Asymptotic expansions for eigenvalues of the Stokes system in the presence of small deformable inclusions

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February 7, 2020

Abstract

In this article, we provide a rigorous derivation of an asymptotic formula for the perturbation of eigenvalues associated to the Stokes eigenvalue problem with Dirichlet conditions and in the presence of small deformable inclusions. Taking advantage of the small sizes of the inclusions immersed in an incompressible Newtonian fluid having kinematic viscosity different from the background one, we show that our asymptotic formula can be expressed in terms of the eigenvalue in the absence of the inclusions and in terms of the so-called viscous moment tensor (VMT). We believe that our results are ambitious tools for determining the locations and/or shapes of small inhomogeneities by taking eigenvalue measurements.

Key words. Stokes eigenvalue problem, small deformable inclusion, asymptotic expansions, viscous moment tensor, oscillating boundary

2010 AMS subject classifications. 35P05, 35Q35, 35R30.

1 Introduction

Let Ω be a bounded open C^2-domain of $\mathbf{R}^d$ ($d = 2$ or 3) containing a Newtonian incompressible fluid. We suppose that Ω contains a finite number N of small deformable inclusions, each of the geometry form

$$B_\alpha^i := z_i + \alpha B_i, \quad 1 \leq i \leq N, \tag{1.1}$$

where α is the shared diameter and $B_i \subset \mathbf{R}^d$ is a bounded C^2-star-shaped domain containing the origin (e.g., it can represent the rescaled shape of a droplet). The total collection of inclusions thus takes the form $\mathcal{B}_\alpha = \bigcup_{i=1}^{N} B_\alpha^i$ and the points $z_i \in \Omega, i = 1, \ldots, N$, that determine the locations of the inclusions are assumed to satisfy

$$\begin{array}{ll} 0 < d_0 \leq |z_i - z_l| & \forall i \neq l \ , \\ 0 < d_0 \leq \mathrm{dist}(z_i, \partial\Omega) & \text{for } i = 1, \cdots, N, \end{array} \tag{1.2}$$

with d_0 is a predefined positive constant.

We denote by $\mu(x)$ the smooth background coefficient of kinematic viscosity of the fluid and we suppose that $0 < c_0 \leq \mu(x) \leq c_1 < +\infty, \forall x \in \Omega$, for some fixed constants c_0 and c_1. For

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simplicity, we assume that $\mu \in C^\infty(\overline{\Omega})$, but this condition could be considerably weakened. We assume that the subdomain B_α^i contains an isotropic Stokes fluid with kinematic viscosity μ_i such that $\mu(x) \neq \mu_i$, for $x \in \Omega$, and we set

$$\mu_\alpha(x) := \begin{cases} \mu(x) & x \in \Omega \setminus \overline{\mathcal{B}_\alpha}, \\ \mu_i & x \in B_\alpha^i, \end{cases} \tag{1.3}$$

where $\mu_i, i = 1, \ldots, N$ is a set of positive constants.

Here and in all the rest of this paper we denote by $\mathbf{I}$ the unit matrix in $R^{d\times d}$ and for a given two $(d \times d)$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ we denote by $A : B$ the contraction, i.e., $A : B = \sum_{ij} a_{ij} b_{ij}$.

Now, it is useful to introduce the stress tensor as follows

$$\sigma(u,p) := -p\mathbf{I} + 2\mu\mathbf{e}(u); \quad (u,p) \in H^1(\Omega)^d \times L^2(\Omega) \tag{1.4}$$

where u denotes the velocity fluid, the scalar function p is the pressure, μ is a given parameter representing the kinematic viscosity of the fluid and $\mathbf{e}(u)$ is the linear strain tensor for the flow:

$$\mathbf{e}(u) := \frac{1}{2}(\nabla u + (\nabla u)^T) = \Big(\frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})\Big)_{1\le i,j\le d}; \quad u = (u_1, \cdots, u_d) \in R^d. \tag{1.5}$$

Here and throughout the paper T denotes the transpose. Moreover, we assume that the divergence of a matrix-valued function A is denoted $\mathrm{Div}(A)$, while the divergence of a vector-valued function u is denoted $\mathrm{div}(u)$.

We consider the following eigenvalue problem for the Stokes system (see, for example,[18, 31]):

$$\begin{cases} -\mathrm{Div}(2\mu\mathbf{e}(v_0)) + \nabla p_0 = \lambda_0 v_0 & \text{in } \Omega, \\ \mathrm{div}(v_0) = 0 & \text{in } \Omega, \\ v_0 = 0 & \text{in } \partial\Omega, \\ \|v_0\| = 1 \end{cases} \tag{1.6}$$

where $(v_0, p_0, \lambda_0) \in (H^2(\Omega))^d \times H^1(\Omega) \times R_+^*$. Here we refer the reader to Boyer and Fabrie ([18], Theorem IV.5.8) for more details about the regularity properties. On the other hand, if S mens the Stokes operator, it is well know that (see for example, [7, 18, 29, 36]) there exist an orthogonal projection (the Leray projection) P such that:

$$S(v) = P(-\Delta v), \quad \forall v \in (H^2(\Omega))^d \cap H(\Omega),$$

where

$$H(\Omega) := \{v \in (H_0^1(\Omega))^d : \quad \mathrm{div}(v) = 0 \text{ in } \Omega\}.$$

In consequence, the Stokes operator enters the general framework about the well known spectral properties of $-\Delta$.
It is well known that the eigenvalue problem (1.6) admits a sequence of no decreasing positive eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$ tending to infinity as $n \to +\infty$.
The associated eigenfunctions $\{v_n\}_{n\ge1} \subset (H_0^1(\Omega))^d$ and eigenpressures $\{p_n\}_{n\ge1} \subset L^2(\Omega)$ may be taken so that $\{v_n\}_{n\ge1}$ constitutes an orthonormal basis of $H(\Omega)$. The pressure p is determined up to an additive constant.
On the other hand, it is well known that the spectrum of the Stokes operator is not always simple, as denoted by Ortega and Zuazua in [31]. This for instance is the case when the domain is a ball as justified by Watson ([37], pp. 124-127).

Let now μ_α be given by (1.3), our main purpose in this work is to develop rigourously asymptotic behaviors of the eigenvalues associated to the following (perturbed) eigenvalue problem with the Dirichlet boundary conditions:

$$\begin{cases} -\mathrm{Div}(2\mu_\alpha \mathbf{e}(v_\alpha)) + \nabla p_\alpha = \lambda_\alpha v_\alpha & \text{in } \Omega, \\ \mathrm{div}(v_\alpha) = 0 & \text{in } \Omega, \\ v_\alpha = 0 & \text{in } \partial\Omega, \\ \|v_\alpha\| = 1 \end{cases} \tag{1.7}$$

where $(v_\alpha, p_\alpha, \lambda_\alpha) \in (H^2(\Omega))^d \times H^1(\Omega) \times R_+^*$. The eigenvalue problem with the Neumann boundary conditions is of equal interest. The asymptotic results for this case can be obtained with only minor modifications of the techniques presented here.

The properties of eigenvalue problems under shape deformations have been a subject of comprehensive studies [1, 2, 11, 15, 14, 16, 23, 25, 27, 31, 33, 34, 35] and the area continues to carry great importance as in [3, 4, 5, 10, 24, 28, 26, 21, 20] and others. A significant portion of these investigations discusses the possibility of using asymptotic expansions of eigenvalues as an aid in identifying unknown small objects.
To the best of our knowledge, the present paper is an appropriate extension of those already derived in [11, 12, 15] for eigenvalues of Laplace operator or of Lamé system. It extends also the main achievements done for Laplace eigenvalue problem in a domain with oscillating boundary [2] and for the case of biharmonic operator in a domain with highly indented boundary [28].

