

# $\oplus - g$ -RADICAL SUPPLEMENTED MODULES

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## Abstract

In this work  $\oplus - g$ -radical supplemented modules are defined and some properties of these modules are investigated. It is proved that the finite direct sum of  $\oplus - g$ -radical supplemented modules is also  $\oplus - g$ -radical supplemented. Let  $M$  be a  $\oplus - g$ -radical supplemented  $R$ -module. If  $Rad_g M \ll_g M$ , then  $M$  is  $\oplus - g$ -supplemented. If  $Rad_g M \ll M$ , then  $M$  is  $\oplus$ -supplemented.

**Key words:** Essential Submodules, Small Submodules,  $g$ -Supplemented Modules,  $g$ -Radical Supplemented Modules.

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## 1 INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let  $R$  be a ring and  $M$  be an  $R$ -module. We will denote a submodule  $N$  of  $M$  by  $N \leq M$ . Let  $M$  be an  $R$ -module and  $N \leq M$ . If  $L = M$  for every submodule  $L$  of  $M$  such that  $M = N + L$ , then  $N$  is called a *small submodule* of  $M$  and denoted by  $N \ll M$ . Let  $M$  be an  $R$ -module and  $N \leq M$ . If there exists a submodule  $K$  of  $M$  such that  $M = N + K$  and  $N \cap K = 0$ , then  $N$  is called a *direct summand* of  $M$  and it is denoted by  $M = N \oplus K$ . For any  $R$ -module  $M$ , we have  $M = M \oplus 0$ . The intersection of all maximal submodules of  $M$  is called the *radical* of  $M$  and denoted by  $RadM$ . If  $M$  have no maximal submodules, then it is defined  $RadM = M$ .  $M$  is said to be *semilocal* if  $M/RadM$  is semisimple. A submodule  $N$  of an  $R$ -module  $M$  is called an *essential submodule* of  $M$  and denoted by  $N \leq_e M$  in case  $K \cap N \neq 0$  for every

