\left(\varepsilon^2\right)$$

if $\varepsilon > 0$ is sufficiently small. If $\lambda_\pm = (-\alpha\omega^2 \pm \sqrt{3\alpha^2\omega^4 - 2\,\gamma^2})/\left(2\omega^3\right)$ then, then this periodic orbit is stable when $\mathrm{Re}(\lambda_\pm) < 0$, and unstable if $\mathrm{Re}(\lambda_+) > 0$ or $\mathrm{Re}(\lambda_-) > 0$.

2. Preliminaries

2.1. **Averaging theory.** In this subsection we present some basic results on the averaging theory, which will be used in the proof of Theorem 2. For a general introduction to the averaging theory see for instance the book of Sanders, Verhulst and Murdock [6].

Consider the following initial value problem

$$\dot{x} = \varepsilon F(t,x) + \varepsilon^2 G(t,x,\varepsilon), \quad x\,(0) = x_0, \tag{4}$$

and the averaged differential equation

$$\dot{y} = \varepsilon f(y), \quad y(0) = x_0. \tag{5}$$

In equations (4) and (5), $x, y \in D$, where $D \subset \mathbb{R}^n$ is an open set, $t \in [0,\infty)$ and ε is a small positive parameter. The functions F :

$[0,\infty) \times D \to \mathbb{R}^n$ and $G : [0,\infty) \times D \times (0,\varepsilon_0] \to \mathbb{R}^n$ are assumed be periodic of period T in the variable t, and $f : D \to \mathbb{R}^n$ is given by

$$f(y) = \frac{1}{T}\int_0^T F(t,y)\,dt. \tag{6}$$

The next theorem establishes that, under certain conditions, the equilibrium points of the averaged equation (5) correspond to T-periodic solutions of system (4). See [7] for a proof.

Theorem 3. Consider the initial value problems (4) and (5) and suppose that F, its Jacobian D_xF, its Hessian $D_{xx}F$, G and its Jacobian D_xG are continuous and bounded by a constant independent of ε in $[0,\infty) \times D$ and $\varepsilon \in (0,\varepsilon_0]$. Further we assume that F and G are $T-$periodic in t, with T independent of ε. Then the following statements hold.

(a) For $t \in [0, 1/\varepsilon]$ we have $x(t) - y(t) = O(\varepsilon)$ as $\varepsilon \to 0$.

(b) If p is an equilibrium point of system (5) such that

$$\det D_y f(p) \neq 0; \tag{7}$$

then there exists a periodic solution $x(t,\varepsilon)$ of period T of system (4) such that $x(0,\varepsilon) - p = O(\varepsilon)$ as $\varepsilon \to 0$.

(c) If all the real parts of the eigenvalues of the matrix $D_y f(p)$ are negative, then the periodic solution $x(t,\varepsilon)$ is stable. If some real part of the eigenvalues is positive, then the periodic solution $x(t,\varepsilon)$ is unstable.

3. Proof of Proposition 1 and Theorem 2

Proof of Proposition 1.

We saw that the characteristic polynomial of the linear part of system (3) at the equilibrium point $x_c = (c,0,0)$ is $q(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c$. We want to find the parameter values for which the polynomial q has a zero eigenvalue and a pair of purely imaginary eigenvalues, that is the parameter values for which q is of the form $-\lambda\left(\lambda^2 + b\right)$ with $b > 0$. In order to simplify the expressions, we will put $b = \omega^2$, with $\omega > 0$. Thus, imposing the condition $q(\lambda) = -\lambda\left(\lambda^2 + \omega^2\right)$, we obtain that $a = c = 0$ and $b > 0$. Hence, when $a = c = 0$ and $b > 0$ there is a unique zero-Hopf equilibrium point at the origin of coordinates. Moreover, if we put $b = \omega^2$, with $\omega > 0$, then the eigenvalues are 0 and $\pm i\omega$. This completes the proof of **Proposition 1**.

Proof of Theorem 2.