The main objective of this paper is to find an asymptotic expansion for the Stokes operator eigenvalues of such a domain which has not been established before this work, with the aim of using the formula as help to locate the unknown inclusions. In other words, we would like to find a method similar to those elaborated in [4, 11, 12] to determine the locations, shape, and/or size of the small inclusions by taking eigenvalue measurements.
Note that the idea, to get the asymptotic formulas for the eigenvalues, is to use a result which provides estimates for the convergence of the eigenvalues of a sequence of self-adjoint compact operators. To the best of readers, this result may be at once obtained by applying a theorem of Osborn [32] to self-adjoint operators, with substantial simplifications. One can also see the works of Kato [23], for more details about the convergence of eigenvalues of a sequence of self-adjoint compact operators.

In this work we focus our attention to the eigenvalues of the Dirichlet boundary value problem in the presence of small isotropic inclusions, while the rigorous derivation of similar asymptotic formulas for the case of anisotropic subdomains requires further work. The general schematic way, presented in this paper, can be extended to other eigenvalue problems such as, eigenvalue problem for Dirichlet biharmonic operator which describes the characteristic vibrations of a clamped plate. Moreover, it can be extended to the quad-curl eigenvalue problem which has a fundamental importance for the analysis and computation of the electromagnetic interior transmission eigenvalues.

This paper is organized as follows. In Section 1, we introduce the adopted notations, we describe the geometry of the domain with its constitutive parameters and we formulate the perturbed Stokes eigenvalue problem with Dirichlet boundary conditions. In Section 2, we present the main results of this paper. In Section 3, we consider an associated Stokes boundary value problem in the presence of deformable inclusions B_α^i and using (for example) the Korn's inequality, we prove some convergence estimates where one of them is measured in a rescaled domain $\tilde{\Omega}$. These results will be useful to prove our main result. In Section 4, we end our paper by applying a theorem of Osborn [32] to establish the result which gives the asymptotic formula for the eigenvalues involving the viscous moment tensors. All these analyzes are of course taken when the shared diameter α approaches 0, and the deformable inclusions are separated from each other and do not touch $\partial\Omega$.

2 Main results

In this paper, we allow inhomogeneities with more general shapes such that deformable inclusions modeling droplets in Stokes flow and /or inclusions with highly oscillating boundaries. To state our main result, we introduce the fourth-order, symmetric, positive definite matrix $\mathbf{V}^{(i)}$ associated with the ith inhomogeneity, called viscous moment tensor [6]. For all $i = 1, \cdots, N$, the coefficients $\mathbf{V}^{(i)}_{klpq}$ of $\mathbf{V}^{(i)}$ are given by

$$\mathbf{V}^{(i)}_{klpq} := 2(\mu(z_i) - \mu_i)\Big(\frac{|B_i|}{2}(\delta_{kp}\delta_{lq} + \delta_{kq}\delta_{lp}) + \int_{\partial B_i} \mathbf{e}(\hat{\mathbf{v}}^{(i)}_{pq})_{kl} dy\Big); \quad 1 \le k, l, p, q \le d, \tag{2.8}$$

where for $1 \le p, q \le d$ the corrector $(\hat{\mathbf{v}}^{(i)}_{pq}, \hat{\pi}^{(i)})$ is the unique solution of:

$$\begin{cases} -\mathrm{Div}(2\mu\mathbf{e}(\hat{\mathbf{v}}^{(i)}_{pq})) + \nabla\hat{\pi}^{(i)} & = & 0 \quad \text{in} \quad \mathbf{R}^d \setminus \overline{B_i}, \\ -\mathrm{Div}(2\mu_i\mathbf{e}(\hat{\mathbf{v}}^{(i)}_{pq})) + \nabla\hat{\pi}^{(i)} & = & 0 \quad \text{in} \quad B_i, \\ \mathrm{div}(\hat{\mathbf{v}}^{(i)}_{pq}) & = & 0 \quad \text{in} \quad \mathbf{R}^d, \\ \hat{\mathbf{v}}^{(i)}_{pq}|_+ - \hat{\mathbf{v}}^{(i)}_{pq}|_- & = & 0 \quad \text{on} \quad \partial B_i, \\ (\hat{\pi}^{(i)}\nu + \mu_i\frac{\partial \hat{\mathbf{v}}^{(i)}_{pq}}{\partial\nu})\,|_- = (\hat{\pi}^{(i)}\nu + \mu(z_i)\frac{\partial \hat{\mathbf{v}}^{(i)}_{pq}}{\partial\nu})\,|_+ & & \text{on} \quad \partial B_i, \\ \hat{\mathbf{v}}^{(i)}_{pq}(y) = O(|y|^{-1}) & & \text{as} \quad |y| \to \infty, \\ \hat{\pi}^{(i)}(y) = O(|y|^{-2}) & & \text{as} \quad |y| \to \infty. \end{cases} \tag{2.9}$$

Here ν denotes the outward unit normal to ∂B_i; superscript $+$ and $-$ indicate the limiting values as we approach ∂B_i from outside B_i, and from inside B_i, respectively. δ_{pq} means the Kronecker's index and the conormal derivative $\frac{\partial u}{\partial \nu}$ is given by

$$\frac{\partial u}{\partial \nu} := (\nabla u + \nabla u^T)\nu. \tag{2.10}$$

Note that the viscous moment tensor depends on the coefficient of kinematic viscosity, size, and shape of the inclusion.

The first main result of this paper is the following derivation of the asymptotic expansion of $\overline{\lambda}^j_\alpha - \overline{\lambda}_0$ as $\alpha \to 0$, where Ω is a bounded region outside the total collection of inclusions $\mathcal{B}_\alpha$.

Theorem 1 *Let Ω be a bounded domain of R^d ($d = 2$ or 3) with boundary of class C^2. Suppose that we have (1.2)-(1.3) and let λ_0 be an eigenvalue of multiplicity m of (1.6), with an L^2 orthonormal basis of eigenfunctions $\{v^j_0\}_{1 \le j \le m}$. Suppose (λ^j_α) are eigenvalues of (1.7) which converge to λ_0. If we define each $\boldsymbol{V}^{(i)}$ by (2.8) for $i = 1 \dots N$, the following asymptotic expansion holds:*

$$\overline{\lambda}^j_\alpha - \overline{\lambda}_0 = \alpha^d \frac{1}{2\lambda^2} \sum_{j=1}^m \sum_{i=1}^N \boldsymbol{e}(v^j_0)(z_i) : \boldsymbol{V}^{(i)} \boldsymbol{e}(v^j_0)(z_i) + o(\alpha^d), \tag{2.11}$$

where $\overline{\lambda}^j_\alpha := \dfrac{1}{m} \displaystyle\sum_{j=1}^m \frac{1}{\lambda^j_\alpha}$ is the average of the set (λ^j_α), $\overline{\lambda}_0 = \dfrac{1}{\lambda_0}$ and the term $o(\alpha^d)$ depends on the separation d_0 but is otherwise independent of the location of the set of points $(z_i)^N_{i=1}$.

Suppose now that Ω contains single, bounded and smooth inhomogeneity B with C^∞-boundary ∂B but this regularity condition could be considerably weakened. In a neighborhood $\mathcal{O}$ of ∂B we introduce a system of natural curvilinear coordinates (n, s), where n is the distance to ∂B, taken with the minus sign inside of B, and s is the arclength on ∂B. We assume now that the background viscosity μ, defined in (1.3), is a constant function and equals to the positive constant μ_0. For $d = 2$, we suppose that Ω enclosed a deformable inhomogeneity B_α with rapidly oscillating boundary (see Fig. 1) defined by

$$\partial B_\alpha = \{x \in \mathcal{O} \subset \Omega : s \in \partial B, \quad n = \alpha^\gamma H(\frac{s}{\alpha}, s)\}, \tag{2.12}$$

where H is a profile function that is smooth relative to both variables, the *slow* variable s and the *fast* variable $\eta = s/\alpha$, and 1-periodic relative to η. Moreover α is a small parameter and γ is a quantity measuring the regularity of the boundary ∂B_α according to Kozlov and Nazarov [28].