submodule  $K \neq 0$ , or equivalently,  $K = 0$  for every  $K \leq M$  with  $N \cap K = 0$ . Let  $M$  be an  $R$ -module and  $K$  be a submodule of  $M$ .  $K$  is called a *generalized small* (or briefly, *g-small*) submodule of  $M$  if for every essential submodule  $T$  of  $M$  with the property  $M = K + T$  implies that  $T = M$ , then we write  $K \ll_g M$  (in [13], it is called an *e-small submodule* of  $M$  and denoted by  $K \ll_e M$ ). It is clear that every small submodule is a generalized small submodule but the converse is not true in general. Let  $M$  be an  $R$ -module.  $M$  is called a *hollow module* if every proper submodule of  $M$  is small in  $M$ .  $M$  is called a *generalized hollow* (or briefly, *g-hollow*) module if every proper submodule of  $M$  is g-small in  $M$ . Here it is clear that every hollow module is generalized hollow. The converse of this statement is not always true.  $M$  is called a *local module* if  $M$  has the largest submodule, i.e. a proper submodule which contains all other proper submodules.  $M$  is called a *generalized local* (briefly, *g-local*) if  $M$  has a large proper essential submodule which contain all proper essential submodules of  $M$  or  $M$  have no proper essential submodules. Let  $U$  and  $V$  be submodules of  $M$ . If  $M = U + V$  and  $V$  is minimal with respect to this property, or equivalently,  $M = U + V$  and  $U \cap V \ll V$ , then  $V$  is called a *supplement* of  $U$  in  $M$ .  $M$  is said to be *supplemented* if every submodule of  $M$  has a supplement in  $M$ . If every submodule of  $M$  has a supplement that is a direct summand in  $M$ , then  $M$  is called a  $\oplus$ -*supplemented module*. Let  $M$  be an  $R$ -module and  $U, V \leq M$ . If  $M = U + V$  and  $M = U + T$  with  $T \leq V$  implies that  $T = V$ , or equivalently,  $M = U + V$  and  $U \cap V \ll_g V$ , then  $V$  is called a *g-supplement* of  $U$  in  $M$ .  $M$  is said to be *g-supplemented* if every submodule of  $M$  has a g-supplement in  $M$ .  $M$  is said to be  $\oplus$ -*g-supplemented* if every submodule of  $M$  has a g-supplement that is a direct summand in  $M$  (see [10]). A module  $M$  is said to have the *Summand Sum Property* (*SSP*) if the sum of two direct summands of  $M$  is again a direct summand of  $M$  (see [12, Exercise 39.17 (3)]). We say that a module  $M$  has (*D3*) property if  $M_1 \cap M_2$  is a direct summand of  $M$  for every direct summands  $M_1$  and  $M_2$  of  $M$  with  $M = M_1 + M_2$  (see [8]). Let  $M$  be an  $R$ -module and  $U, V \leq M$ . If  $M = U + V$  and  $U \cap V \leq \text{Rad} V$ , then  $V$  is called a *generalized (radical) supplement* (briefly, *Rad-supplement*) of  $U$  in  $M$ .  $M$  is said to be *generalized (radical) supplemented* (briefly, *Rad-supplemented*) if every submodule of  $M$  has a Rad-supplement in  $M$ .  $M$  is said to be *generalized (radical)  $\oplus$ -supplemented* (briefly, *Rad- $\oplus$ -supplemented*) if every submodule of  $M$  has a Rad-supplement that is a direct summand in  $M$ . The intersection of all essential maximal submodules of an  $R$ -module  $M$  is called the *generalized radical* of  $M$  and denoted by  $\text{Rad}_g M$  (in [13], it is denoted by  $\text{Rad}_e M$ ). If  $M$  have no essential maximal submodules, then we denote  $\text{Rad}_g M = M$ . An  $R$ -module  $M$  is said to be *g-semilocal* if  $M/\text{Rad}_g M$  is semisimple (see [9]). Let  $M$  be an  $R$ -module and  $U, V \leq M$ . If  $M = U + V$  and  $U \cap V \leq \text{Rad}_g V$ , then  $V$  is called a *generalized radical supplement* (or briefly, *g-radical supplement*) of  $U$  in  $M$ .  $M$  is said to be *generalized radical supplemented* (briefly, *g-radical supplemented*) if every submodule of  $M$  has a g-radical supplement in  $M$ .

More informations about supplemented modules are in [1], [8] and [12]. More results about  $\oplus$ -supplemented modules are in [4], [5] and [8]. More details about generalized (radical) supplemented modules are in [11]. More details

about generalized (radical)  $\oplus$ -supplemented modules are in [2] and [3]. More informations about g-supplemented modules are in [6]. More informations about g-radical supplemented modules are in [7].

Now we will give some important properties of the generalized radical of any module.

**Lemma 1.1** *Let  $M$  be an  $R$ -module. The following conditions are hold.*

- (1)  $Rad_g M = \sum_{L \ll_g M} L$ .
- (2)  $Rm \ll_g M$  for every  $m \in Rad_g M$ .
- (3) If  $N \leq M$ , then  $Rad_g N \leq Rad_g M$ .
- (4) If  $K, L \leq M$ , then  $Rad_g K + Rad_g L \leq Rad_g (K + L)$ .
- (5) Let  $N$  be an  $R$ -module and  $f : M \longrightarrow N$  be an  $R$ -module homomorphism. Then  $f(Rad_g M) \leq Rad_g N$ .
- (6) If  $K, L \leq M$ , then  $\frac{Rad_g K + L}{L} \leq Rad_g \frac{K + L}{L}$ .
- (7) If  $M = \bigoplus_{i \in I} M_i$ , then  $Rad_g M = \bigoplus_{i \in I} Rad_g M_i$ .

