Received: 00 Month 0000 | Revised: 00 Month 0000 | Accepted: 00 Month 0000

DOI: xxx/xxxx

RESEARCH ARTICLE

On Boolean elements and derivations in 2-dimension linguistic lattice implication algebras

Hua Zhu[1,2] | Jianbin Zhao*[1,2] | Hairui Jia[3]

[1]School of Mathematics and Statistics, Zhengzhou University, Henan,450001, China
[2] Henan Key Laboratory of Financial Engineering, Zhengzhou, Henan,450001, China
[3]China Academy of Finance Research, Zhejiang University of Finance and Economics, Zhejiang, 310018, China

Correspondence
*Jianbin Zhao, Email: zhaojianbin@zzu.edu.cn

Abstract

A 2-dimension linguistic lattice implication algebra (2DL-LIA) can build a bridge for logical algebra and 2-dimension fuzzy linguistic information. In this paper, the notion of a Boolean element is proposed in a 2DL-LIA and some properties of Boolean elements are discussed. Then derivations on 2DL-LIAs are introduced and the related properties of derivations are investigated. Moreover, it proves that the derivations on 2DL-LIAs can be constructed by Boolean elements.

KEYWORDS:
Derivation; Boolean element, Lattice implication algebra (LIA), 2-dimension linguistic lattice implication algebra (2DL-LIA), 2-dimension fuzzy linguistic information

1 | INTRODUCTION

In real life, human's intelligent activities are often associated with fuzziness and incomparability. As two kinds of uncertainty[37], fuzziness and incomparability exist not only in the processed object itself, but also in the course of the object being dealt with. Lattice-valued logic, as an important non-classical logic, has been extensively studied to establish the logical foundation for uncertainty inference[32,33,34,35]. Accordingly, in order to provide algebraic semantics with lattice-valued logic, Xu et al.[35] proposed the concept of lattice implication algebras (LIAs). By using of the algebraic structures of LIAs, we can describe the relationships between uncertain information, especially for incomparable relationships. A lot of literatures[14,15,17,47,48] have been researched algebraic structures and properties of LIAs. Meanwhile, LIAs have been extended to lattice implication ordered semigroups[25], residuated lattices[13], linguistic truth-valued intuitionistic fuzzy lattices[50], linguistic truth-valued lattice implication algebras (L-LIAs)[36] and 2-dimension linguistic lattice implication algebras (2DL-LIAs)[46].

Zadeh[39] put forward the notion of fuzzy linguistic information, which usually takes as a tool for describing qualitative attributes such as low, medium and high. For precisely representing fuzzy linguistic information, Zhu et al.[49] proposed the concept of 2-dimension fuzzy linguistic information. The 2-dimension fuzzy linguistic information includes two common linguistic labels: one describes the evaluation result of alternatives, the other describes the self-assessment of the decision maker on the reliability of the given evaluation result. Further, aiming to precisely describe the relationships between 2-dimension fuzzy linguistic information, especially for incomparable relationship, Zhu et al.[46] gave the notion of a 2-dimension linguistic lattice implication algebra (2DL-LIA). Under the structure of 2DL-LIA, some important decision making methods are proposed to deal with 2-dimension linguistic information[45,42,43]. A 2DL-LIA has not only the features of logical algebra but also the features of evaluation sets for fuzzy linguistic information. Therefore, it can build a bridge for logical algebras and 2-dimension fuzzy linguistic information.

The notion of derivation, which comes from the analytic theory, is also helpful for investigating algebraic structures and properties of various kinds of algebras. The derivation in a prime ring $(R;+,\cdot)$ has been proposed by Posner[26], which is a mapping

[0]**Abbreviations:** H. Zhu, J.B. Zhao, H.R. Jia

$d : R \to R$ such that two conditions (1) $d(x+y) = d(x) + d(y)$ and (2) $d(x \cdot y) = d(x) \cdot y + x \cdot d(y)$ for all $x, y \in R$. After that, derivations on rings and near rings have been investigated by many researchers[3,8,9,23,29]. In 2004, derivations on BCI-algebras have been introduced by Jun et al.[18] and further studied in[1,12,21,22,24,40]. Besides, derivations on regular algebras[4], derivations on CSL-algebras[41], derivations on f-algebras[19], derivations on basic algebras[20], derivations on LIAs[48] and derivations on L-LIAs[44] have been studied by different researchers. Moreover, derivations on lattices have been discussed in[6,28,30,31]. Furthermore, derivations on MV-algebras and GMV-algebras have been investigated in[2,7,27,38]. Especially, a derivation on a residuated lattice $(L, \wedge, \vee, \odot, \to, 0, 1)$ is proposed by He et al.[10], which is a mapping $d : L \to L$ satisfying the conditions $d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y))$ for all $x, y \in L$.

Inspired by the above-mentioned work, especially by derivations on rings[26] and derivations on residuated lattices[10], derivations on 2DL-LIAs are proposed in this paper. This paper is organized as follows: Section 2 reviews some basic concepts about LIAs and 2DL-LIAs. In Section 3, a Boolean element is proposed in a 2DL-LIA, and then some properties of Boolean elements are investigated. Section 4 introduces derivations on 2DL-LIAs and discusses some properties of derivations. The conclusions are drawn in Section 5.

2 | PRELIMINARIES

This section gives some results about lattice implication algebras (LIAs) and 2-dimension linguistic lattice implication algebras (2DL-LIAs).

2.1 | Lattice implication algebras (LIAs)

For a lattice implication algebra (LIA)[35], we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution $'$, in which 0 and 1 are the smallest and the greatest element of L respectively, and a binary operation $\to$ satisfying the following axioms:
(I_1) $x \to (y \to z) = y \to (x \to z)$;
(I_2) $x \to x = 1$;
(I_3) $x \to y = y' \to x'$;
(I_4) if $x \to y = y \to x = 1$, then $x = y$;
(I_5) $(x \to y) \to y = (y \to x) \to x$;
(L_1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$;
(L_2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$;
for all $x, y, z \in L$.