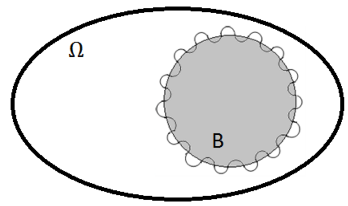

Figure 1: The domain Ω contains an inclusion with rapidly oscillating boundary. The initial inclusion B is tinted.

The viscosity of the subdomain B_α is assumed to be a positive constant $\mu_1 > 0$, and then introduce $\mu_{B_\alpha} := \mu_0 \chi_{(\Omega \setminus B_\alpha)} + \mu_1 \chi_{B_\alpha}$ where χ is the characteristic function.

The following result holds

Theorem 2 *Let $\Omega \subset R^2$ be a bounded, smooth domain containing the inhomogeneity B_α with rapidly oscillating boundary defined by (2.12). Let λ_0 be a simple eigenvalue of the problem (1.6) and associated to the eigenfunction v_0. Let $v_0^e = v_0|_{\Omega \setminus B}$ and $v_0^i = v_0|_{\overline{B}}$. Then, replacing μ_α by μ_{B_α}, there exists a simple eigenvalue of problem (1.7), denoted by λ_α such that $\lambda_\alpha \to \lambda_0$ as $\alpha \to 0$, and*

$$\lambda_\alpha - \lambda_0 = \alpha(1 - \frac{\mu_0}{\mu_1}) \int_{\partial B} \widetilde{H}(s) \boldsymbol{e}(v_0^e)(s,0) : \Big((\mu_1 \boldsymbol{e}(v_0^i)\tau) \otimes \tau + (\mu_0 \boldsymbol{e}(v_0^e)\nu) \otimes \nu \Big)(s,0) ds \qquad (2.13)$$

$$+ O(\alpha^{1+\beta})$$

for some positive β dependant on γ, and where $\widetilde{H}(s) = \int_0^1 H(\eta, s) d\eta$. Here, ν, τ are respectively the outward normal vector and the tangent vector to ∂B and the term $O(\alpha^{1+\beta})$ depends on the separation d_0, the quantity γ but is otherwise independent of H and the location of B_α.

Using the parameterization (n, s) of the neighborhood of the boundary ∂B, the Laplacian operator can be written as

$$\Delta_x := J(n,s)^{-1} \Big(\partial_n J(n,s) \partial_n + \partial_s J(n,s)^{-1} \partial_s \Big),$$

where $J(n,s) = 1 + n\kappa(s)$ is the Jacobian, and $\kappa(s)$ is the curvature of ∂B at the point s. Let $\hat{n}$ and $\hat{s}$ be the unit vectors of the axes n and s respectively, then the gradient operator can be given by

$$\nabla_x \cdot := \partial_n \cdot \hat{n} + J(n,s)^{-1} \partial_s \cdot \hat{s}.$$

For more details one can refer, for example, to [8, 28, 22]. We omit in this article the detailed proof of the Theorem 2, while the proof of the main result stated by Theorem 1 will be entirely given in Section 4.

3 Estimates for an associated boundary value problem

In this section we suppose that the small deformable inclusions having the shape (1.1) are immersed in the viscous fluid, and around which the fluid is flowing in a greater bounded domain Ω. It is well known that we make the assumption of small size of the objects to use asymptotic formulae which allows us to get a simple and precise reconstruction method. In connection with this idea, one can see the main works [9, 10, 13, 17, 30] for example. Eigenvalue expansions in the presence of small inclusions with Dirichlet or Neumann boundary conditions have been developed by Ammari et al. in a series of papers [11, 12, 15] for the Laplace operator and in [14] for the full Maxwell equations. The case of elastic eigenvalue problem it is attentively studied by Ammari et al. in [4] in terms of elastic moment tensors (EMT). We will use a similar approach, along with estimates for the convergence of the eigenvalues of a sequence of self-adjoint compact operators, to develop an asymptotic expansion for the eigenvalues in terms of viscous moment tensor (VMT).

Let f be in $L^2_0(\Omega)^d$. In this section we assume that the velocity field, v_α satisfies the following homogeneous Stokes problem, with Dirichlet boundary conditions

$$\begin{cases} -\mathrm{Div}(2\mu_\alpha \mathbf{e}(u_\alpha)) + \nabla p_\alpha = f & \text{in } \Omega \\ \mathrm{div}(u_\alpha) = 0 & \text{in } \Omega \\ u_\alpha = 0 & \text{on } \partial\Omega \end{cases} \tag{3.14}$$

Here the coefficient of kinematic viscosity μ_α is given by (1.3).

We know that as α approaches zero, the velocity field u_α and the pressure p_α converge respectively to the background velocity field u_0 and the pressure p_0 which satisfy:

$$\begin{cases} -\mathrm{Div}(2\mu \mathbf{e}(u_0)) + \nabla p_0 = f & \text{in } \Omega \\ \mathrm{div}(u_0) = 0 & \text{in } \Omega \\ u_0 = 0 & \text{in } \partial\Omega. \end{cases} \tag{3.15}$$

In the case of N star-shaped imperfections of type $z_i + \alpha B_i$ which are sufficiently separated from each other and the boundary, it has been shown in [17] (one can see [19] for the case of conductivity imperfections), that the following asymptotic formula holds when $d = 2$ or 3:

$$(u_\alpha - u_0)(y)_i = \alpha^d \sum_{j=1}^{N} \mathbf{e}_x(G_i)(z_j, y) : V^{(j)} \mathbf{e}_x(u_0)(z_j) + o(\alpha^{d+\frac{1}{2}}), \quad \text{for } 1 \le i \le d, \tag{3.16}$$

where z_j is the center of the ith inhomogeneity, and we have denoted by G_i the ith row of G, where $(G, F) \in H^1(\Omega)^{d^2} \times L^2_0(\Omega)^d$ denote the Green tensors associated with the reference system (3.15). Precisely, for a fixed $z \in \Omega$, $G(\cdot, z) = (G_{ij}(\cdot, z))_{1\le i,j\le d}$ and $F(\cdot, z) = (F_i(\cdot, z))_{1\le i\le d}$ are the solutions to

$$\begin{cases} -\mu_0 \Delta_x G_{ij}(x, z) + \dfrac{\partial F_i}{\partial x_j}(x, z) = \delta_{ij}\delta_z(x) & \text{in } \Omega \\ \displaystyle\sum_{j=1}^{d} \frac{\partial G_{ij}}{\partial x_j}(x, z) = 0 & \text{in } \Omega \\ G_{ij}(x, z) = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.17}$$

where of course δ_z is the Dirac mass at the point z while δ_{ij} is the Kronecker delta.

The remainder $o(\alpha^d)$ in (3.16) is independent of the set of points $(z_i)_{i=1}^N$ and signifies a term which goes to zero faster than α^d uniformly in y for y bounded away from the inhomogeneities. For more details we refer the readers to [17]. For the conductivity case, similar results have been shown rigorously in [9, 19]. Notice that the achievements in this section will help us to prove our main result in this paper, then we introduce firstly the following remark.