**Proof.** See [7, Lemma 2, Lemma 3 and Lemma 4]. ■

## 2 $\oplus - g$ -RADICAL SUPPLEMENTED MODULES

**Definition 2.1** *Let  $M$  be an  $R$ -module. If every submodule of  $M$  has a g-radical supplement that is a direct summand of  $M$ , then  $M$  is called a  $\oplus - g$ -generalized radical supplemented (briefly,  $\oplus - g$ -radical supplemented) module.*

Clearly we can see that every  $\oplus - g$ -supplemented module is  $\oplus - g$ -radical supplemented, but the converse is not true in general (See Example 2.20 and Example 2.21). We also clearly can see that every  $\oplus$ -supplemented and every g-hollow modules are  $\oplus - g$ -radical supplemented. Let  $M$  be  $\oplus - g$ -radical supplemented  $R$ -module. Then  $M$  is g-radical supplemented and by [7, Theorem 1],  $M$  is g-semilocal.

**Lemma 2.2** *Let  $M$  be an  $R$ -module and  $M = M_1 \oplus M_2$ . If  $M_1$  and  $M_2$  are  $\oplus - g$ -radical supplemented, then  $M$  is also  $\oplus - g$ -radical supplemented.*

**Proof.** Let  $U$  be any submodule of  $M$ . Since  $M_2$  is  $\oplus - g$ -radical supplemented,  $(M_1 + U) \cap M_2$  has a g-radical supplement  $X$  that is a direct summand of  $M_2$ . Since  $X$  is a g-radical supplement of  $(M_1 + U) \cap M_2$  in  $M_2$ ,  $M_2 = (M_1 + U) \cap M_2 + X$  and  $(M_1 + U) \cap X = (M_1 + U) \cap M_2 \cap X \leq Rad_g X$ . By  $M_2 = (M_1 + U) \cap M_2 + X$ ,  $M = M_1 \oplus M_2 = M_1 + (M_1 + U) \cap M_2 + X = M_1 + U + X$ . Since  $M_1$  is  $\oplus - g$ -radical supplemented,  $(U + X) \cap M_1$  has a g-radical supplement  $Y$  that is a direct summand of  $M_1$ . Since  $Y$  is a g-radical supplement of  $(U + X) \cap M_1$  in  $M_1$ ,  $M_1 = (U + X) \cap M_1 + Y$  and  $(U + X) \cap Y = (U + X) \cap M_1 \cap Y \leq Rad_g Y$ . By  $M_1 = (U + X) \cap M_1 + Y$ ,  $M = M_1 + U + X = (U + X) \cap M_1 + Y + U + X =$

$U+X+Y$ . Since  $(M_1 + U) \cap X \leq \text{Rad}_g X$  and  $(U + X) \cap Y \leq \text{Rad}_g Y$ , by Lemma 1.1,  $U \cap (X + Y) \leq (U + Y) \cap X + (U + X) \cap Y \leq (M_1 + U) \cap X + (U + X) \cap Y \leq \text{Rad}_g X + \text{Rad}_g Y \leq \text{Rad}_g (X + Y)$ . Hence  $X + Y$  is a  $g$ -radical supplement of  $U$  in  $M$ . Since  $X$  is a direct summand of  $M_2$  and  $Y$  is a direct summand of  $M_1$ ,  $X + Y$  is a direct summand of  $M = M_1 \oplus M_2$ . Hence  $M$  is  $\oplus - g$ -radical supplemented. ■

**Corollary 2.3** *Let  $M$  be an  $R$ -module and  $M = M_1 \oplus M_2 \oplus \dots \oplus M_n$ . If  $M_i$  is  $\oplus - g$ -radical supplemented for every  $i = 1, 2, \dots, n$ , then  $M$  is also  $\oplus - g$ -radical supplemented.*

**Proof.** Clear from Lemma 2.2. ■

**Proposition 2.4** *Let  $M$  be a  $\oplus - g$ -radical supplemented  $R$ -module. If  $\text{Rad}_g M \ll_g M$ , then  $M$  is  $\oplus - g$ -supplemented.*

