RAD$-\oplus-$SUPPLEMENTED LATTICES

Çiğdem Biçer[1] and Celil Nebiyev[2]

Department of Mathematics, Ondokuz Mayıs University,
55270 *Kurupelit-Atakum/Samsun/TÜRKİYE-TURKEY*
[1] *cigdem_bicer184@hotmail.com*, [2] *cnebiyev@omu.edu.tr*

Abstract

In this work, we define Rad$-\oplus-$supplemented and strongly Rad$-\oplus-$supplemented lattices and give some properties of these lattices. We generalize some properties of Rad$-\oplus-$supplemented modules to lattices. Let L be a lattice and $1 = a_1 \oplus a_2 \oplus ... \oplus a_n$ with $a_1, a_2, ..., a_n \in L$. If $a_i/0$ is Rad$-\oplus-$supplemented for every $i = 1, 2, ..., n$, then L is also Rad$-\oplus-$supplemented. Let L be a distributive Rad$-\oplus-$supplemented lattice. Then $1/u$ is Rad$-\oplus-$supplemented for every $u \in L$. We also define completely Rad$-\oplus-$supplemented lattices and prove that every Rad$-\oplus-$supplemented lattice with SSP property is completely Rad$-\oplus-$supplemented.

Key words: Lattices, Radical, Supplemented Lattices, Generalized (Radical) Supplemented Lattices.

2010 Mathematics Subject Classification: 06C05, 06C15.

1 INTRODUCTION

In this paper, every lattice is complete modular lattice with the smallest element 0 and the greatest element 1. Let L be a lattice, $x, y \in L$ and $x \leq y$. A sublattice $\{a \in L | x \leq a \leq y\}$ is called a *quotient sublattice* and denoted by y/x. An element y of a lattice L is called a *complement* of x in L if $x \wedge y = 0$ and $x \vee y = 1$, this case we denote $1 = x \oplus y$ (in this case we call x and y are *direct summands* of L). L is said to be *complemented* if each element has at least one complement in L. An element x of L is said to be *small* or *superfluous* and denoted by $x \ll L$ if $y = 1$ for every $y \in L$ such that $x \vee y = 1$. The meet of all the maximal ($\neq 1$) elements of a lattice L is called the *radical* of L and denoted by $r(L)$. An element a of L is called a *supplement* of b in L if it is minimal for $a \vee b = 1$. a is a supplement of b in a lattice L if and only if $a \vee b = 1$ and $a \wedge b \ll a/0$. A lattice L is called a *supplemented lattice* if every element of L has a supplement in L. If every element of L has a supplement that is a direct summand in L, then L is called a *$\oplus-$supplemented lattice*. We say that an element y of L *lies above* an element x of L if $x \leq y$ and $y \ll 1/x$. L is said to be *hollow* if every element distinct from 1 is superfluous in L, and L is said to be *local* if L has the greatest element ($\neq 1$). An element $x \in L$ has *ample supplements* in L if for every $y \in L$ with $x \vee y = 1$, x has a supplement z in L with $z \leq y$. L is said to be *amply supplemented*, if every element of L has ample supplements in L. It is clear that every amply supplemented lattice is supplemented. A lattice L is said to be *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for every $x, y, z \in L$. An element y of a lattice L is called a *generalized* (*radical*) *supplement* (or briefly, *Rad-supplement*) of x in L if $1 = x \vee y$ and $x \wedge y \leq r(y/0)$. A lattice L is said to be *generalized* (*radical*) supplemented (or briefly, *Rad-supplemented*) if every element of L has a generalized (radical) supplement in L.

Let L be a lattice. Consider the following conditions.

($D1$) For every element x of L, there exist $x_1, x_2 \in L$ such that $1 = x_1 \oplus x_2$, $x_1 \leq x$ and $x_2 \wedge x \ll x_2/0$.

($D3$) If x_1 and x_2 are direct summands of L and $1 = x_1 \vee x_2$, then $x_1 \wedge x_2$ is also a direct summand of L.