Remark 3.17 *Let v_0 and v_α solutions to (1.6), (1.7) respectively. Then by replacing f in (3.15) (resp. in (3.14)) by $\lambda_0 v_0$ (resp. by $\lambda_\alpha v_\alpha$), the estimation properties of $v_\alpha - v_0$ can be easily deduced from those of $u_\alpha - u_0$.*

Next we will prove, using Korn's inequality, the following lemma which describes the H^1-norm convergence of u_α to u_0.

Lemma 1 *Assume $d = 2$ or 3 and suppose that we have (1.2)-(1.3). Let u_α be the solution to (3.14) and u_0 the solution to (3.15) for a given $f \in L^2(\Omega)^d$. Then for some constant C, depending on u_0 but independent of α and the set of points $(z_i)_{i=1}^N$, the following estimate holds:*

$$\|u_\alpha - u_0\|_{H^1(\Omega)^d} \leq C\alpha^{\frac{d}{2}}.$$

Proof. Let u_α be the solution to (3.14) and u_0 the solution to (3.15) respectively. Expanding the following

$$\int_\Omega 2\mu_\alpha \mathbf{e}(u_0) : \mathbf{e}(u_\alpha - u_0)dx = \int_\Omega 2\mu(x)\mathbf{e}(u_0) : \mathbf{e}(u_\alpha - u_0)dx - \int_\Omega 2(\mu(x) - \mu_\alpha)\mathbf{e}(u_0) : \mathbf{e}(u_\alpha - u_0)dx.$$

An integration by parts yields

$$\begin{aligned} \int_\Omega 2\mu_\alpha \mathbf{e}(u_0) : \mathbf{e}(u_\alpha - u_0)dx &= \int_{\partial\Omega} (2\mu(x)\mathbf{e}(u_0) - p_0\mathbf{I})\nu_x \cdot (u_\alpha - u_0)d\sigma(x) \\ &\quad -2\sum_{i=1}^N \int_{B_\alpha^i} (\mu(x) - \mu_i)\mathbf{e}(u_0) : \mathbf{e}(u_\alpha - u_0)dx \\ &= -2\sum_{i=1}^N \int_{B_\alpha^i} (\mu(x) - \mu_i)\mathbf{e}(u_0) : \mathbf{e}(u_\alpha - u_0)dx. \end{aligned} \tag{3.18}$$

On the other hand, choosing $u_\alpha - u_0$ as a test function in (3.14) yields

$$\int_\Omega 2\mu_\alpha \mathbf{e}(u_\alpha) : \mathbf{e}(u_\alpha - u_0)dx = \int_{\partial\Omega} (2\mu_\alpha \mathbf{e}(u_\alpha) - p_\alpha \mathbf{I})\nu_x \cdot (u_\alpha - u_0)d\sigma(x) = 0. \tag{3.19}$$

The last relation was deduced by using the Dirichelet conditions on $\partial\Omega$ and the fact that $\mathrm{div}(u_\alpha) = 0$.
Using equalities (3.18) and (3.19), we immediately obtain

$$\begin{aligned} \int_\Omega 2\mu_\alpha |\mathbf{e}(u_\alpha - u_0)|^2 dx &= -\int_\Omega 2\mu_\alpha \mathbf{e}(u_0) : \mathbf{e}(u_\alpha - u_0)dx + \int_\Omega 2\mu_\alpha \mathbf{e}(u_\alpha) : \mathbf{e}(u_\alpha - u_0)dx \\ &= 2\sum_{i=1}^N \int_{B_\alpha^i} (\mu(x) - \mu_i)\mathbf{e}(u_0) : \mathbf{e}(u_\alpha - u_0)dx. \end{aligned}$$

But, the Cauchy-Schwarz inequality and the fact that vector valued function u_0 is bounded in Ω shows that

$$\begin{aligned} \int_\Omega 2\mu_\alpha |\mathbf{e}(u_\alpha - u_0)|^2 dx &\leq \sum_{i=1}^N \int_{B_\alpha^i} |\mu(x) - \mu_i| \|\mathbf{e}(u_0)\|_{L^2(B_\alpha^i)^{d^2}} \|\mathbf{e}(u_\alpha - u_0)\|_{L^2(B_\alpha^i)^{d^2}} \\ &\leq C\alpha^{d/2} \|\mathbf{e}(u_\alpha - u_0)\|_{L^2(\Omega)^{d^2}}, \end{aligned}$$

which, by invoking the Korn inequality [29], yields the desired result. □

Now we make the change of variables $y := \frac{x-x_i}{\alpha}$, and we set $\widetilde{\Omega} = \frac{1}{\alpha}\Omega$. Note, here we use similar approach as done in [12, 19] for the conductivity problems and in [17] for the case of a dilute suspension of droplets with interfacial tension. Let $(V,Q) \in H^1(R^d)^d_{\text{loc}} \times L^2(R^d)^d_{\text{loc}}$ be the unique solution to

$$\begin{cases} -\mathrm{Div}_y(2\mu \mathbf{e}_y(V)) + \nabla_y Q & = 0 \quad \text{in} \quad \mathbf{R}^n \setminus \overline{B_i}, \\ -\mathrm{Div}_y(2\mu_i \mathbf{e}_y(V)) + \nabla_y Q & = 0 \quad \text{in} \quad B_i, \\ \mathrm{div}_y(V) & = 0 \quad \text{in} \quad \mathbf{R}^n, \\ \quad V|_+ - V|_- & = 0 \quad \text{on} \quad \partial B_i, \\ (Q\nu + \mu(z_i)\frac{\partial V}{\partial \nu}) \mid_+ - (Q\nu + \mu_i \frac{\partial V}{\partial \nu}) \mid_- & = -(\mu(z_i) - \mu_i)\mathbf{e}_x(u_0(z_i))\nu \qquad \text{on} \ \partial B_i, \\ \quad V(y) & = O(|y|^{-1}) \text{ as } |y| \to \infty, \\ \quad Q(y) & = O(|y|^{-2}) \text{ as } |y| \to \infty, \end{cases} \tag{3.20}$$

where the subscripts $+$ and $-$ indicate the limits from outside and inside of B_i, respectively.

The reader may be referred, for example, to [17] for the existence and uniqueness of V by using single layer potentials with suitably chosen densities.

The following result will be useful to develop the asymptotic expansion of the eigenvalues.

Theorem 3 *Assume $d = 2$ or 3 and suppose that we have (1.2)-(1.3). Let u_α be the solution to (3.14) and u_0 the solution to (3.15) for a given $f \in L^2(\Omega)^d$. Then for some constant C, depending on u_0 but independent of α and the set of points $(z_i)_{i=1}^N$, the following estimate holds:*

$$\| \boldsymbol{e}_y \left(u_\alpha(z_i + \alpha y) - u_0(z_i + \alpha y) - \alpha V(y) \right) \|_{L^2(\tilde{\Omega})^{d^2}} \leq C \alpha^{\frac{3}{2}}. \tag{3.21}$$

Proof. In order to verify the above estimate, we introduce firstly $(V_\alpha, Q_\alpha) \in H^1(\widetilde{\Omega})^d \times L^2_0(\widetilde{\Omega})$ to be the unique solution to

$$\begin{cases} -\mathrm{Div}_y(2\mu \mathbf{e}_y(V_\alpha)) + \nabla_y Q_\alpha & = 0 \quad \text{in} \quad \widetilde{\Omega}, \\ \mathrm{div}(V_\alpha) & = 0 \quad \text{in} \quad \widetilde{\Omega}, \\ V_\alpha & = 0 \quad \text{on} \quad \partial\widetilde{\Omega}. \end{cases} \tag{3.22}$$