**Proof.** Let  $U \leq M$ . Since  $M$  is  $\oplus - g$ -radical supplemented,  $U$  has a  $g$ -radical supplement  $V$  that is a direct summand in  $M$ . Here  $M = U + V$  and  $U \cap V \leq \text{Rad}_g V$ . By Lemma 1.1,  $\text{Rad}_g V \leq \text{Rad}_g M \ll_g M$ . Since  $V$  is a direct summand of  $M$ , we can see that  $\text{Rad}_g V \ll_g V$ . Hence  $U \cap V \ll_g V$  and  $V$  is a  $g$ -supplement of  $U$  in  $M$ . Therefore,  $M$  is  $\oplus - g$ -supplemented. ■

**Proposition 2.5** *Let  $M$  be a  $\oplus - g$ -radical supplemented  $R$ -module. If  $\text{Rad}_g M \ll M$ , then  $M$  is  $\oplus -$ supplemented.*

**Proof.** Similar to proof of Proposition 2.4. ■

**Lemma 2.6** *Let  $M = M_1 \oplus M_2$  and  $X, Y \leq M_2$ . Then  $Y$  is a  $g$ -radical supplement of  $X$  in  $M_2$  if and only if  $Y$  is a  $g$ -radical supplement of  $M_1 + X$  in  $M$ .*

**Proof.** ( $\Rightarrow$ ) Since  $Y$  is a  $g$ -radical supplement of  $X$  in  $M_2$ ,  $M_2 = X + Y$  and  $X \cap Y \leq \text{Rad}_g Y$ . Then  $M = M_1 + M_2 = M_1 + X + Y$  and by Modular Law,  $(M_1 + X) \cap Y = (M_1 + X) \cap M_2 \cap Y = (M_1 \cap M_2 + X) \cap Y = (0 + X) \cap Y = X \cap Y \leq \text{Rad}_g Y$ . Hence  $Y$  is a  $g$ -supplement of  $M_1 + X$  in  $M$ .

( $\Leftarrow$ ) Since  $Y$  is a  $g$ -radical supplement of  $M_1 + X$  in  $M$ ,  $M = M_1 + X + Y$  and  $(M_1 + X) \cap Y \leq \text{Rad}_g Y$ . Then by Modular Law,  $M_2 = M_2 \cap M = M_2 \cap (M_1 + X + Y) = M_1 \cap M_2 + X + Y = 0 + X + Y = X + Y$  and  $X \cap Y \leq (M_1 + X) \cap Y \leq \text{Rad}_g Y$ . Hence  $Y$  is a  $g$ -radical supplement of  $X$  in  $M_2$ . ■

**Proposition 2.7** *Let  $M$  be a  $\oplus - g$ -radical supplemented module. If every  $g$ -radical supplement submodule in  $M$  is a direct summand of  $M$ , then every direct summand of  $M$  is  $\oplus - g$ -radical supplemented.*

**Proof.** Let  $N$  be a direct summand of  $M$  and  $M = N \oplus K$  with  $K \leq M$ . Since  $M$  is  $g$ -radical supplemented, by [7, Lemma 9],  $M/K$  is  $g$ -radical supplemented. By  $\frac{M}{K} = \frac{N \oplus K}{K} \cong \frac{N}{N \cap K} = \frac{N}{0} \cong N$ ,  $N$  is also  $g$ -radical supplemented. Let  $X \leq N$  and  $Y$  be a  $g$ -radical supplement of  $X$  in  $N$ . Since  $M = N \oplus K$ , by Lemma 2.6,

$Y$  is a  $g$ -radical supplement of  $K + X$  in  $M$ . Since every  $g$ -radical supplement submodule in  $M$  is a direct summand of  $M$ ,  $Y$  is a direct summand of  $M$ . By  $Y \leq N$ ,  $Y$  is also a direct summand of  $N$ . Hence  $N$  is  $\oplus - g$ -radical supplemented. ■

**Lemma 2.8** *Let  $M$  be a  $\oplus - g$ -radical supplemented  $R$ -module and  $K \leq M$ . If  $\frac{X+K}{K}$  is a direct summand of  $\frac{M}{K}$  for every direct summand  $X$  of  $M$ , then  $\frac{M}{K}$  is  $\oplus - g$ -radical supplemented.*