A LIA L is called a lattice H implication algebra (LHIA), if for all $x, y, z \in L$, $x \vee y \vee ((x \wedge y) \to z) = 1$.

A lattice implication homomorphism is a mapping $f : L_1 \to L_2$ from LIAs L_1 to L_2, for any $x, y \in L_1$,

$$\begin{aligned} f(x \to y) &= f(x) \to f(y), \\ f(x \vee y) &= f(x) \vee f(y), \\ f(x \wedge y) &= f(x) \wedge f(y), \\ f(x') &= (f(x))'. \end{aligned}$$

Let L be a LIA, the binary operators $\otimes$ and $\oplus$ are defined as follows: for all $x, y \in L$,

$$x \otimes y = (x \to y')', x \oplus y = x' \to y.$$

Theorem 1. Let L be a LIA, then L is a LHIA if and only if for all $x \in L$, $x \oplus x = x$, $x \otimes x = x$.

Example 2.1. (Łukasiewicz implication algebra on a finite chain L_n)[35]. Let L be a finite chain, $L = \{a_i | i = 1, 2, \cdots, n\}$ and $0 = a_1 \leq a_2 \leq \cdots \leq a_n = 1$, for any $a_i, a_j \in L$, define operations $\vee, \wedge, \to$ and $'$ as follows:

$$\begin{aligned} a_i \vee a_j &= a_{max\{i,j\}}, \\ a_i \wedge a_j &= a_{min\{i,j\}}, \\ a_i \to a_j &= a_{min\{n-i+j,n\}}, \\ (a_i)' &= a_{n-i+1}. \end{aligned}$$

Then $(L, \vee, \wedge, ', \rightarrow, a_1, a_n)$ is a LIA, denoted by L_n.

Definition 1. [5,35] Let L_{m+1}, L_{n+1} be two Łukasiewicz implication algebras and $L_{m+1} = \{a_0, a_1, \cdots, a_m\} : a_0 \leq a_1 \leq \cdots \leq a_m$, $L_{n+1} = \{b_0, b_1, \cdots, b_n\} : b_0 \leq b_1 \leq \cdots \leq b_n$. Define the direct product of L_{m+1} and L_{n+1} as follows: $L_{m+1} \times L_{n+1} = \{(a, b) | a \in L_{m+1}, b \in L_{n+1}\}$. The operations on $L_{m+1} \times L_{n+1}$ are defined respectively as follows: for any $(a_i, b_k), (a_j, b_l) \in L_{m+1} \times L_{n+1}$,

$$
\begin{aligned}
(a_i, b_k) \vee (a_j, b_l) &= (a_i \vee a_j, b_k \vee b_l) = (a_{\max\{i,\, j\}}, b_{\max\{k,\, l\}}), \\
(a_i, b_k) \wedge (a_j, b_l) &= (a_i \wedge a_j, b_k \wedge b_l) = (a_{\min\{i,\, j\}}, b_{\min\{k,\, l\}}), \\
(a_i, b_k) \rightarrow (a_j, b_l) &= (a_i \rightarrow a_j, b_k \rightarrow b_l) = (a_{\min\{m-i+j,\, m\}}, b_{\min\{n-k+l,\, n\}}), \\
(a_i, b_k)' &= (a_i', b_k') = (a_{m-i}, b_{n-k}),
\end{aligned}
$$

then $(L_{m+1} \times L_{n+1}, \vee, \wedge, ', \rightarrow, (a_0, b_0), (a_m, b_n))$ is a LIA, denoted by $L_{(m+1)\times(n+1)}$.

Let L be a LIA, for all $x, y, z \in L$, define the partial relation $\leq$ in L as $x \leq y \Longleftrightarrow x \rightarrow y = 1$, then the followings hold[35]:

(1) $x \rightarrow 0 = x'$;

(2) $x \vee y = (x \rightarrow y) \rightarrow y$;

(3) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;

(4) $x \otimes y \leq x \wedge y \leq x \vee y \leq x \oplus y$;

(5) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z),\ x \otimes (y \wedge z) = (x \otimes y) \wedge (x \otimes z)$;

(6) $x \oplus (y \vee z) = (x \oplus y) \vee (x \oplus z),\ x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$.

For more details of LIAs, we refer to the monograph[35].

2.2 | 2-dimension linguistic lattice implication algebras (2DL-LIAs)

Firstly, linguistic label sets and 2-dimension fuzzy linguistic information are reviewed as follows:

Let $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic label set with the cardinality $g + 1$. For any $s_i, s_j \in S$, where $i, j \in \{0, 1, \cdots, g\}$, the following properties should hold[11,16]:

(1) if $i \leq j$, then $s_i \leq s_j$;

(2) $(s_i)' = s_{g-i}$;

(3) if $s_i \leq s_j$, then $\max(s_i, s_j) = s_j$;

(4) if $s_i \leq s_j$, then $\min(s_i, s_j) = s_i$.

In some real decision making environments, a decision maker needs to provide the evaluation result of alternatives by using linguistic labels as well as his (or her) self-appraisal. For example, when an expert is invited to express his (or her) opinions on a submitted journal paper, there are always two linguistic label sets provided, where one linguistic label set is supplied to evaluate the submitted paper, the other is supplied to evaluate the familiar degree of the expert with the contents of the submitted paper. Aiming to describe such phenomena, Zhu et al.[49] introduced the notion of 2-dimension fuzzy linguistic information, which is reviewed as follows.

Definition 2. [49] Let $S = \{s_0, s_1, \cdots, s_g\}$ and $H = \{h_0, h_1, \cdots, h_t\}$ be two linguistic label sets, where $g + 1$ is the cardinality of S and $t + 1$ is the cardinality of H. $\hat{r}=(s_i, h_j)$ is called a 2-dimension linguistic label (2DLL), in which $h_j \in H$ represents the assessment information about the alternative given by the decision maker, and $s_i \in S$ represents the self-assessment of the decision maker.