We set $R_\alpha(y) := u_\alpha(z_i + \alpha y) - u_0(z_i + \alpha y) - \alpha V_\alpha(y)$ for $y \in \widetilde{\Omega}$. Integration by parts gives

$$\begin{aligned} \int_{\widetilde{\Omega}} \mu |\mathbf{e}_y(R_\alpha(y))|^2 dy \quad = \quad & - \int_{\widetilde{\Omega}} \mu \mathbf{e}_y(R_\alpha(y)) : \mathbf{e}_y(u_0(z_i + \alpha y)) dy \\ & + \int_{\widetilde{\Omega}} \mu \mathbf{e}_y(R_\alpha(y)) : \mathbf{e}_y(u_\alpha(z_i + \alpha y)) dy \\ & - \alpha \int_{\widetilde{\Omega}} \mu \mathbf{e}_y(R_\alpha(y)) : \mathbf{e}_y(V_\alpha(y)) dy. \end{aligned} \tag{3.23}$$

For the second term, in the right-hand side of the previous equality, one may use change of variables and integration by parts to find that

$$\begin{aligned} - \int_{\widetilde{\Omega}} \mu \mathbf{e}_y(R_\alpha(y)) : \mathbf{e}_y(u_0(z_i + \alpha y)) dy \quad & = \quad -\alpha^{2-d} \int_{\Omega} \mu \mathbf{e}_x(R_\alpha(\frac{x - z_i}{\alpha})) : \mathbf{e}_x(u_0(x)) dx \\ & = \quad -\alpha^{2-d} \int_{\Omega} \mu(z_i) \mathbf{e}_x(R_\alpha(\frac{x - z_i}{\alpha})) : \mathbf{e}_x(u_0(x)) dx \\ & \qquad -\alpha^{2-d} \sum_{i=1}^N \int_{B_\alpha^i} \big(\mu_i - \mu(z_i)\big) \mathbf{e}_x(R_\alpha(\frac{x - z_i}{\alpha})) : \mathbf{e}_x(u_0(x)) dx \\ & = \quad \alpha^2 \sum_{i=1}^N \int_{B_i} \big(\mu(z_i) - \mu_i\big) \mathbf{e}_y(R_\alpha(y)) : \mathbf{e}_x(u_0(z_i + \alpha y)) dy. \end{aligned} \tag{3.24}$$

On the other hand, since R_α is divergence free and and $R_\alpha(\frac{x-z_i}{\alpha})|_{\partial\Omega} = 0$, we may use integration by parts to obtain

$$\begin{aligned}
\int_{\widetilde{\Omega}} \mu \mathbf{e}_y(R_\alpha(y)) : \mathbf{e}_y(u_\alpha(z_i+\alpha y))dy &= \alpha^{2-d}\int_{\Omega} \mu \mathbf{e}_x(R_\alpha(\frac{x-z_i}{\alpha})) : \mathbf{e}_x(u_\alpha(x))dx \\
&= \alpha^{2-d}\int_{\partial\Omega} \mu \mathbf{e}_x(R_\alpha(\frac{x-z_i}{\alpha}))\cdot(2\mu\mathbf{e}_x(u_\alpha) - p_\alpha \mathbf{I})\nu_x d\sigma(x) \\
&= 0. \qquad (3.25)
\end{aligned}$$

Using now R_α as a test function in (3.20), we may use integration by parts again to find that

$$-\alpha\int_{\widetilde{\Omega}} \mu \mathbf{e}_y(R_\alpha(y)) : \mathbf{e}_y(V_\alpha(y))dy = 0. \qquad (3.26)$$

Inserting all equalities (3.24), (3.25) and (3.26) into (3.23), the following holds

$$\int_{\widetilde{\Omega}} \mu|\mathbf{e}_y(R_\alpha(y))|^2 dy = \alpha^2\sum_{i=1}^{N}\int_{B_i}\big(\mu(z_i)-\mu_i\big)\mathbf{e}_y(R_\alpha(y)) : \mathbf{e}_x(u_0(z_i+\alpha y))dy. \qquad (3.27)$$

But u_0 is a bounded vector-valued function in Ω, therefore the Cauchy-Schwarz inequality applied to (3.27) immediately gives

$$\|\mathbf{e}_y(u_\alpha(z_i+\alpha y) - u_0(z_i+\alpha y) - \alpha V_\alpha(y))\|_{L^2(\widetilde{\Omega})^d} := \|\mathbf{e}_y(R_\alpha)\|_{L^2(\widetilde{\Omega})^{d^2}} \le C\alpha^2. \qquad (3.28)$$

Next, if we set

$$U_\alpha := V_\alpha - V \quad \text{and } \Pi_\alpha := Q_\alpha - Q,$$

where V_α, V are the solutions of (3.22), (3.20) respectively, we get $(U_\alpha, \Pi_\alpha) \in H^1(\widetilde{\Omega})^d \times L^2_0(\widetilde{\Omega})$ and

$$\left\{\begin{array}{lcll}
-\mathrm{Div}_y(2\mu\mathbf{e}_y(U_\alpha)) + \nabla_y \Pi_\alpha &=& 0 & \text{in } \widetilde{\Omega}, \\
\mathrm{div}(U_\alpha) &=& 0 & \text{in } \widetilde{\Omega}, \\
U_\alpha &=& -V & \text{on } \partial\widetilde{\Omega}.
\end{array}\right. \qquad (3.29)$$

Integration by parts immediately gives

$$\begin{aligned}
\int_{\widetilde{\Omega}} 2\mu\mathbf{e}_y(U_\alpha(y)) : \mathbf{e}_y(U(y))dy &= \int_{\partial\widetilde{\Omega}} V\cdot(2\mu\mathbf{e}_y(U_\alpha) - \Pi_\alpha\mathbf{I})\nu_y d\sigma(y) \\
&= -\alpha^{1-d}\int_{\partial\Omega} V(\frac{x-z_i}{\alpha})\cdot(2\alpha\mu\mathbf{e}_x(U_\alpha)(\frac{x-z_i}{\alpha}) \\
&\quad -\Pi_\alpha(\frac{x-z_i}{\alpha})\mathbf{I})\nu_x d\sigma(x). \qquad (3.30)
\end{aligned}$$

Using Cauchy-Schwarz inequality, and the corresponding decay of V (see for example Appendix B in [17]) one may compute that

$$\Big|\int_{\widetilde{\Omega}} 2\mu\mathbf{e}_y(U_\alpha(y)) : \mathbf{e}_y(U(y))dy\Big| \le C\alpha^{1/2}\|\mathbf{e}_y(U_\alpha(y))\|_{L^2(\widetilde{\Omega})^{d^2}}.$$

Thus,

$$\|\mathbf{e}_y(V_\alpha - V)\|_{L^2(\widetilde{\Omega})^{d^2}} = \|\mathbf{e}_y(U_\alpha(y))\|_{L^2(\widetilde{\Omega})^{d^2}} \le C\alpha^{1/2}. \qquad (3.31)$$