**Proof.** Let  $U/K$  be any submodule of  $M/K$ . Since  $M$  is  $\oplus - g$ -radical supplemented,  $U$  has a  $g$ -radical supplement  $X$  in  $M$  that is a direct summand in  $M$ . Since  $X$  is a  $g$ -radical supplement of  $U$  in  $M$  and  $K \leq U$ , by [7, Lemma 8],  $\frac{X+K}{K}$  is a  $g$ -radical supplement of  $U/K$  in  $M/K$ . Since  $X$  is a direct summand of  $M$ , by hypothesis,  $\frac{X+K}{K}$  is a direct summand of  $M/K$ . Hence  $M/K$  is  $\oplus - g$ -radical supplemented. ■

**Lemma 2.9** *Let  $M$  be a distributive and  $\oplus - g$ -radical supplemented  $R$ -module. Then every factor module of  $M$  is  $\oplus - g$ -radical supplemented.*

**Proof.** Let  $K \leq M$  and  $X$  be a direct summand of  $M$ . Since  $X$  is a direct summand of  $M$ , there exists  $Y \leq M$  such that  $M = X \oplus Y$ . Since  $M = X \oplus Y$ ,  $\frac{M}{K} = \frac{X+K}{K} + \frac{Y+K}{K}$ . Since  $M$  is distributive,  $(X+K) \cap (Y+K) = K$  and  $\frac{X+K}{K} \cap \frac{Y+K}{K} = \frac{(X+K) \cap (Y+K)}{K} = \frac{K}{K} = 0$ . Hence  $\frac{M}{K} = \frac{X+K}{K} \oplus \frac{Y+K}{K}$  and by Lemma 2.8,  $M/K$  is  $\oplus - g$ -radical supplemented. ■

**Corollary 2.10** *Let  $M$  be a distributive and  $\oplus - g$ -radical supplemented  $R$ -module. Then every homomorphic image of  $M$  is  $\oplus - g$ -radical supplemented.*

**Proof.** Clear from Lemma 2.9. ■

**Lemma 2.11** *Let  $M$  be an  $R$ -module and  $K$  be a direct summand of  $M$ . Then  $Rad_g K = K \cap Rad_g M$ .*

**Proof.** Since  $K$  is a direct summand of  $M$ , there exists  $T \leq M$  such that  $M = K \oplus T$ . By Lemma 1.1,  $Rad_g M = Rad_g K \oplus Rad_g T$ . Then by Modular Law,  $K \cap Rad_g M = K \cap (Rad_g K \oplus Rad_g T) = Rad_g K \oplus K \cap Rad_g T = Rad_g K \oplus 0 = Rad_g K$ , as desired. ■

**Lemma 2.12** *Let  $M$  be a  $\oplus - g$ -radical supplemented  $R$ -module with (D3) property. Then every direct summand of  $M$  is  $\oplus - g$ -radical supplemented.*

**Proof.** Let  $K$  be any direct summand of  $M$ . Then there exists  $T \leq M$  such that  $M = K \oplus T$ . Let  $U \leq K$ . Since  $M$  is  $\oplus - g$ -radical supplemented,  $U$  has a  $g$ -radical supplement  $X$  that is a direct summand in  $M$ . Here  $M = U + X$  and  $U \cap X \leq Rad_g X$ . Since  $U \leq K$ ,  $M = U + X = K + X$  and since  $M$  has (D3) property,  $K \cap X$  is a direct summand of  $M$ . Then there exists  $Y \leq M$  such that  $M = (K \cap X) \oplus Y$ . Here  $K = (K \cap X) \oplus (K \cap Y)$ . Since  $M = U + X$  and  $U \leq K$ , by Modular Law,  $K = U + (K \cap X)$ . Since  $U \cap X \leq Rad_g X \leq Rad_g M$ ,