In order to precisely describe the relationships between 2-dimension fuzzy linguistic information, a 2-dimension linguistic lattice implication algebra (2DL-LIA) is constructed by combining two linguistic label sets with a LIA structure, which is reviewed as follows.

Definition 3. [46] Let $S = \{s_0, s_1, \cdots, s_g\}$: $s_0 \leq s_1 \leq \cdots \leq s_g$ and $H = \{h_0, h_1, \cdots, h_t\}$: $h_0 \leq h_1 \leq \cdots \leq h_t$ be two linguistic label sets, $L_{(g+1)\times(t+1)}$ be a LIA as defined in Definition 1. Let a mapping $f : S \times H \rightarrow L_{(g+1)\times(t+1)}$ be defined such that $f((s_i, h_j)) = (a_i, b_j)$, where $i \in \{0, 1, \cdots, g\}$, $j \in \{0, 1, \cdots, t\}$, then f is a bijection, denoted its inverse mapping as f^{-1}. For any

$(s_i, h_k), (s_j, h_l) \in S \times H$, define

$$\begin{aligned}(s_i, h_k) \vee (s_j, h_l) &= f^{-1}(f((s_i, h_k)) \vee f((s_j, h_l)))\\ (s_i, h_k) \wedge (s_j, h_l) &= f^{-1}(f((s_i, h_k)) \wedge f((s_j, h_l)))\\ (s_i, h_k) \rightarrow (s_j, h_l) &= f^{-1}(f((s_i, h_k)) \rightarrow f((s_j, h_l)))\\ (s_i, h_k)' &= f^{-1}((f(s_i, h_k))'),\end{aligned}$$

then it is obvious to verify that $(S \times H, \vee, \wedge, \rightarrow, ', (s_0, h_0), (s_g, h_t))$ is a LIA, which is called a 2-dimension linguistic lattice implication algebra (2DL-LIA), whose Hasse Diagram is shown in Figure 1.

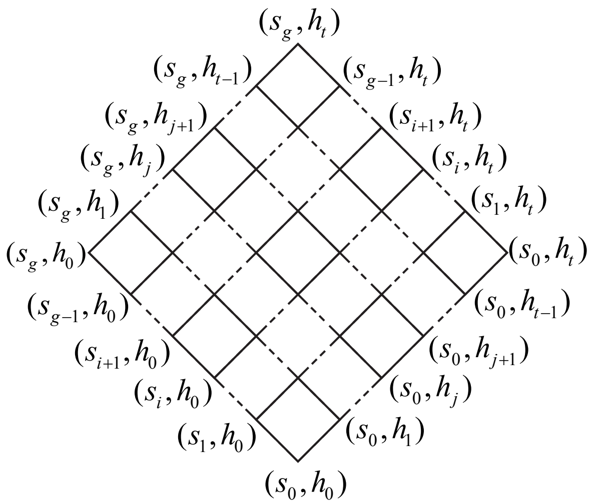

FIGURE 1 Hasse Diagram of 2DL-LIA

In the following, $S \times H$ is always denoted as a 2DL-LIA, where $S = \{s_0, s_1, \cdots, s_g\}$, $H = \{h_0, h_1, \cdots, h_t\}$ be two linguistic label sets.

Then, we give the notion of a 2-dimension linguistic lattice H implication algebra (2DL-LHIA), which will be mentioned in next section.

Definition 4. Let $S \times H$ be a 2DL-LIA, if for all $(s_{i1}, h_{j1}), (s_{i2}, h_{j2}), (s_{i3}, h_{j3}) \in S \times H$,

$$(s_{i1}, h_{j1}) \vee (s_{i2}, h_{j2}) \vee (((s_{i1}, h_{j1}) \wedge (s_{i2}, h_{j2})) \rightarrow s_{i3}, h_{j3}) = (s_g, h_t),$$

where $i1, i2, i3 \in \{0, 1, \cdots g\}$, $j1, j2, j3 \in \{0, 1, \cdots, t\}$, then $S \times H$ is called 2-dimension linguistic lattice H implication algebra (2DL-LHIA).

Next by using of the indexes of linguistic labels in $S \times H$, some operations including $\vee, \wedge, \rightarrow, '$ can make direct computations in the following theorem.

Theorem 2. [45] Let $(S \times H, \vee, \wedge, \rightarrow, ')$ be a 2DL-LIA, (s_g, h_t) and (s_0, h_0) are the maximal element and minimal element of $S \times H$, $(s_{i1}, h_{i2}), (s_{j1}, h_{j2}) \in S \times H$. Then

$$\begin{aligned}(s_{i1}, h_{j1}) \vee (s_{i2}, h_{j2}) &= (s_{\max\{i1,i2\}}, h_{\max\{j1,j2\}})\\ (s_{i1}, h_{j1}) \wedge (s_{i2}, h_{j2}) &= (s_{\min\{i1,i2\}}, h_{\min\{j1,j2\}})\\ (s_{i1}, h_{j1}) \rightarrow (s_{i2}, h_{j2}) &= (s_{\min\{g-i1+i2,\ g\}}, h_{\min\{t-j1+j2,\ t\}})\\ (s_{i1}, h_{j1})' &= (s_{g-i1}, h_{t-j1}).\end{aligned}$$

Now, mainly for aggregation of 2-dimension fuzzy linguistic information, two logical operators $\oplus$ and $\otimes$ can be defined in a 2DL-LIA $S \times H$ as follows: for all $(s_{i1}, h_{j1}), (s_{i1}, h_{j1}) \in S \times H$,

$$(s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2}) = (s_{i1}, h_{j1})' \to (s_{i2}, h_{j2})$$
$$(s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2}) = ((s_{i1}, h_{j1}) \to (s_{i2}, h_{j2})')'$$

Similarly, the computational methods of $\oplus$ and $\otimes$ are provided by using of the indexes of linguistic labels in a 2DL-LIA.