More informations about (amply) supplemented lattices are in [1], [2] and [7]. The definition of $\oplus-$supplemented lattices and some informations about these lattices are in [4]. More results about (amply) supplemented modules are in [11]. The definition of generalized supplemented lattices and some important properties of them are in [3]. Some important properties of Rad$-\oplus-$supplemented modules are in [6], [8]and [10]. The definition of β_* relation on lattices and some properties of this relation are in [9]. The definition of β^* relation on modules and some properties of this relation are in [5].

Lemma 1.1 *Let L be a lattice, $a, b \in L$ and a be a Rad-supplement of b in L. Then $r(a/0) = a \wedge r(L)$.*

Proof. See [3, Lemma 2(b)]. ■

Lemma 1.2 *Let L be a lattice and y be a Rad-supplement of x in L. Then for $a \leq x$, $a \vee y$ is a Rad-supplement of x in $1/a$.*

Proof. See [3, Lemma 5]. ■

Lemma 1.3 *Let L be a lattice and $a, b \in L$. If x is a Rad-supplement of $a \vee b$ in L and y is a Rad-supplement of $a \wedge (b \vee x)$ in $a/0$, then $x \vee y$ is a Rad-supplement of b in L.*

Proof. See [3, the proof of Lemma 7]. ■

2 RAD$-\oplus-$SUPPLEMENTED LATTICES

Definition 2.1 *Let L be a lattice. If every element of L has a Rad-supplement that is a direct summand in L, then L is called a Rad$-\oplus-$supplemented (or generalized $\oplus-$supplemented) lattice.*

It is clear that every Rad$-\oplus-$supplemented lattice is Rad-supplemented, but the converse is not true in general (see Example 2.19 and Example 2.20). It is also clear that every $\oplus-$supplemented lattice is Rad$-\oplus-$supplemented, but the converse is not true in general (See Example 2.21). Hence Rad$-\oplus-$supplemented lattices are more general than $\oplus-$supplemented lattices. Hollow and local lattices are Rad$-\oplus-$supplemented.

Lemma 2.2 *Let L be a lattice, $a_1, a_2 \in L$ and $1 = a_1 \oplus a_2$. If $a_1/0$ and $a_2/0$ are Rad$-\oplus-$supplemented, then L is also Rad$-\oplus-$supplemented.*

Proof. Let x be any element of L. Then 0 is a Rad-supplement of $a_1 \vee a_2 \vee x$ in L. Since $a_1/0$ is Rad$-\oplus-$supplemented, $a_1 \wedge (a_2 \vee x)$ has a Rad-supplement y that is a direct summand in $a_1/0$. Then by Lemma 1.3, $y = y \vee 0$ is a Rad-supplement of $a_2 \vee x$ in L. Since $a_2/0$ is Rad$-\oplus-$supplemented, $a_2 \wedge (x \vee y)$ has a Rad-supplement z that is a direct summand in $a_2/0$. Then by Lemma 1.3, $y \vee z$ is a Rad$-\oplus-$supplement of x in L. Since y is a direct summand of $a_1/0$ and z is a direct summand of $a_2/0$ and $1 = a_1 \oplus a_2$, $y \vee z = y \oplus z$ is a direct summand of L. Hence L is Rad$-\oplus-$supplemented. ■

Corollary 2.3 *Let $a_1, a_2, ..., a_n \in L$ and $1 = a_1 \oplus a_2 \oplus ... \oplus a_n$. If $a_i/0$ is Rad$-\oplus-$supplemented for every $i = 1, 2, ..., n$, then L is also Rad$-\oplus-$supplemented.*

Proof. Clear form Lemma 2.2. ■

Lemma 2.4 *Let L be a Rad$-\oplus-$supplemented lattice, $u \in L$ and $u = (u \wedge a) \vee (u \wedge b)$ for every $a, b \in L$ with $1 = a \oplus b$. Then;*

(i) $1/u$ is Rad$-\oplus-$supplemented.

(ii) If u is a direct summand of L, then $u/0$ is also Rad$-\oplus-$supplemented.