We shall use the averaging theory of first order described in subsection 2.1 (see **Theorem 3**) in order to study if from the double zero-Hopf equilibrium point located at the origin of coordinates, it bifurcates some periodic orbit by moving the parameters a, b, c of system (3). Thus, let the parameters a, b, c of system (3) be given by $a = \varepsilon\alpha, b = \omega^2 + \varepsilon\beta, c = \varepsilon\gamma$ with $\varepsilon > 0$ a sufficiently small parameter. Then, the system (3) becomes

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -\varepsilon\gamma x - (\omega^2 + \varepsilon\beta)\, y - \varepsilon\alpha z + x^2 - x^4 \end{cases} \tag{8}$$

The first step in order to write our differential system (8) in the normal form for applying the averaging theory is to write the linear part at the origin of system (8) when $\varepsilon = 0$ into its real Jordan normal form, that is into the form

$$\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

To do this, we apply the linear change of variables $(x, y, z) \to (X, Y, Z)$ where

$$x = \frac{Z - \omega X}{\omega^2}, y = Y, z = \omega X. \tag{9}$$

In the new variables (X, Y, Z), system (8) becomes

$$\begin{aligned} \dot{X} &= (-\omega^4 X^4 + \omega^6 X^2 - Y\omega^{10} + \omega^4 Z^2 - Z^4 + 4\,\omega^3 X^3 Z - 2\,\omega^5 XZ + 4\,\omega\, XZ^3 \\ &\quad -6\,\omega^2 X^2 Z^2)/\omega^9 + \left(\gamma\,\omega^7 X - \gamma\,\omega^6 Z - Y\omega^8\beta - \alpha\,\omega^9 X\right)\varepsilon/\omega^9, \\ \dot{Y} &= \omega X, \\ \dot{Z} &= (\omega^6 X^2 - 2\,\omega^5 XZ + \omega^4 Z^2 - \omega^4 X^4 + 4\,\omega^3 X^3 Z - 6\,\omega^2 X^2 Z^2 + 4\,\omega\, XZ^3 - Z^4)/\omega^8 \\ &\quad + \left(\gamma\,\omega^7 X - \gamma\,\omega^6 Z - Y\omega^8\beta - \alpha\,\omega^9 X\right)\varepsilon/\omega^8. \end{aligned} \tag{10}$$

Now we re-scale the variables (X, Y, Z) as follows $(X, Y, Z) \to (\varepsilon u, \varepsilon v, \varepsilon w)$.Then system (10) becomes

$$\begin{aligned} \dot{u} &= -v\omega + (\gamma\,\omega^7 u - \gamma\,\omega^6 w - \alpha\,\omega^9 u + \omega^6 u^2 - v\omega^8\beta + \omega^4 w^2 - 2\,\omega^5 uw)\varepsilon/\omega^9 \\ &\quad +(-\omega^4 u^4 - w^4 + 4\,\omega^3 u^3 w + 4\,\omega\, uw^3 - 6\,\omega^2 u^2 w^2)\varepsilon^3/\omega^9, \\ \dot{v} &= \omega u, \\ \dot{w} &= \left(\gamma\,\omega^7 u - \gamma\,\omega^6 w - v\omega^8\beta - \alpha\,\omega^9 u + \omega^6 u^2 - 2\,\omega^5 uw + \omega^4 w^2\right)\varepsilon/\omega^8, \\ &\quad + \left(-\omega^4 u^4 - w^4 + 4\,\omega^3 u^3 w + 4\,\omega\, uw^3 - 6\,\omega^2 u^2 w^2\right)\varepsilon^3/\omega^8. \end{aligned} \tag{11}$$

Now we pass the differential system (11) to cylindrical coordinates (r, θ, w) defined by $u = r\cos\theta$ and $v = r\sin\theta$ and we obtain