Theorem 3. Let $S \times H$ be a 2DL-LIA, then for all $(s_{i1}, h_{j1}), (s_{i2}, h_{j2}) \in S \times H$, where $i1, i2 \in \{0, 1, \cdots, g\}, j1, j2 \in \{0, 1, \cdots, t\}$, we have:
(1) $(s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2}) = (s_{\min\{i1+i2,g\}}, h_{\min\{j1+j2,t\}})$;
(2) $(s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2}) = (s_{\max\{i1+i2-g,0\}}, h_{\max\{j1+j2-t,0\}})$.

Proof. (1) Since $(s_{i1}, h_{j1})' \to (s_{i2}, h_{j2}) = (s_{g-i1}, h_{t-j1}) \to (s_{i2}, h_{j2}) = (s_{\min\{i1+i2,g\}}, h_{\min\{j1+j2,t\}})$ by Theorem 2, we obtain that $(s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2}) = (s_{\min\{i1+i2,g\}}, h_{\min\{j1+j2,t\}})$;
(2) Because $(s_{i1}, h_{j1}) \to (s_{i2}, h_{j2})' = (s_{i1}, h_{j1}) \to (s_{g-i2}, h_{t-j2}) = (s_{\min\{2g-i1-i2,g\}}, h_{\min\{2t-j1-j2,t\}})$ and $(s_{\min\{2g-i1-i2,g\}}, h_{\min\{2t-j1-j2,t\}})' = (s_{\max\{i1+i2-g,0\}}, h_{\max\{j1+j2-t,0\}})$ by Theorem 2, we have $(s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2}) = (s_{\max\{i1+i2-g,0\}}, h_{\max\{j1+j2-t,0\}})$.

Finally, the relationships among $\vee$, $\oplus$, $\otimes$ and $\wedge$ are discussed in a 2DL-LIA.

Theorem 4. Let $S \times H$ be a 2DL-LIA, then for all $(s_{i1}, h_{j1}), (s_{i2}, h_{j2}) \in S \times H$, we have:
(1) $(s_{i1}, h_{j1}) \vee (s_{i2}, h_{j2}) = ((s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2})') \oplus (s_{i2}, h_{j2})$;
(2) $(s_{i1}, h_{j1}) \wedge (s_{i2}, h_{j2}) = ((s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2})') \otimes (s_{i2}, h_{j2})$.

Proof. (1) Since $(s_{i1}, h_{j1}) \vee (s_{i2}, h_{j2}) = ((s_{i1}, h_{j1}) \to (s_{i2}, h_{j2})) \to (s_{i2}, h_{j2})$, then we have $(s_{i1}, h_{j1}) \vee (s_{i2}, h_{j2}) = ((s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2})') \oplus (s_{i2}, h_{j2})$.
(2) Since $((s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2})') \otimes (s_{i2}, h_{j2}) = (((s_{i1}, h_{j1})' \to (s_{i2}, h_{j2})') \to (s_{i2}, h_{j2})')'$, then we have $((s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2})') \otimes (s_{i2}, h_{j2}) = ((s_{i1}, h_{j1})' \vee (s_{i2}, h_{j2})')' = (s_{i1}, h_{j1}) \wedge (s_{i2}, h_{j2})$.

3 | BOOLEAN ELEMENTS OF 2DL-LIAS

In this section, a Boolean element is defined in a 2DL-LIA, then some properties of Boolean elements are investigated. Finally, logical operator $\oplus$ is discussed in a 2DL-LIA, which can build a bridge for 2-dimension fuzzy linguistic information aggregations and logical algebras.

Definition 5. Let $S \times H$ be a 2DL-LIA, (s_g, h_t) and (s_0, h_0) are the maximal element and minimal element of $S \times H$ respectively, $(s_i, h_j) \in S \times H$. If $(s_i, h_j) \vee (s_i, h_j)' = (s_g, h_t)$, (or equivalently $(s_i, h_j) \wedge (s_i, h_j)' = (s_0, h_0)$), then (s_i, h_j) is called a boolean element of $S \times H$.

In the following, denote $B(S \times H)$ be the set of boolean elements in $S \times H$.

Example 3.1. Let $S_1 \times H_1$ be a 2DL-LIA, whose Hasse Diagram is shown in Figure 2, where $S_1 = \{s_0, s_1\}$, $H_1 = \{h_0, h_1\}$.

The operations $'$ and $\to$ can be computed by Theorem 2 as follows: $(s_0, h_0)' = (s_1, h_1), (s_0, h_1)' = (s_1, h_0), (s_1, h_0)' = (s_0, h_1)$, $(s_1, h_1)' = (s_0, h_0)$ and

$\to$	(s_0, h_0)	(s_0, h_1)	(s_1, h_0)	(s_1, h_1)
(s_0, h_0)	(s_1, h_1)	(s_1, h_1)	(s_1, h_1)	(s_1, h_1)
(s_0, h_1)	(s_1, h_0)	(s_1, h_1)	(s_1, h_0)	(s_1, h_1)
(s_1, h_0)	(s_0, h_1)	(s_0, h_1)	(s_1, h_1)	(s_1, h_1)
(s_1, h_1)	(s_0, h_0)	(s_0, h_1)	(s_1, h_0)	(s_1, h_1)

We can check that $(s_0, h_0), (s_0, h_1), (s_1, h_0), (s_1, h_1)$ are all boolean elements, that is $(s_0, h_0), (s_0, h_1), (s_1, h_0), (s_1, h_1) \in B(S_1 \times H_1)$.

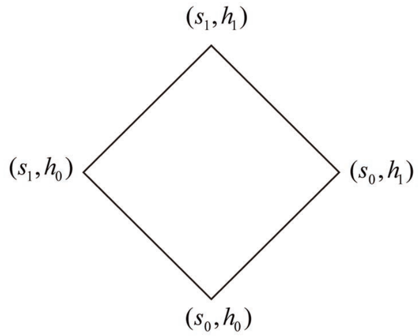

FIGURE 2 Hasse Diagram of $S_1 \times H_1$

Now, some properties of boolean elements are investigated in a 2DL-LIA.