Proof. (i) Let $x \in 1/u$. Since L is Rad$-\oplus-$supplemented, x has a Rad-supplement y that is a direct summand in L. By Lemma 1.2, $y \vee u$ is a Rad-supplement of x in $1/u$. Since y is a direct summand of L, there exists $z \in L$ such that $1 = y \oplus z$. Here $1 = (y \vee u) \vee (z \vee u)$. Since $u = (u \wedge y) \vee (u \wedge z)$, $(y \vee u) \wedge (z \vee u) = (y \vee (u \wedge y) \vee (u \wedge z)) \wedge (z \vee (u \wedge y) \vee (u \wedge z)) = (y \vee (u \wedge z)) \wedge (z \vee (u \wedge y)) = (y \wedge (z \vee (u \wedge y))) \vee (u \wedge z) = (y \wedge z) \vee (u \wedge y) \vee (u \wedge z) = 0 \vee (u \wedge y) \vee (u \wedge z) = (u \wedge y) \vee (u \wedge z) = u$. Hence $1/u$ is Rad$-\oplus-$supplemented.

(ii) Let u is a direct summand of L and $x \in u/0$. Since L is Rad$-\oplus-$supplemented, there exist $y, z \in L$ such that $1 = x \vee y$, $x \wedge y \leq r(y/0) \leq r(L)$ and $1 = y \oplus z$. By hypothesis $u = (u \wedge y) \oplus (u \wedge z)$. Since u is a direct summand of L, $u \wedge y$ is also a direct summand of L. Since $1 = x \vee y$ and $x \leq u$, by modularity, $u = x \vee (u \wedge y)$. Since $u \wedge y$ is a direct summand of L, by Lemma 1.1, $r((u \wedge y)/0) = u \wedge y \wedge r(L)$. Since $x \wedge u \wedge y = x \wedge y \leq r(L)$ and $x \wedge u \wedge y \leq u \wedge y$, $x \wedge u \wedge y \leq u \wedge y \wedge r(L) = r((u \wedge y)/0)$. Hence $u/0$ is Rad$-\oplus-$supplemented. ■

Corollary 2.5 *Let L be a distributive and Rad$-\oplus-$supplemented lattice. Then $1/u$ Rad$-\oplus-$supplemented for every $u \in L$.*

Proof. Clear from Lemma 2.4. ■

Lemma 2.6 *Let L be a Rad$-\oplus-$supplemented lattice with $(D3)$ property. Then for every direct summand u of L, $u/0$ is Rad$-\oplus-$supplemented.*

Proof. Let u is a direct summand of L and $x \in u/0$. Since L is Rad$-\oplus-$supplemented, there exists a direct summand y of L such that $1 = x \vee y$ and $x \wedge y \leq r(y/0)$. Since $u \vee y = 1$ and L has $(D3)$ property, $u \wedge y$ is a direct summand of L. Hence $u \wedge y$ is a direct summand of $u/0$. By modularity, $u = x \vee (u \wedge y)$. Since $x \wedge u \wedge y = x \wedge y \leq r(y/0) \leq r(L)$ and $x \wedge u \wedge y \leq u \wedge y$, $x \wedge u \wedge y \leq u \wedge y \wedge r(L)$. Since $u \wedge y$ is a direct summand of L, by Lemma 1.1, $u \wedge y \wedge r(L) = r((u \wedge y)/0)$. Therefore, $u/0$ is Rad$-\oplus-$supplemented. ■

Proposition 2.7 *Let $1 = a \oplus b$ with $a, b \in L$. Then $b/0$ is Rad$-\oplus-$supplemented if and only if for every $x \in 1/a$, there exists a direct summand y of L such that $y \in b/0$, $1 = x \vee y$ and $x \wedge y \leq r(L)$.*

Proof. $(\Longrightarrow)$ Let $x \in 1/a$. Then $x \wedge b \in b/0$ and since $b/0$ is Rad$-\oplus-$supplemented, $x \wedge b$ has a Rad-supplement y that is a direct summand in $b/0$. Here $b = (x \wedge b) \vee y$ and $x \wedge y = x \wedge b \wedge y \leq r(y/0) \leq r(L)$. Since y is a direct summand of $b/0$, there exists $z \in b/0$ such that $b = y \oplus z$. Then $1 = a \oplus b = a \oplus y \oplus z$ and y is a direct summand of L. Since $a \leq x$ and $b = (x \wedge b) \vee y$, $1 = a \vee b = x \vee b = x \vee (x \wedge b) \vee y = x \vee y$.