To establish the theorem, it now suffices to remark that

$$u_\alpha(z_i+\alpha y) - u_0(z_i+\alpha y) - \alpha V(y) = u_\alpha(z_i+\alpha y) - u_0(z_i+\alpha y) - \alpha V_\alpha(y) + \alpha(V_\alpha - V)(y)$$

and to use both relations (3.28) and (3.31). $\square$

4 Asymptotic formulas for the eigenvalues

In this section we may detail the proof of Theorem 1, while the proof of Theorem 2 is dedicated to a separately work [22]. Our method is deeply based on the theorem of Osborn [32] which gives estimates for the convergence of the eigenvalues of a sequence of self-adjoint compact operators. For simplicity we will not state the theorem in its full generality, and we refer the reader to [12, 25] for example, for considerable simplifications and applications of this theorem. A simplified statement of the Osborn's Theorem is as follows: Let $\mathcal{H}$ be a (real) Hilbert space and suppose we have a compact, self-adjoint linear operator $T : \mathcal{H} \to \mathcal{H}$ along with a sequence of compact, self-adjoint linear operators $T_\alpha : \mathcal{H} \to \mathcal{H}$ such that $T_\alpha \to T$ pointwise as $\alpha \to 0$ and the sequence $\{T_\alpha\}$ is collectively compact. Let ω_0 be a nonzero eigenvalue of T of multiplicity m. Then we know that for small α, each T_α has a set of m eigenvalues counted according to multiplicity, $\{\omega_\alpha^1, \dots, \omega_\alpha^m\}$ such that for each j, $\omega_\alpha^j \to \omega_0$ as $\alpha \to 0$. Define the average

$$\overline{\omega}_\alpha = \frac{1}{m}\sum_{j=1}^{m} \omega_\alpha^j.$$

If $\psi^1, \psi^2, \dots, \psi^m$ is an orthonormal basis of eigenfunctions associated with the eigenvalue ω_0, then there exists a constant C such that for $j = 1, \dots, m$ the following estimate holds:

$$\big|\omega_0 - \overline{\omega}_\alpha - \frac{1}{m}\sum_{j=1}^{m}\langle (T - T_\alpha)\psi^j, \psi^j\rangle\big| \le C \|(T - T_\alpha)|_{\mathrm{span}\{\psi^j\}_{1\le j\le m}}\|^2, \tag{4.32}$$

where $(T - T_\alpha)|_{\mathrm{span}\{\psi^j\}_{1\le j\le m}}$ denotes the restriction of $(T - T_\alpha)$ to the m-dimensional vector space spanned by $\{\psi^j\}_{1\le j\le m}$. The above estimate (4.32) is slightly different from what is stated in the theorem in [32].

For our case, we let $\mathcal{H}$ be $L^2(\Omega)^d$ with the standard inner product. For any $f \in L^2(\Omega)^d$, define

$$T_\alpha f = u_\alpha, \tag{4.33}$$

where v_α is the solution of (1.7), and

$$Tf = v_0, \tag{4.34}$$

where v_0 is the solution to (1.6). Clearly $T (:= T_0)$ and $\{T_\alpha\}_{\alpha>0}$ are linear and self-adjoint operators from $L^2(\Omega)^d$ to $L^2(\Omega)^d$. Using Korn and Poincar'e inequalities, we may prove the next result which is useful to apply Osborn's Theorem in order to prove Theorem 1. We recall the reader the properties of $\{T_\alpha\}$ are proved before in [9] but for the Stokes operator, it is introduced as follows for the first time.

Proposition 4.1 *Let $\alpha > 0$ and T_α, T be defined by (4.33), (4.34) respectively. Then the family of operators $\{T_\alpha\}$ is collectively compact such that $T_\alpha \to T$ pointwise as $\alpha \to 0$.*

Proof. To prove that T_α is a compact operator, we may use standard energy estimates based on Korn and Poincaré inequalities. For all $\alpha \ge 0$ we have

$$\|T_\alpha f\|_{H^1(\Omega)^d} = \|v_\alpha\|_{H^1(\Omega)^d} \le C\|\mathbf{e}(v_\alpha)\|_{L^2(\Omega)^{d^2}} \le C\|f\|_{L^2(\Omega)^d},$$

where the constant C is independent of α. Moreover, due to the fact that the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact, we conclude that the perturbed operator T_α is compact. The compactness of T follows exactly by same arguments.

Now, since the constant C is independent of α, the sequence of operators $(T_\alpha)_{\alpha\ge 0}$ is collectively compact. To show that $T_\alpha \to T$ pointwise as $\alpha \to 0$, one may prove that $T_\alpha f$ converges to Tf in $L^2(\Omega)^d$ for every $f \in L^2(\Omega)^d$. In the presence of the total collection $\mathcal{B}_\alpha$ satisfying (1.1)-(1.2), we set $u_\alpha = T_\alpha f$ and $u_0 = Tf$. For any $w \in H_0^1(\Omega)^d$, we have:

$$\int_\Omega \mathrm{Div}(\mu_\alpha \mathbf{e}(u_\alpha) - p_\alpha \mathbf{I}).w = \int_\Omega f.w.$$

So,

$$\int_{\Omega}(\mu_\alpha \mathbf{e}(u_\alpha) - p_\alpha \mathbf{I}) : \mathbf{e}(w) = -\int_{\Omega} f.w.$$

Similarly,

$$\int_{\Omega}(\mu \mathbf{e}(u_0) - p_0 \mathbf{I}) : \mathbf{e}(w) = -\int_{\Omega} f.w.$$

Consequently, choosing $w = u_\alpha - u_0$ and subtracting these two equations we get

$$\int_{\Omega}\big(\mu_\alpha \mathbf{e}(u_\alpha) - \mu \mathbf{e}(u_0) - (p_\alpha - p_0)\mathbf{I}\big) : \mathbf{e}(u_\alpha - u_0) = 0,$$

which gives

$$\int_{\Omega} \mu_\alpha \mathbf{e}(u_\alpha - u_0) : \mathbf{e}(u_\alpha - u_0) = -\int_{\Omega}(\mu_\alpha - \mu)\mathbf{e}(u_0) : \mathbf{e}(u_\alpha - u_0).$$

Hence, by using successively Korn inequality and Hölder's one, we may obtain

$$\|\mathbf{e}(u_\alpha - u_0)\|_{L^2(\Omega)^{d^2}} \leq C\|\mathbf{e}(u_0)\|_{L^2(\mathcal{B}_\alpha)^{d^2}}.$$

It then follows by the Poincaré inequality that

$$\|T_\alpha f - Tf\|_{H^1(\Omega)^d} = \|u_\alpha - u_0\|_{H^1(\Omega)^d} \leq C\|\mathbf{e}(u_0)\|_{L^2(\mathcal{B}_\alpha)^{d^2}}.$$

Consequently, using the last inequality and the fact that $|\mathcal{B}_\alpha| \to 0$ as $\alpha \to 0$ and that u_0 is a smooth vector-valued function in Ω, we obtain that $T_\alpha \to T$ pointwise as $\alpha \to 0$ in $L^2(\Omega)^d$. $\square$

Now we can proceed with the proof of Theorem 1 as follows.

Proof of Theorem 1. By means of Proposition 4.1, we see that the family of operators $\{T_\alpha\}$ is collectively compact. For this reason all hypotheses hold for writing (4.32). Let (v_0, λ_0), and $(v_\alpha, \lambda_\alpha)$ be normalized eigenpairs of (1.6) and (1.7) respectively. Then according to (4.34), we have

$$\begin{cases} -\mathrm{Div}(2\mu \mathbf{e}(v_0)) + \nabla p_0 = f & \text{in } \Omega \\ \mathrm{div}(v_0) = 0 & \text{in } \Omega \\ v_0 = 0 & \text{in } \partial\Omega, \end{cases} \tag{4.35}$$

and by (4.33), we have

$$\begin{cases} -\mathrm{Div}(2\mu_\alpha \mathbf{e}(v_\alpha)) + \nabla p_\alpha = f & \text{in } \Omega \\ \mathrm{div}(v_\alpha) = 0 & \text{in } \Omega \\ v_\alpha = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.36}$$

Notice that, the boundary value problems (4.36) and (4.35) are similar to those (3.14) and (3.15) respectively. Then all estimates established for the difference $u_\alpha - u_0$, in Section 3, hold exactly for $v_\alpha - v_0$.