$$
\begin{aligned}
\dot{r} &= \cos(\theta)\,\varepsilon\,(-\varepsilon^2\omega^4 r^4 (\cos(\theta))^4 - \varepsilon^2 w^4 + 4\,\varepsilon^2\omega^3 r^3 (\cos(\theta))^3\, w + 4\,\varepsilon^2\omega\, r\cos(\theta)\, w^3 \\
&\quad -6\,\varepsilon^2\omega^2 r^2 (\cos(\theta))^2\, w^2 + \gamma\,\omega^7 r\cos(\theta) - \gamma\,\omega^6 w - \alpha\,\omega^9 r\cos(\theta) + \omega^6 r^2 (\cos(\theta))^2 \\
&\quad -r\sin(\theta)\,\omega^8\beta + \omega^4 w^2 - 2\,\omega^5 r\cos(\theta)\, w)/\omega^9, \\
\dot{\theta} &= -(-\sin(\theta)\,\varepsilon^3\omega^4 r^4 (\cos(\theta))^4 - \sin(\theta)\,\varepsilon^3 w^4 + 4\,\sin(\theta)\,\varepsilon^3\omega^3 r^3 (\cos(\theta))^3\, w \\
&\quad +4\,\sin(\theta)\,\varepsilon^3\omega\, r\cos(\theta)\, w^3 - 6\,\sin(\theta)\,\varepsilon^3\omega^2 r^2 (\cos(\theta))^2\, w^2 \\
&\quad +\sin(\theta)\,\omega^7\varepsilon\,\gamma\, r\cos(\theta) - \sin(\theta)\,\gamma\,\omega^6\varepsilon\, w - \sin(\theta)\,\omega^9\varepsilon\,\alpha\, r\cos(\theta) \\
&\quad +\sin(\theta)\,\omega^6\varepsilon\, r^2 (\cos(\theta))^2 + \sin(\theta)\,\omega^4\varepsilon\, w^2 - 2\,\sin(\theta)\,\omega^5\varepsilon\, r\cos(\theta)\, w \\
&\quad +\omega^8\varepsilon r\beta\,(\cos(\theta))^2 - \omega^8\varepsilon\, r\beta - r\omega^{10})/r\omega^9 \\
\dot{w} &= \varepsilon(-\varepsilon^2\omega^4 r^4 (\cos(\theta))^4 - \varepsilon^2 w^4 + 4\,\varepsilon^2\omega^3 r^3 (\cos(\theta))^3\, w + 4\,\varepsilon^2\omega\, r\cos(\theta)\, w^3 \\
&\quad -6\,\epsilon^2\omega^2 r^2 (\cos(\theta))^2\, w^2 + \gamma\,\omega^7 r\cos(\theta) - \gamma\,\omega^6 w - \alpha\,\omega^9 r\cos(\theta) + \omega^6 r^2 (\cos(\theta))^2 \\
&\quad -r\sin(\theta)\,\omega^8\beta + \omega^4 w^2 - 2\,\omega^5 r\cos(\theta)\, w)/\omega^8
\end{aligned}
\tag{12}
$$

In system (12) we take as the new independent variable, and we get

$$
\begin{aligned}
\frac{dr}{d\theta} &= \varepsilon F_1(\theta, r, w) \\
\frac{dw}{d\theta} &= \varepsilon F_2(\theta, r, w)
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
F_1(\theta, r, w) &= \cos(\theta)\,(\gamma\,\omega^3 r\cos(\theta) - \gamma\,\omega^2 w - \alpha\,\omega^5 r\cos(\theta) \\
&\quad +\omega^2 r^2 (\cos(\theta))^2 - r\sin(\theta)\,\omega^4\beta + w^2 - 2\,\omega\, r\cos(\theta))w/\omega^6 \\
F_2(\theta, r, w) &= \gamma\,\omega^3 r\cos(\theta) - \gamma\,\omega^2 w - \alpha\,\omega^5 r\cos(\theta) + \omega^2 r^2 (\cos(\theta))^2 \\
&\quad -r\sin(\theta)\,\omega^4\beta + w^2 - 2\,\omega\, r\cos(\theta)\, w/\omega^5
\end{aligned}
$$

Using the notation of subsection 2.1, we have $t = \theta, T = 2\pi, x = (r, w)^T$ and

$$
F(\theta, r, w) = \begin{pmatrix} F_1(\theta, r, w) \\ F_2(\theta, r, w) \end{pmatrix} \qquad \text{and} \qquad f(r, w) = \begin{pmatrix} f_1(r, w) \\ f_2(r, w) \end{pmatrix}.
$$

It is immediate to check that system (13) satisfies all the assumptions of Theorem 4.

Now we compute the integrals (6). We obtain that

$$
\begin{aligned}
f_1(r, w) &= \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta, r, w)\, d\theta = \frac{r\,(-2\,w + \gamma\,\omega^2 - \alpha\,\omega^4)}{2\omega^5}, \\
f_2(r, w) &= \frac{1}{2\pi}\int_0^{2\pi} F_2(\theta, r, w)\, d\theta = -\frac{2\,\gamma\,\omega^2 w - \omega^2 r^2 - 2\,w^2}{2\omega^5}.
\end{aligned}
$$

The system $f_1(r,w) = f_2(r,w) = 0$ has a unique solution (r,w) with $r^* > 0$, namely

$$r^* = \frac{\sqrt{-2\,\omega^4\alpha^2 + 2\,\gamma^2\omega}}{2}, w^* = \frac{\omega^2\,(\gamma - \alpha\,\omega^2)}{2}$$