$U \cap K \cap X \leq K \cap X \cap \text{Rad}_g M$ . Since  $K \cap X$  is a direct summand of  $M$ , by Lemma 2.11,  $U \cap K \cap X \leq K \cap X \cap \text{Rad}_g M = \text{Rad}_g(K \cap X)$ . Hence  $K \cap X$  is a  $g$ -radical supplement of  $U$  that is a direct summand in  $K$ . Therefore,  $K$  is  $\oplus - g$ -radical supplemented. ■

**Corollary 2.13** *Let  $M$  be a  $\oplus - g$ -radical supplemented  $R$ -module with (D3) property. Then  $M/X$  is  $\oplus - g$ -radical supplemented for every direct summand  $X$  of  $M$ .*

**Proof.** Let  $X$  be any direct summand of  $M$ . Then there exists  $Y \leq M$  such that  $M = X \oplus Y$ . By Lemma 2.12,  $Y$  is  $\oplus - g$ -radical supplemented. Then by  $\frac{M}{X} = \frac{X+Y}{X} \cong \frac{Y}{X \cap Y} = \frac{Y}{0} \cong Y$ ,  $M/X$  is also  $\oplus - g$ -radical supplemented. ■

**Corollary 2.14** *Let  $M$  be a  $\oplus - g$ -radical supplemented  $R$ -module with (D3) property and  $f : M \rightarrow N$  be an  $R$ -module epimorphism with  $N$  is an  $R$ -module and  $\text{Ker}(f)$  is a direct summand of  $M$ . Then  $N$  is  $\oplus - g$ -radical supplemented.*

**Proof.** Clear from Corollary 2.13, since  $M/\text{Ker}(f) \cong \text{Im}(f) = N$ . ■

**Lemma 2.15** *Let  $M$  be a  $\oplus - g$ -radical supplemented  $R$ -module,  $K \leq M$  and  $K = (K \cap M_1) \oplus (K \cap M_2)$  for every  $M_1, M_2 \leq M$  with  $M = M_1 \oplus M_2$ . Then  $M/K$  is  $\oplus - g$ -radical supplemented.*

**Proof.** Let  $U/K \leq M/K$ . Since  $M$  is  $\oplus - g$ -radical supplemented,  $U$  has a  $g$ -radical supplement  $V$  that is a direct summand in  $M$ . Here there exists  $X \leq M$  such that  $M = V \oplus X$ . By hypothesis,  $K = (K \cap V) \oplus (K \cap X)$ . Since  $V$  is a  $g$ -radical supplement of  $U$  in  $M$  and  $K \leq U$ , by [7, Lemma 8],  $\frac{V+K}{K}$  is a  $g$ -radical supplement of  $U/K$  in  $M/K$ . Since  $M = V \oplus X$ ,  $\frac{M}{K} = \frac{V+K}{K} + \frac{X+K}{K}$ . Here  $\frac{V+K}{K} \cap \frac{X+K}{K} = \frac{(V+K) \cap (X+K)}{K} = \frac{(V+K) \cap X + K}{K} = \frac{(V+K \cap V + K \cap X) \cap X + K}{K} = \frac{(V+K \cap X) \cap X + K}{K} = \frac{V \cap X + K \cap X + K}{K} = \frac{0 + K}{K} = \frac{K}{K} = 0$ . Hence  $\frac{M}{K} = \frac{V+K}{K} \oplus \frac{X+K}{K}$ . Thus  $M/K$  is  $\oplus - g$ -radical supplemented. ■

**Corollary 2.16** *Let  $M$  be a  $\oplus - g$ -radical supplemented  $R$ -module,  $f : M \rightarrow N$  be an  $R$ -module epimorphism with  $N$  be an  $R$ -module and  $\text{Ker}(f) = (\text{Ker}(f) \cap M_1) \oplus (\text{Ker}(f) \cap M_2)$  for every  $M_1, M_2 \leq M$  with  $M = M_1 \oplus M_2$ . Then  $N$  is  $\oplus - g$ -radical supplemented.*

**Proof.** Clear from Lemma 2.15, since  $M/\text{Ker}(f) \cong \text{Im}(f) = N$ . ■

**Lemma 2.17** *Let  $M$  be a  $\oplus - g$ -radical supplemented  $R$ -module with SSP property. Then  $M/K$  is  $\oplus - g$ -radical supplemented for every direct summand  $K$  of  $M$ .*