Proposition 1. Let $S \times H$ be a 2DL-LIA, then $(s_i, h_j) \in B(S \times H)$ if and only if $(s_i, h_j)' \in B(S \times H)$.

Proof. It is obvious by Definition 5.

Proposition 2. Let $S \times H$ be a 2DL-LIA, $(s_i, h_j) \in S \times H$, then we have:
(1) $(s_i, h_j) \in B(S \times H)$ if and only if $(s_i, h_j) \oplus (s_i, h_j) = (s_i, h_j)$;
(2) $(s_i, h_j) \in B(S \times H)$ if and only if $(s_i, h_j) \otimes (s_i, h_j) = (s_i, h_j)$.

Proof. (1) Suppose $(s_i, h_j) \in B(S \times H)$, then $(s_i, h_j) \vee (s_i, h_j)' = (s_g, h_t)$. Because $((s_i, h_j) \oplus (s_i, h_j)) \to (s_i, h_j) = ((s_i, h_j)' \to (s_i, h_j)) \to (s_i, h_j) = (s_i, h_j)' \vee (s_i, h_j) = (s_g, h_t)$, we have $(s_i, h_j) \oplus (s_i, h_j) \leq (s_i, h_j)$. It is obvious that $(s_i, h_j) \leq (s_i, h_j) \oplus (s_i, h_j)$. Therefore $(s_i, h_j) \oplus (s_i, h_j) = (s_i, h_j)$.

On the other hand, assume $(s_i, h_j) \oplus (s_i, h_j) = (s_i, h_j)$, then we have $(s_i, h_j)' \vee (s_i, h_j) = ((s_i, h_j)' \to (s_i, h_j)) \to (s_i, h_j) = ((s_i, h_j) \oplus (s_i, h_j)) \to (s_i, h_j) = (s_i, h_j) \to (s_i, h_j) = (s_g, h_t)$.

(2) The conclusion can be obtained analogously.

Proposition 3. Let $S \times H$ be a 2DL-LIA, (s_g, h_t) and (s_0, h_0) are the maximal element and minimal element of $S \times H$ respectively, if $(s_i, h_j) \in B(S \times H)$, then for all $(s_{i1}, h_{j1}) \in S \times H$, we have:
(1) $(s_{i1}, h_{j1}) \otimes (s_i, h_j) = (s_{i1}, h_{j1}) \wedge (s_i, h_j)$;
(2) $(s_{i1}, h_{j1}) \oplus (s_i, h_j) = (s_{i1}, h_{j1}) \vee (s_i, h_j)$.

Proof. (1) We only need to prove $(s_{i1}, h_{j1}) \wedge (s_i, h_j) \leq (s_{i1}, h_{j1}) \otimes (s_i, h_j)$.

Suppose $(s_i, h_j) \in B(S \times H)$, then we have $(s_i, h_j) \vee (s_i, h_j)' = (s_g, h_t)$ by Definition 5.

Since $(s_{i1}, h_{j1}) \wedge (s_i, h_j) \to (s_{i1}, h_{j1}) \otimes (s_i, h_j) = ((s_{i1}, h_{j1}) \to (s_{i1}, h_{j1}) \otimes (s_i, h_j)) \vee ((s_i, h_j) \to (s_{i1}, h_{j1}) \otimes (s_i, h_j)) = (((s_{i1}, h_{j1}) \to (s_i, h_j)') \to (s_{i1}, h_{j1})') \vee (((s_{i1}, h_{j1}) \to (s_i, h_j)') \to (s_i, h_j)') = (s_i, h_j) \vee (s_{i1}, h_{j1})' \vee (s_{i1}, h_{j1}) \vee (s_i, h_j)' \geq (s_i, h_j) \vee (s_i, h_j)' = (s_g, h_t)$, that is $(s_{i1}, h_{j1}) \wedge (s_i, h_j) \to (s_{i1}, h_{j1}) \otimes (s_i, h_j) = (s_g, h_t)$, then we have $(s_{i1}, h_{j1}) \wedge (s_i, h_j) \leq (s_{i1}, h_{j1}) \otimes (s_i, h_j)$.

(2) We only need to prove that $(s_{i1}, h_{j1}) \oplus (s_i, h_j) \leq (s_{i1}, h_{j1}) \vee (s_i, h_j)$.

Suppose $(s_i, h_j) \in B(S \times H)$, then we have $(s_i, h_j) \vee (s_i, h_j)' = (s_g, h_t)$ by Definition 5.

Since $(s_{i1}, h_{j1}) \oplus (s_i, h_j) \to (s_{i1}, h_{j1}) \vee (s_i, h_j) = ((s_{i1}, h_{j1}) \oplus (s_i, h_j) \to (s_{i1}, h_{j1})) \vee ((s_{i1}, h_{j1}) \oplus (s_i, h_j) \to (s_i, h_j)) = (((s_i, h_j)' \to (s_{i1}, h_{j1})) \to (s_{i1}, h_{j1})) \vee (((s_{i1}, h_{j1})' \to (s_i, h_j)) \to (s_i, h_j)) = (s_i, h_j)' \vee (s_{i1}, h_{j1}) \vee (s_{i1}, h_{j1})' \vee (s_i, h_j) \geq (s_i, h_j)' \vee (s_i, h_j) = (s_g, h_t)$, that is $(s_{i1}, h_{j1}) \oplus (s_i, h_j) \to (s_{i1}, h_{j1}) \vee (s_i, h_j) = (s_g, h_t)$, then we have $(s_{i1}, h_{j1}) \oplus (s_i, h_j) \leq (s_{i1}, h_{j1}) \vee (s_i, h_j)$.