$(\Longleftarrow)$ Let $x \in b/0$. Then $a \vee x \in 1/a$ and by hypothesis, there exists a direct summand y of L such that $y \in b/0$, $1 = a \vee x \vee y$ and $(a \vee x) \wedge y \leq r(L)$. Then we have $b = b \wedge 1 = b \wedge (a \vee x \vee y) = (a \wedge b) \vee x \vee y = x \vee y$ and $x \wedge y \leq (a \vee x) \wedge y \leq r(L)$. Since y is a direct summand of L, there exists $z \in L$ with $1 = y \oplus z$. Here $b = b \wedge 1 = b \wedge (y \oplus z) = y \oplus (b \wedge z)$ and y is a direct summand of $b/0$. By Lemma 1.1, $r(y/0) = y \wedge r(L)$. Since $x \wedge y \leq r(L)$ and $x \wedge y \leq y$, $x \wedge y \leq y \wedge r(L) = r(y/0)$. Hence y is a Rad-supplement of x in $b/0$ and $b/0$ is Rad$-\oplus-$supplemented. ■

Proposition 2.8 *Let L be a Rad$-\oplus-$supplemented lattice, a be a direct summand of L and for every direct summand t of L with $1 = t \vee a$, $t \wedge a$ be a direct summand of $a/0$. Then $a/0$ is Rad$-\oplus-$supplemented.*

Proof. Since a is a direct summand of L, there exists $b \in L$ with $1 = a \oplus b$. Let $x \in a/0$. Since L is Rad$-\oplus-$supplemented, there exist $y, z \in L$ such that $1 = x \vee y$, $x \wedge y \leq r(y/0)$ and $1 = y \oplus z$. By $x \leq a$, $1 = x \vee y = a \vee y$. By hypothesis, $a \wedge y$ is a direct summand of $a/0$ and since a is a direct summand of L, $a \wedge y$ is a direct summand of L. By Lemma 1.1, $r((a \wedge y)/0) = a \wedge y \wedge r(L)$. Here $x \wedge y \leq y \wedge r(L)$ and $x \wedge a \wedge y \leq a \wedge y \wedge r(L) = r((a \wedge y)/0)$. Since $1 = x \vee y$ and $x \leq a$, $a = a \wedge 1 = a \wedge (x \vee y) = x \vee (a \wedge y)$. Hence $a/0$ is Rad$-\oplus-$supplemented. ■

Let $x, y \in L$. It is defined a relation β_* on the elements of L by $x \beta_* y$ if and only if for every $t \in L$ with $x \vee t = 1$ then $y \vee t = 1$ and for every $k \in L$ with $y \vee k = 1$ then $x \vee k = 1$. (See [9, Definition 1])

Lemma 2.9 *Let L be a Rad-supplemented lattice. If every Rad-supplement element in L is β_* equivalent to a direct summand of L, then L is Rad$-\oplus-$supplemented.*

Proof. Let x be any element of L and y be a Rad-supplement of x in L. By hypothesis, there exists a direct summand a of L such that $y\beta_* a$. Since $x \vee y = 1$, $x \vee a = 1$. Assume $x \wedge a \nleq r(L)$. Then there exists a maximal ($\neq 1$) element t of L with $x \wedge a \nleq t$. Here $(x \wedge a) \vee t = 1$. By [9, Lemma 2], $a \vee (x \wedge t) = 1$ and since $y\beta_* a$, $y \vee (x \wedge t) = 1$. Since $x \vee t = 1$, by [9, Lemma 2], $(x \wedge y) \vee t = 1$. Since $x \wedge y \leq r(y/0) \leq r(L) \leq t$, $t = (x \wedge y) \vee t = 1$. This contradicts with $t \neq 1$. Hence $x \wedge a \leq r(L)$. Since a is a direct summand of L, by Lemma 1.1, $x \wedge a \leq a \wedge r(L) = r(a/0)$. Hence a is a Rad-supplement of x in L and L is Rad$-\oplus-$supplemented. ■

Corollary 2.10 *Let L be a Rad-supplemented lattice. If every Rad-supplement element in L lies above a direct summand of L, then L is Rad$-\oplus-$supplemented.*

Proof. Clear from Lemma 2.9. ■

Definition 2.11 *Let L be a lattice. If $a/0$ is Rad$-\oplus-$supplemented for every direct summand a of L, then L is called a completely Rad$-\oplus-$supplemented lattice.*

Clearly we can see that every completely Rad$-\oplus-$supplemented lattice is Rad$-\oplus-$supplemented.