**Proof.** Let  $K$  be any direct summand of  $M$  and  $U/K \leq M/K$ . Since  $M$  is  $\oplus - g$ -radical supplemented,  $U$  has a  $g$ -radical supplement  $V$  in  $M$  such that  $V$  is a direct summand of  $M$ . By [7, Lemma 8],  $\frac{V+K}{K}$  is a  $g$ -radical supplement of  $U/K$  in  $M/K$ . Since  $K$  and  $V$  are direct summands of  $M$  and  $M$  has SSP

property,  $K + V$  is also a direct summand of  $M$ . Hence there exists  $T \leq M$  such that  $M = (K + V) \oplus T$ . Since  $M = (K + V) \oplus T$ ,  $\frac{M}{K} = \frac{K+V+T}{K} = \frac{V+K}{K} + \frac{T+K}{K}$ . Since  $(V + K) \cap T = 0$ ,  $\frac{V+K}{K} \cap \frac{T+K}{K} = \frac{(V+K) \cap (T+K)}{K} = \frac{(V+K) \cap T + K}{K} = \frac{0+K}{K} = 0$ . Hence  $\frac{M}{K} = \frac{V+K}{K} \oplus \frac{T+K}{K}$  and  $M/K$  is  $\oplus - g$ -radical supplemented. ■

**Corollary 2.18** *Let  $M$  be a  $\oplus - g$ -radical supplemented  $R$ -module with SSP property. Then every direct summand of  $M$  is  $\oplus - g$ -radical supplemented.*

**Proof.** Let  $T$  be any direct summand of  $M$ . Then there exists a submodule  $K$  of  $M$  such that  $M = T \oplus K$ . By Lemma 2.17,  $M/K$  is  $\oplus - g$ -radical supplemented. Since  $\frac{M}{K} = \frac{T+K}{K} \cong \frac{T}{T \cap K} = \frac{T}{0} \cong T$ ,  $T$  is also  $\oplus - g$ -radical supplemented. ■

**Remark 2.19** *Let  $M$  be an  $R$ -module which has only four proper submodules  $0, A, B, C$  with  $C \leq A, C \leq B, A \not\leq B$  and  $B \not\leq A$ . Then  $M$  is  $g$ -radical supplemented but not  $\oplus - g$ -radical supplemented.*

**Example 2.20** *Consider the  $\mathbb{Z}$ -module  $\mathbb{Q}$ . Since  $\text{Rad}_g \mathbb{Q} = \text{Rad} \mathbb{Q} = \mathbb{Q}$ ,  ${}_Z \mathbb{Q}$  is  $\oplus - g$ -radical supplemented. But, since  ${}_Z \mathbb{Q}$  is not supplemented and every nonzero submodule of  ${}_Z \mathbb{Q}$  is essential in  ${}_Z \mathbb{Q}$ ,  ${}_Z \mathbb{Q}$  is not  $g$ -supplemented and hence  ${}_Z \mathbb{Q}$  is not  $\oplus - g$ -supplemented.*

**Example 2.21** *Consider the  $\mathbb{Z}$ -module  $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$  for a prime  $p$ . It is easy to check that  $\text{Rad}_g \mathbb{Z}_{p^2} \neq \mathbb{Z}_{p^2}$ . By Lemma 1.1,  $\text{Rad}_g (\mathbb{Q} \oplus \mathbb{Z}_{p^2}) = \text{Rad}_g \mathbb{Q} \oplus \text{Rad}_g \mathbb{Z}_{p^2} \neq \mathbb{Q} \oplus \mathbb{Z}_{p^2}$ . Since  $\mathbb{Q}$  and  $\mathbb{Z}_{p^2}$  are  $\oplus - g$ -radical supplemented, by Lemma 2.2,  $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$  is  $\oplus - g$ -radical supplemented. But  $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$  is not  $\oplus - g$ -supplemented.*

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