Proposition 4. Let $S \times H$ be a 2DL-LIA, if $(s_i, h_j) \in B(S \times H)$, then for all $(s_{i1}, h_{j1}), (s_{i2}, h_{j2}) \in S \times H$, we have:
(1) $((s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2})) \wedge (s_i, h_j) = ((s_{i1}, h_{j1}) \wedge (s_i, h_j)) \oplus ((s_{i2}, h_{j2}) \wedge (s_i, h_j))$;
(2) $((s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2})) \vee (s_i, h_j) = ((s_{i1}, h_{j1}) \vee (s_i, h_j)) \oplus ((s_{i2}, h_{j2}) \vee (s_i, h_j))$;
(3) $((s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2})) \wedge (s_i, h_j) = ((s_{i1}, h_{j1}) \wedge (s_i, h_j)) \otimes ((s_{i2}, h_{j2}) \wedge (s_i, h_j))$;
(4) $((s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2})) \vee (s_i, h_j) = ((s_{i1}, h_{j1}) \vee (s_i, h_j)) \otimes ((s_{i2}, h_{j2}) \vee (s_i, h_j))$;

(5) $((s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2})) \otimes (s_i, h_j) = ((s_{i1}, h_{j1}) \otimes (s_i, h_j)) \oplus ((s_{i2}, h_{j2}) \otimes (s_i, h_j))$;
(6) $((s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2})) \oplus (s_i, h_j) = ((s_{i1}, h_{j1}) \oplus (s_i, h_j)) \otimes ((s_{i2}, h_{j2}) \oplus (s_i, h_j))$.

Proof. We only prove (1) and (2).

(1) Suppose $(s_i, h_j) \in B(S \times H)$, then we have $(s_i, h_j) \oplus (s_i, h_j) = (s_i, h_j)$ by Proposition 2 (1).

Since $((s_{i1}, h_{j1}) \wedge (s_i, h_j)) \oplus ((s_{i2}, h_{j2}) \wedge (s_i, h_j)) = ((s_{i1}, h_{j1}) \wedge (s_i, h_j))' \to ((s_{i2}, h_{j2}) \wedge (s_i, h_j)) = (s_{i1}, h_{j1})' \vee (s_i, h_j)' \to (s_{i2}, h_{j2}) \wedge (s_i, h_j) = ((s_{i1}, h_{j1})' \to (s_{i2}, h_{j2})) \wedge ((s_{i1}, h_{j1})' \to (s_i, h_j)) \wedge ((s_i, h_j)' \to (s_{i2}, h_{j2})) \wedge ((s_i, h_j)' \to (s_i, h_j)) = ((s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2})) \wedge ((s_{i1}, h_{j1}) \oplus (s_i, h_j)) \wedge ((s_i, h_j) \oplus (s_{i2}, h_{j2})) \wedge ((s_i, h_j) \oplus (s_i, h_j))$ and $((s_{i1}, h_{j1}) \oplus (s_i, h_j)) \wedge ((s_i, h_j) \oplus (s_{i2}, h_{j2})) \wedge ((s_i, h_j) \oplus (s_i, h_j)) = (s_i, h_j)$, then we have $((s_{i1}, h_{j1}) \wedge (s_i, h_j)) \oplus ((s_{i2}, h_{j2}) \wedge (s_i, h_j)) = ((s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2})) \wedge (s_i, h_j)$.

(2) Suppose $(s_i, h_j) \in B(S \times H)$, then we have $(s_i, h_j) \oplus (s_i, h_j) = (s_i, h_j)$ by Proposition 2 (1).

Since $((s_{i1}, h_{j1}) \vee (s_i, h_j)) \oplus ((s_{i2}, h_{j2}) \vee (s_i, h_j)) = ((s_{i1}, h_{j1}) \vee (s_i, h_j))' \to (s_{i2}, h_{j2}) \vee (s_i, h_j) = ((s_{i1}, h_{j1})' \to (s_{i2}, h_{j2})) \vee ((s_{i1}, h_{j1})' \to (s_i, h_j)) \vee ((s_i, h_j)' \to (s_{i2}, h_{j2})) \vee ((s_i, h_j)' \to (s_i, h_j)) = ((s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2})) \vee ((s_{i1}, h_{j1}) \oplus (s_i, h_j)) \vee ((s_i, h_j) \oplus (s_{i2}, h_{j2})) \vee ((s_i, h_j) \oplus (s_i, h_j))$ and $((s_{i1}, h_{j1}) \oplus (s_i, h_j)) \vee ((s_i, h_j) \oplus (s_{i2}, h_{j2})) \vee ((s_i, h_j) \oplus (s_i, h_j)) = (s_{i1}, h_{j1}) \vee (s_{i2}, h_{j2}) \vee (s_i, h_j)$ by Proposition 3 (2), then we have $((s_{i1}, h_{j1}) \vee (s_i, h_j)) \oplus ((s_{i2}, h_{j2}) \vee (s_i, h_j)) = ((s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2})) \vee (s_i, h_j)$.

Proposition 5. Let $S \times H$ be a 2DL-LIA, (s_g, h_t) and (s_0, h_0) are the maximal element and minimal element of $S \times H$ respectively. If $(s_{i1}, h_{j1}), (s_{i2}, h_{j2}) \in B(S \times H)$, then we have $(s_{i1}, h_{j1}) \wedge (s_{i2}, h_{j2}), (s_{i1}, h_{j1}) \vee (s_{i2}, h_{j2}), (s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2}), (s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2}) \in B(S \times H)$.

Proof. Firstly, we prove $(s_{i1}, h_{j1}) \wedge (s_{i2}, h_{j2}) \in B(S \times H)$ and $(s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2}) \in B(S \times H)$.

Suppose $(s_{i1}, h_{j1}), (s_{i2}, h_{j2}) \in B(S \times H)$, we have $(s_{i1}, h_{j1}) \wedge (s_{i1}, h_{j1})' = (s_0, h_0)$ and $(s_{i2}, h_{j2}) \wedge (s_{i2}, h_{j2})' = (s_0, h_0)$.