Now if we set

$$\omega_0 = \frac{1}{\lambda_0} \quad \text{and } \omega_\alpha = \frac{1}{\lambda_\alpha},$$

then according to problems (4.36) and (4.35) one can see that $(v_\alpha, \omega_\alpha)$ and (v_0, ω_0) are the normalized eigenpairs of T_α and T respectively. Throughout this proof we suppose that λ_0 is an eigenvalue of (1.6) with multiplicity m and with a corresponding set of orthonormal eigenfunctions $\{v_0^j\}_{1\leq j\leq m}$. Since the set $\{v_0^j\}$ is orthonormal,

$$\|(T - T_\alpha)|_{\mathrm{span}\{v_0^j\}_{1\leq j\leq m}}\| = \max_j \|Tv_0^j - T_\alpha v_0^j\|_{L^2(\Omega)^d}. \tag{4.37}$$

Now we need to express the term $T_\alpha v_0^j$, but one may use the definition of T_α to see that there exists a vector-valued function v_α^j solving the following boundary value problem

$$\left\{\begin{array}{ll} -\mathrm{Div}(2\mu_\alpha \mathbf{e}(v_\alpha^j)) + \nabla p_\alpha = v_0^j & \text{in } \Omega \\ \mathrm{div}(v_\alpha^j) = 0 & \text{in } \Omega \\ v_\alpha^j = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{4.38}$$

such that

$$T_\alpha v_0^j = v_\alpha^j.$$

Then (4.37) it transforms to,

$$\|(T - T_\alpha)|_{\mathrm{span}\{v_0^j\}_{1\le j\le m}}\| = \max_j \|\frac{1}{\lambda_0} v_0^j - v_\alpha^j\|_{L^2(\Omega)^d}.$$

On the other hand, since v_0^j satisfies (1.6), we can use directly Lemma 1 to obtain the estimate

$$\|\frac{1}{\lambda_0} v_0^j - v_\alpha^j\|_{L^2(\Omega)^d} \le C\alpha^{\frac{d}{2}}.$$

Consequently, inserting all of this information into the relation (4.32), the following holds

$$\frac{1}{\lambda_0} - \frac{1}{m}\sum_{j=1}^m \frac{1}{\lambda_\alpha^j} = \frac{1}{m}\sum_{j=1}^m \langle \frac{1}{\lambda_0} v_0^j - v_\alpha^j, v_0^j\rangle + O(\alpha^d). \tag{4.39}$$

We focus our attention now on the term

$$\langle \frac{1}{\lambda_0} v_0^j - v_\alpha^j, v_0^j\rangle := \int_\Omega \left(\frac{1}{\lambda_0} v_0^j - v_\alpha^j\right) v_0^j dx.$$

Since v_0^j solves (1.6), we immediately get

$$\begin{aligned} \langle \frac{1}{\lambda_0} v_0^j - v_\alpha^j, u^j\rangle &= \int_\Omega \left(\frac{1}{\lambda_0} v_0^j - v_\alpha^j\right) \nabla\cdot\gamma\nabla\left(\frac{1}{\lambda_0} v_0^j\right) dx \qquad (4.40) \\ &= \int_{\Omega\setminus\mathcal{B}_\alpha} \left(\frac{1}{\lambda_0} v_0^j - v_\alpha^j\right) \nabla\cdot\gamma\nabla\left(\frac{1}{\lambda_0} v_0^j\right) dx + \int_{\mathcal{B}_\alpha} \left(\frac{1}{\lambda_0} v_0^j - v_\alpha^j\right) \nabla\cdot\gamma\nabla\left(\frac{1}{\lambda_0} v_0^j\right) dx. \end{aligned}$$

Let's use integration by parts twice on $\Omega\setminus B_\alpha$ and on B_α successively, taking into account that v_0^j and v_α^j are already two solutions of (1.6) and (4.38) respectively, and recall that $\mathcal{B}_\alpha := \bigcup_{i=1}^N (z_i + \alpha B_i) = \bigcup_{i=1}^N B_\alpha^i$. Then, by carefully calculus, the following holds

$$\begin{aligned} \langle \frac{1}{\lambda_0} v_0^j - v_\alpha^j, v_0^j\rangle &= \frac{1}{\lambda_0}\sum_{i=1}^N \int_{\partial B_\alpha^i} \left(\left(2\mu(z_i)\mathbf{e}(v_\alpha^j) - p\mathbf{I}\right)\nu_x|_- - \left(2\mu(x)e(v_\alpha^j) - p\mathbf{I}\right)\nu_x|_+\right)\cdot v_0^j d\sigma_x \\ &+ \frac{1}{\lambda_0}\sum_{i=1}^N \int_{B_\alpha^i} (\mu(x) - \mu(z_i))\,\mathbf{e}(v_\alpha^j) : \mathbf{e}(v_0^j) dx \\ &+ \sum_{i=1}^N \frac{\mu_i - \mu(z_i)}{\mu_i\lambda_0} \int_{B_\alpha^i} |v_0^j|^2 dx. \end{aligned}$$

Now, by the change of variables $y = \frac{x-z_i}{\alpha}$ and the fact that v_0^j is bounded, we can see easily that the last term above is of the order α^{d+1}. Hence by the conjugation condition for v_α^j this gives us

$$\begin{aligned}\langle\frac{1}{\lambda_0}v_0^j-v_\alpha^j,v_0^j\rangle &= \frac{1}{\lambda_0}\sum_{i=1}^N\int_{\partial B_\alpha^i}(\mu(z_i)-\mu_i)\sigma(v_\alpha^j,p)\nu_x|_-\cdot v_0^j d\sigma_x\\ &+ \frac{1}{\lambda_0}\sum_{i=1}^N\int_{B_\alpha^i}(\mu(x)-\mu(z_i))\,\mathbf{e}(v_\alpha^j):\mathbf{e}(v_0^j)dx\\ &+ O(\alpha^d). \end{aligned} \tag{4.41}$$

Now we introduce the following vector-valued function

$$r_\alpha(x)=v_\alpha^j(x)-\frac{1}{\lambda_0}v_0^j(x)-\alpha V(\frac{x-z_i}{\alpha}),$$

where $V(y)$ is the solution of (3.20), (with $\frac{1}{\lambda_0}v^j$ in place of u). Inserting this into the above formula, we immediately get

$$\begin{aligned}\langle\frac{1}{\lambda_0}v_0^j-v_\alpha^j,v_0^j\rangle &= \frac{1}{\lambda_0}\sum_{i=1}^N\int_{B_\alpha^i}(\mu(x)-\mu(z_i))\,\mathbf{e}(\frac{1}{\lambda_0}v_0^j+\alpha V(\frac{x-z_i}{\alpha})):\mathbf{e}(v_0^j)dx\\ &+ \frac{1}{\lambda_0}\sum_{i=1}^N\int_{B_\alpha^i}(\mu(x)-\mu(z_i))\,\mathbf{e}(r_\alpha):\mathbf{e}(v_0^j)dx\\ &+ \frac{1}{\lambda_0}\sum_{i=1}^N\int_{\partial B_\alpha^i}(\mu(z_i)-\mu_i)\sigma(r_\alpha,p)\nu_x|_-\cdot v_0^j d\sigma_x\\ &+ \frac{1}{\lambda_0}\sum_{i=1}^N\int_{\partial B_\alpha^i}(\mu(z_i)-\mu_i)\big(\sigma(\frac{1}{\lambda_0}v_0^j,p)\nu_x+\sigma(V,Q)\nu_y|_-\big)\cdot v_0^j d\sigma_x\\ &+ O(\alpha^d). \end{aligned} \tag{4.42}$$