The Jacobian (7) at (r,w) takes the value

$$\det \left.\frac{\partial\,(f_1,f_2)}{(r,w)}\right|_{(r,w)=(r^*,w^*)} = \frac{\gamma^2 - \alpha^2\omega^4}{2\omega^6}$$

which is nonzero by hypothesis. Moreover the eigenvalues of the Jacobian matrix

$$\left.\frac{\partial\,(f_1,f_2)}{(r,w)}\right|_{(r,w)=(r^*,w^*)}$$

are given by

$$(-\alpha\omega^2 \pm \sqrt{3\alpha^2\omega^4 - 2\,\gamma^2})/\left(2\omega^3\right)$$

The rest of the proof of Theorem **2** follows immediately from Theorem **3** if we show that the periodic solution corresponding to (r,w) provides a periodic orbit bifurcating from the origin of coordinates of the differential system (8) at $\varepsilon = 0$.

Theorem **4** garantes for $\varepsilon > 0$ sufficiently small the existence of a periodic solution $(r(\theta,\varepsilon), w(\theta,\varepsilon))$ of system (13) such that

$$(r(0,\varepsilon), w(0,\varepsilon)) \to \left(\frac{\sqrt{-2\,\omega^4\alpha^2 + 2\,\gamma^2\omega}}{2}, \frac{\omega^2\,(\gamma - \alpha\,\omega^2)}{2}\right)$$

when $\varepsilon \to 0$. From the second equation of system (12) we obtain that $\theta(t,\theta) = \omega t + O(\varepsilon)$. Moreover, we have that $(r(t,\varepsilon), \theta\,(t,\varepsilon)\,, w(t,\varepsilon))$ is a periodic solution of system (12) such that

$$(r(0,\varepsilon), \theta\,(0,\varepsilon)\,, w(0,\varepsilon)) \to \left(\frac{\sqrt{-2\,\omega^4\alpha^2 + 2\,\gamma^2\omega}}{2}, 0, \frac{\omega^2\,(\gamma - \alpha\,\omega^2)}{2}\right)$$

when $\varepsilon \to 0$. So for $\varepsilon > 0$ sufficiently small system (11) has the periodic solution

$$(u(t,\varepsilon), v(t,\varepsilon), w(t,\varepsilon)) = (r(t,\varepsilon)\cos(t,\varepsilon), r(t,\varepsilon)\sin(t,\varepsilon), w(t,\varepsilon)),$$

such that

$$(u(0,\varepsilon), v\,(0,\varepsilon)\,, w(0,\varepsilon)) \to \left(\frac{\sqrt{-2\,\omega^4\alpha^2 + 2\,\gamma^2\omega}}{2}, 0, \frac{\omega^2\,(\gamma - \alpha\,\omega^2)}{2}\right)$$

when $\varepsilon \to 0$. This periodic solution in the di erential system (10) writes as $(X(t,\varepsilon), Y(t,\varepsilon), Z(t,\varepsilon)) = (\varepsilon u(t,\varepsilon), \varepsilon v(t,\varepsilon), \varepsilon w(t,\varepsilon))$, and it satisfies that

$$(X(0,\varepsilon), Y(0,\varepsilon), Z(0,\varepsilon)) \to \left(\frac{\sqrt{-2\,\omega^4\alpha^2 + 2\,\gamma^2}\omega\varepsilon}{2}, 0, \frac{\omega^2\left(\gamma - \alpha\,\omega^2\right)\varepsilon}{2} \right)$$

when $\varepsilon \to 0$. Finally, we have that system (8) has the periodic solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ obtained from solution $(X(t,\varepsilon), Y(t,\varepsilon), Z(t,\varepsilon))$ through the change of variables (9). It satis es that $(x(0,\varepsilon), y\,(0,\varepsilon)\,, z(0,\varepsilon))$ is

$$\varepsilon \left(\frac{\left(\gamma - \alpha\,\omega^2\right)}{2} - \frac{\sqrt{-2\,\omega^4\alpha^2 + 2\,\gamma^2}}{2}, 0, \frac{\omega^2\left(\gamma - \alpha\,\omega^2\right)}{2} \right) + O\left(\varepsilon^2\right)$$

if ε is sufficiently small. Thus $(x(0,\varepsilon), y(0,\varepsilon), z(0,\varepsilon)) \to (0,0,0)$ when $\varepsilon \to 0$. Therefore, it is a periodic solution starting at the zero-Hopf equilibrium point located at the origin of coordinates when $\varepsilon = 0$. This completes the proof of Theorem 2.

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