Because $((s_{i1}, h_{j1}) \wedge (s_{i2}, h_{j2})) \wedge ((s_{i1}, h_{j1}) \wedge (s_{i2}, h_{j2}))' = ((s_{i1}, h_{j1}) \wedge (s_{i2}, h_{j2})) \wedge ((s_{i1}, h_{j1})' \vee (s_{i2}, h_{j2})') = ((s_{i1}, h_{j1}) \wedge (s_{i2}, h_{j2}) \wedge (s_{i1}, h_{j1})') \vee ((s_{i1}, h_{j1}) \wedge (s_{i2}, h_{j2}) \wedge (s_{i2}, h_{j2})') = (s_0, h_0)$, then we have $(s_{i1}, h_{j1}) \wedge (s_{i2}, h_{j2}) \in B(S \times H)$ by Definition 5, which implies $(s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2}) \in B(S \times H)$ by Proposition 3 (1).

Next, we prove $(s_{i1}, h_{j1}) \vee (s_{i2}, h_{j2}) \in B(S \times H)$ and $(s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2}) \in B(S \times H)$.

Suppose $(s_{i1}, h_{j1}), (s_{i2}, h_{j2}) \in B(S \times H)$, we have $(s_{i1}, h_{j1}) \vee (s_{i1}, h_{j1})' = (s_g, h_t)$ and $(s_{i2}, h_{j2}) \vee (s_{i2}, h_{j2})' = (s_g, h_t)$.

Because $((s_{i1}, h_{j1}) \vee (s_{i2}, h_{j2})) \vee ((s_{i1}, h_{j1}) \vee (s_{i2}, h_{j2}))' = ((s_{i1}, h_{j1}) \vee (s_{i2}, h_{j2})) \vee ((s_{i1}, h_{j1})' \wedge (s_{i2}, h_{j2})') = ((s_{i1}, h_{j1}) \vee (s_{i2}, h_{j2}) \vee (s_{i1}, h_{j1})') \wedge ((s_{i1}, h_{j1}) \vee (s_{i2}, h_{j2}) \vee (s_{i2}, h_{j2})') = (s_g, h_t)$, then we have $(s_{i1}, h_{j1}) \vee (s_{i2}, h_{j2}) \in B(S \times H)$ by Definition 5, which implies $(s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2}) \in B(S \times H)$ by Proposition 3 (2).

Theorem 5. Let $S \times H$ be a 2DL-LIA, if for all $(s_i, h_j) \in S \times H$, $(s_i, h_j) \in B(S \times H)$, then $S \times H$ is a 2DLH-LIA.

Proof. Suppose $\forall (s_i, h_j) \in S \times H, (s_i, h_j) \in B(S \times H)$, then we have $(s_i, h_j) \otimes (s_i, h_j) = (s_i, h_j)$ and $(s_i, h_j) \oplus (s_i, h_j) = (s_i, h_j)$ by Proposition 2. Therefore $S \times H$ is a 2DLH-LIA by Theorem 1.

Finally, we focus on logical operator $\oplus$ and its application in a 2DL-LIA, which can build a bridge for 2-dimension fuzzy linguistic information aggregations and logical algebras.

As we know, in real decision making environment, the weights of 2-dimension fuzzy linguistic information are critical for decision makers to aggregate 2-dimension fuzzy linguistic information. Therefore we define operations between constants and 2-dimension linguistic labels (2DLLs) in a 2DL-LIA as follows.

Definition 6. Let $S \times H$ be a 2DL-LIA, $S = \{s_0, s_1, \cdots, s_g\}$, $H = \{h_0, h_1, \cdots, t\}$, then for all $(s_i, h_j) \in S \times H, \lambda \in R^+$, $\lambda(s_i, h_j) = (s_{\min\{\lambda i, g\}}, h_{\min\{\lambda j, t\}})$.

Example 3.2. Let $S \times H$ be a 2DL-LIA, where $S = \{s_0, s_1, s_2\} = \{\text{very familiar, familiar, not familiar}\}$, $H = \{h_0, h_1, h_2, h_3\} = \{\text{very low, low, medium, high, very high}\}$ be two linguistic label sets. Suppose there are two reviewers who are invited to evaluate the same submitted manuscript by using of 2DLLs in $S \times H$. One reviewer's opinion is (s_2, h_3), and the other is (s_2, h_2).

If the weight vector of two reviewers is $(0.6, 0.4)$, then we can obtain the final opinion of this manuscript is

$$0.6(s_2, h_3) \oplus 0.4(s_2, h_2) = (s_2, h_{2.6}).$$

Remark. By using of the operation $\oplus$ and operation between constants and 2DLLs, we can aggregate 2DLLs provided by decision makers to obtain the collective one. Therefore the operation $\oplus$ builds a bridge between logical operators and aggregation operators in a certain sense.

Moreover, as in Example 3.2, we can compute that $(s_1, h_2) \oplus (s_1, h_2) = (s_2, h_4)$. This outcome may be unreasonable in some real conditions, and we hope that $\oplus$ satisfies idempotent property, that is $(s_i, h_j) \oplus (s_i, h_j) = (s_i, h_j)$. As we know, if

$(s_i, h_j) \in B(S \times H)$, then $(s_i, h_j) \oplus (s_i, h_j) = (s_i, h_j)$. However, it is a pity that there are few Boolean elements in many 2DL-LIAs.

4 | DERIVATIONS ON 2DL-LIAS

In this section, derivations on 2DL-LIAs are introduced and the related properties of derivations are discussed in order to investigate algebraic structures of 2DL-LIAs. Finally, by using of Boolean elements in a 2DL-LIAs, some derivations on 2DL-LIAs can be constructed.

Definition 7. Let $S \times H$ be a 2DL-LIA, $d : S \times H \to S \times H$ be a mapping. For any $(s_{i1}, h_{j1}), (s_{i2}, h_{j2}) \in S \times H$, where $i1, i2 \in \{0, 1, \cdots, g\}, j1, j2 \in \{0, 1, \cdots, t\}$, if

(1) $d((s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2})) = d((s_{i1}, h_{j1})) \oplus d((s_{i2}, h_{j2}))$
(2) $d((s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2})) = (d((s_{i1}, h_{j1})) \otimes (s_{i2}, h_{j2})) \vee ((s_{i1}, h_{j1}) \otimes d((s_{i2}, h_{j2})))$,

then d is called a derivation on $S \times H$.