Note that by change of variables and since $\mathbf{e}(v_0^j)(\alpha y+z_i)$ is uniformly bounded for α small enough, in view of Theorem 3, we get that

$$\begin{aligned}\Big|\sum_{i=1}^N\int_{B_\alpha^i}(\mu(x)-\mu(z_i))\,\mathbf{e}_x(r_\alpha):\mathbf{e}(v_0^j)dx\Big| &= \alpha^d\Big|\sum_{i=1}^N\int_{B_i}(\mu_i-\mu(z_i))\,\mathbf{e}_y(r_\alpha):\mathbf{e}(v_0^j)(\alpha y+z_i)dy\Big|\\ &\le c\alpha^{d+3/2}. \end{aligned} \tag{4.43}$$

At the same time, by a rescaling a Taylor expansion of $v_0^j(\alpha y+z_i)$ about $x=z_i$ and using Lemma 1, we obtain

$$\begin{aligned}&\Big|\sum_{i=1}^N\int_{\partial B_\alpha^i}(\mu(z_i)-\mu_i)\sigma(r_\alpha,p)\nu_x|_-\cdot v_0^j d\sigma_x+\sum_{i=1}^N\int_{\partial B_\alpha^i}(\mu(z_i)-\mu_i)\big(\sigma(\frac{1}{\lambda_0}v_0^j,p)\nu_x+\sigma(V,Q)\nu_y|_-\big)\cdot v_0^j d\sigma_x\Big|\\ &= O(\alpha^d). \end{aligned}$$

Thus, considering the above estimate and (4.43), so from (4.42) we obtain that

$$\begin{aligned}\langle\frac{1}{\lambda_0}v_0^j-v_\alpha^j,v_0^j\rangle &= \frac{1}{\lambda_0}\sum_{i=1}^N\int_{B_\alpha^i}(\mu(x)-\mu(z_i))\,\mathbf{e}(\frac{1}{\lambda_0}v_0^j+\alpha V(\frac{x-z_i}{\alpha})):\mathbf{e}(v_0^j)dx+O(\alpha^d)\\ &= \alpha^d\frac{1}{\lambda_0}\sum_{i=1}^N\int_{B_i}(\mu_i-\mu(z_i))\Big(\mathbf{e}(\frac{1}{\lambda_0}v_0^j)(\alpha y+z_i)+\mathbf{e}_y(V)(y)\Big):\mathbf{e}(v_0^j)(\alpha y+z_i)dy\\ &+ O(\alpha^d) \end{aligned} \tag{4.44}$$

where the term $O(\alpha^d)$ is uniformly bounded by $c\alpha^d$, for some constant c that depends on μ, but is independent of α. On the other hand, a Taylor expansion of $\mathbf{e}(v_0^j)(\alpha y + z_i)$ at $x = z_i$ yields

$$\alpha^d \int_{B_i} (\mu_i - \mu(z_i)) \Big(\mathbf{e}(\frac{1}{\lambda_0} v_0^j)(\alpha y + z_i) + \mathbf{e}_y(V)(y)\Big) : \mathbf{e}(v_0^j)(\alpha y + z_i) dy$$

$$= \alpha^d \int_{B_i} (\mu_i - \mu(z_i)) \Big(\mathbf{e}(\frac{1}{\lambda_0} v_0^j)(z_i) + \mathbf{e}_y(V)(y)\Big) : \mathbf{e}(v_0^j)(z_i) dy + O(\alpha^d).$$

Therefore (4.44) becomes

$$\langle \frac{1}{\lambda_0} v_0^j - v_\alpha^j, v_0^j \rangle = \alpha^d \frac{1}{\lambda_0} \sum_{i=1}^N \int_{B_i} (\mu_i - \mu(z_i))\, \mathbf{e}(v_0^j)(z_i) : \Big(\mathbf{e}(\frac{1}{\lambda_0} v_0^j)(z_i) + \mathbf{e}_y(V)(y)\Big) dy \\ + O(\alpha^d). \tag{4.45}$$

Noting that the pair (V, Q) solution of (3.20) is an affine function of $\mathbf{e}(v_0^j)(z_i)$, and can be rewritten as

$$V(y) = \sum_{k,l=1}^d \mathbf{e}(\frac{1}{\lambda_0} v_0^j)(z_i)_{kl} \hat{\mathbf{v}}_{kl}^{(i)}(y) \quad \text{and } Q(y) = \sum_{k,l=1}^d \mathbf{e}(\frac{1}{\lambda_0} v_0^j)(z_i)_{kl} \hat{\pi}^{(i)}(y), \tag{4.46}$$

where $(\hat{\mathbf{v}}_{kl}^{(i)}, \hat{\pi}^{(i)})$ are solutions to (2.9). Thus, by inserting the first term of (4.46) into relation (4.45) one can obtain that

$$\langle \frac{1}{\lambda_0} v_0^j - v_\alpha^j, v_0^j \rangle = \alpha^d \frac{1}{\lambda_0} \sum_{i=1}^N (\mu_i - \mu(z_i))\, \mathbf{e}(v_0^j)(z_i) : \int_{B_i} \Big(\mathbf{e}(\frac{1}{\lambda_0} v_0^j)(z_i) + \sum_{k,l=1}^d \mathbf{e}(\frac{1}{\lambda_0} v_0^j)(z_i)_{kl} \mathbf{e}(\hat{\mathbf{v}}_{kl}^{(i)})\Big) dy \\ + O(\alpha^d) \tag{4.47}$$

which, by using the notation (2.8), allows us to conclude that

$$\langle \frac{1}{\lambda_0} v_0^j - v_\alpha^j, v_0^j \rangle = \frac{\alpha^d}{2\lambda_0^2} \sum_{i=1}^N \mathbf{e}(v_0^j)(z_i) : \mathbf{V}^{(i)} \mathbf{e}(v_0^j)(z_i) + O(\alpha^d). \tag{4.48}$$

The proof of Theorem 1 is then achieved by inserting (4.48) into (4.39).

Now suppose that $m = 1$, then the following result holds.

Corollary 1 *Suppose that λ_0 is a simple eigenvalue of (1.6) with the corresponding eigenfunction v_0, and suppose that λ_α is the eigenvalue of (1.7) which converge to λ_0. Let $\boldsymbol{V}^{(i)}$ be defined by (2.8) for $i = 1 \ldots N$. Then, there exist a positive constant α_0 such that for $|\alpha| < \alpha_0$, the following asymptotic expansion holds:*

$$\lambda_\alpha - \lambda_0 = \frac{\alpha^d}{2} \sum_{i=1}^N \boldsymbol{e}(v_0)(z_i) : \boldsymbol{V}^{(i)} \boldsymbol{e}(v_0)(z_i) + o(\alpha^d), \tag{4.49}$$

where the term $o(\alpha^d)$ depends on the separation d_0 but is otherwise independent of the location of the set of points $(z_i)_{i=1}^N$.

Proof. The fact that λ_α tends to λ_0 allows us to find a small positive constant α_0 such that

$$\lambda_0 \lambda_\alpha = \lambda_0^2 + o(1), \quad \text{for } |\alpha| < \alpha_0. \tag{4.50}$$

On the other hand rewriting (2.11), given by Theorem 1, with respect to $m = 1$,

$$\frac{1}{\lambda_0} - \frac{1}{\lambda_\alpha} = \frac{\alpha^d}{2\lambda_0^2} \sum_{i=1}^{N} \mathbf{e}(v_0)(z_i) : \mathbf{V}^{(i)} \mathbf{e}(v_0)(z_i) + o(\alpha^d).$$

The proof is then achieved by writing the left-hand side of the above equality as follows

$$\lambda_\alpha - \lambda_0 = \lambda_0 \lambda_\alpha \left(\frac{1}{\lambda_0} - \frac{1}{\lambda_\alpha}\right),$$

and by considering (4.50). □

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