Proposition 2.12 *Let L be a Rad$-\oplus-$supplemented lattice with $(D3)$ property. Then L is completely Rad$-\oplus-$supplemented.*

Proof. Clear from Lemma 2.6. ■

Definition 2.13 *Let L be a lattice. L is said to have SSP property if $a \vee b$ is a direct summand for every direct summands a and b of L.*

Proposition 2.14 *Let L be a Rad$-\oplus-$supplemented lattice with SSP property. Then L is completely Rad$-\oplus-$supplemented.*

Proof. Let a be a direct summand of L. Then there exists $b \in L$ such that $1 = a \oplus b$. let $x \in 1/b$. Since L is Rad$-\oplus-$supplemented, there exists a direct summand y of L such that $x \vee y = 1$ and $x \wedge y \leq r(y/0)$. Here $b \vee y$ is a Rad-supplement of x in $1/b$, by Lemma 1.2. Since b and y are direct summands of L and L has SSP property, $b \vee y$ is a direct summand of L and there exists $z \in L$ such that $1 = (b \vee y) \oplus z$. Here $1 = (b \vee y) \vee (b \vee z)$ and $(b \vee y) \wedge (b \vee z) = b \vee ((b \vee y) \wedge z) = b \vee 0 = b$ and $b \vee y$ is a direct summand of $1/b$. Hence $1/b$ is Rad$-\oplus-$supplemented and since $\frac{a}{0} = \frac{a}{a \wedge b} \cong \frac{a \vee b}{b} = \frac{1}{b}$, $a/0$ also Rad$-\oplus-$supplemented. ■

Definition 2.15 *Let L be a Rad-supplemented lattice. If every Rad-supplement element in L is a direct summand of L, then L is called a strongly Rad$-\oplus$ $-$supplemented lattice.*

It is clear that every strongly $\oplus-$supplemented lattice is strongly Rad$-\oplus$ $-$supplemented. Since every lattice with $(D1)$ property is strongly $\oplus-$supplemented, these lattices are strongly Rad$-\oplus-$supplemented too. Every strongly Rad$-\oplus$ $-$supplemented lattice is Rad$-\oplus-$supplemented, but the converse is not true in general (See Example 2.21).

Lemma 2.16 *Let $1 = a \oplus b$ in L and $x, y \in b/0$. Then y is a Rad-supplement of x in $b/0$ if and only if y is a Rad-supplement of $a \vee x$ in L.*

Proof. ($\Longrightarrow$) Since y is a Rad-supplement of x in $b/0$, $b = x \vee y$ and $x \wedge y \leq r\,(y/0)$. Then $1 = a \oplus b = a \vee x \vee y$ and $(a \vee x) \wedge y = (a \vee x) \wedge b \wedge y = ((a \wedge b) \vee x) \wedge y = x \wedge y \leq r\,(y/0)$. Hence y is a Rad-supplement of $a \vee x$ in L.

($\Longleftarrow$) Since y is a Rad-supplement of $a \vee x$ in L, $1 = a \vee x \vee y$ and $(a \vee x) \wedge y \leq r\,(y/0)$. Then $b = 1 \wedge b = (a \vee x \vee y) \wedge b = (a \wedge b) \vee x \vee y = x \vee y$ and $x \wedge y \leq (a \vee x) \wedge y \leq r\,(y/0)$. Hence y is a Rad-supplement of x in $b/0$. ■

Lemma 2.17 *Let L be a strongly Rad$-\oplus-$supplemented lattice. Then $a/0$ is strongly Rad$-\oplus-$supplemented for every direct summand a of L.*

Proof. Let a be a direct summand of L and $1 = a \oplus b$ with $b \in L$. Let y be a Rad-supplement of x in $a/0$. By Lemma 2.16, y is a Rad-supplement of $b \vee x$ in L. Since L is strongly Rad$-\oplus-$supplemented, y is a direct summand of L. By this, there exists $z \in L$ with $1 = y \oplus z$. By modularity, $a = a \wedge 1 = a \wedge (y \oplus z) = y \oplus (a \wedge z)$. Hence y is a direct summand of $a/0$ and $a/0$ is strongly Rad$-\oplus-$supplemented. ■

Corollary 2.18 *Every strongly Rad$-\oplus-$supplemented lattice is completely Rad$-\oplus-$supplemented.*

Proof. Clear from Lemma 2.17. ■

Example 2.19 *Consider the lattice $L = \{1, a, b, c, 0\}$ given by the following diagram;*

$$\begin{tikzcd} & 1 & \\ b \arrow[ur] & & c \arrow[ul] \\ & a \arrow[ul] \arrow[ur] & \\ & 0 \arrow[u] & \end{tikzcd}$$

Then L is Rad-supplemented but not Rad$-\oplus-$supplemented.

Example 2.20 *Consider the lattice $L=\{1,a,b,c,d,e,0\}$ given by the following diagram;*

$$\begin{tikzcd}
 & 1 & & \\
a \arrow[ur] & & b \arrow[ul] & \\
 & c \arrow[ul] \arrow[ur] & & d \arrow[ul] \\
 & & e \arrow[ul] \arrow[ur] & \\
 & & 0 \arrow[u] &
\end{tikzcd}$$

Then L is Rad-supplemented but not Rad$-\oplus-$supplemented.

Example 2.21 *Consider the interval $[0,1]$ with natural topology. Let P be the set of all closed subsets of $[0,1]$. P is complete modular lattice by the inclusion (See [1, Example 2.10]). Here $\underset{i\in I}{\wedge} C_i = \underset{i\in I}{\cap} C_i$ and $\underset{i\in I}{\vee} C_i = \overline{\underset{i\in I}{\cup} C_i}$ for every $C_i \in P$ $(i\in I)$ $\left(\overline{\underset{i\in I}{\cup} C_i}\right.$ is the closure of $\left.\underset{i\in I}{\cup} C_i\right)$. By [4, Example 3], P is amply supplemented but not $\oplus$-supplemented. Since P is amply supplemented, then it is Rad-supplemented too. Since P is not $\oplus-$supplemented, it is not strongly Rad$-\oplus-$supplemented too. It is clear that $r(P)=[0,1]$ and hence P is Rad$-\oplus-$supplemented.*

References

[1] R. Alizade and E. Toksoy, Cofinitely Weak Supplemented Lattices, *Indian Journal of Pure and Applied Mathematics*, 40 No.5, 337-346 (2009).

[2] R. Alizade and E. Toksoy, Cofinitely Supplemented Modular Lattices, *Arabian Journal for Science and Engineering*, 36, 919-923 (2011).

[3] Ç. Biçer, C. Nebiyev and A. Pancar, Generalized Supplemented Lattices, *Miskolc Mathematical Notes*, 19 No.1, 141-147 (2018).

[4] Ç. Biçer and C. Nebiyev, $\oplus-$Supplemented Lattices, *Miskolc Mathematical Notes*, 20 No.2, 773-780 (2019).

[5] G. F. Birkenmeier, F. T. Mutlu, C. Nebiyev, N. Sokmez and A. Tercan, Goldie*-Supplemented Modules, *Glasgow Mathematical Journal*, 52A, 41–52 (2010).

[6] H. Çalışıcı and E. Türkmen, Generalized $\oplus-$Supplemented Modules, *Algebra and Discrete Mathematics*, 10 No.2, 10-18 (2010).

[7] G. Călugăreanu, *Lattice Concepts of Module Theory*, Kluwer Academic Publisher, Dordrecht, Boston, London, 2000.

[8] Ş. Ecevit, M. T. Koşan and R. Tribak, Rad$-\oplus-$Supplemented Modules and Cofinitely Rad$-\oplus-$Supplemented Modules, *Algebra Colloquium*, 19 No.4, 637-648 (2012).

[9] C. Nebiyev and H. H. Ökten, β_* Relation on Lattices, *Miskolc Mathematical Notes*, 18 No. 2, 993-999 (2017).

[10] Y. Talebi, A. R. M. Hamzekolaei and D. K. Tütüncü, On Rad$-\oplus-$Supplemented Modules, *Hadronic Journal*, 32, 505-512 (2009).

[11] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Philadelphia, 1991.