Now, some examples are given to indicate that there exist some derivations on 2DL-LIAs.

Example 4.1. Let $S \times H$ be a 2DL-LIA, where (s_0, h_0) is the minimal element of $S \times H$. For all $(s_i, h_j) \in S \times H$, define a mapping d on $S \times H$ as $d((s_i, h_j)) = (s_0, h_0)$. Then d is a derivation on 2DL-LIA, which is called a zero derivation.

Example 4.2. Let $S \times H$ be a 2DL-LIA. For all $(s_i, h_j) \in S \times H$, define a mapping d on $S \times H$ as $d((s_i, h_j)) = (s_i, h_j)$. Then d is a derivation on 2DL-LIA, which is called an identity derivation.

Example 4.3. As in Example 3.1, define a mapping $d : S_1 \times H_1 \to S_1 \times H_1$ such that $d((s_0, h_0)) = (s_0, h_0)$, $d((s_0, h_1)) = (s_0, h_0)$, $d((s_1, h_0)) = (s_1, h_0)$, $d((s_1, h_1)) = (s_1, h_0)$ and a mapping $d_1 : S_1 \times H_1 \to S_1 \times H_1$ such that $d_1((s_0, h_0)) = (s_0, h_0)$, $d_1((s_0, h_1)) = (s_0, h_0)$, $d_1((s_1, h_0)) = (s_0, h_1)$, $d_1((s_1, h_1)) = (s_0, h_1)$.
We can check that d and d_1 are two derivations on $S_1 \times H_1$ by Definition 7.

Then some properties of derivations are investigated in a 2DL-LIA.

Proposition 6. Let d be a derivation on $S \times H$, (s_g, h_t) and (s_0, h_0) be the maximal element and minimal element of $S \times H$ respectively. Then for all $(s_{i1}, h_{j1}), (s_{i2}, h_{j2}) \in S \times H$, we have:
(1) $d((s_0, h_0)) = (s_0, h_0)$;
(2) if $(s_{i1}, h_{j1}) \le (s_{i2}, h_{j2})$, then $d((s_{i1}, h_{j1})) \le d((s_{i2}, h_{j2}))$; (i.e. d is istone)
(3) $d((s_g, h_t)) \in B(S \times H)$;
(4) $d((s_{i1}, h_{j1})) \le (s_{i1}, h_{j1})$.

Proof. (1) Suppose d be a derivation on $S \times H$, then $d((s_0, h_0)) = d((s_0, h_0) \otimes (s_0, h_0)) = (d((s_0, h_0)) \otimes (s_0, h_0)) \vee ((s_0, h_0) \otimes d((s_0, h_0))) = (s_0, h_0)$ by Definition 7 (2).
(2) Suppose d be a derivation on $S \times H$. If $(s_{i1}, h_{j1}) \le (s_{i2}, h_{j2})$, then we have $(s_{i2}, h_{j2}) = (s_{i2}, h_{j2}) \vee (s_{i1}, h_{j1}) = ((s_{i2}, h_{j2}) \otimes (s_{i1}, h_{j1})') \oplus (s_{i1}, h_{j1})$ by Theorem 4 (1). Then $d((s_{i2}, h_{j2})) = d((s_{i2}, h_{j2}) \vee (s_{i1}, h_{j1})) = d(((s_{i2}, h_{j2}) \otimes (s_{i1}, h_{j1})') \oplus (s_{i1}, h_{j1})) = d((s_{i2}, h_{j2}) \otimes (s_{i1}, h_{j1})') \oplus d((s_{i1}, h_{j1}))$ by Definition 7 (1), thus $d((s_{i1}, h_{j1})) \le d((s_{i2}, h_{j2}))$. Therefore every derivation on 2DL-LIA is istone.
(3) Suppose d be a derivation on $S \times H$. Because $d((s_g, h_t)) = d((s_g, h_t) \oplus (s_g, h_t)) = d((s_g, h_t)) \oplus d((s_g, h_t))$ by Definition 7 (1), then we have $d((s_g, h_t)) \in B(S \times H)$ by Proposition 2 (1).
(4) Suppose d be a derivation on $S \times H$. Then $\forall (s_{i1}, h_{j1}) \in S \times H$, $(s_0, h_0) = d((s_0, h_0)) = d((s_{i1}, h_{j1}) \otimes (s_{i1}, h_{j1})') = (d((s_{i1}, h_{j1})) \otimes (s_{i1}, h_{j1})') \vee ((s_{i1}, h_{j1}) \otimes d((s_{i1}, h_{j1})')$, thus $d((s_{i1}, h_{j1})) \otimes (s_{i1}, h_{j1})' = (s_0, h_0)$. Because $d((s_{i1}, h_{j1})) \otimes (s_{i1}, h_{j1})' = (d((s_{i1}, h_{j1})) \to (s_{i1}, h_{j1}))' = (s_0, h_0)$, we have $d((s_{i1}, h_{j1})) \to (s_{i1}, h_{j1}) = (s_g, h_t)$. Therefore $d((s_{i1}, h_{j1})) \le (s_{i1}, h_{j1})$.

Proposition 7. Let d be a derivation on $S \times H$, (s_g, h_t) be the maximal element of $S \times H$. Then for all $(s_{i1}, h_{j1}), (s_{i2}, h_{j2}) \in S \times H$,
(1) $d((s_{i1}, h_{j1})) = (s_{i1}, h_{j1}) \wedge d((s_g, h_t)) = (s_{i1}, h_{j1}) \otimes d((s_g, h_t))$;
(2) $d((s_{i1}, h_{j1})^n) = (s_{i1}, h_{j1})^{n-1} \otimes d((s_{i1}, h_{j1}))$, where $(s_{i1}, h_{j1})^n = (s_{i1}, h_{j1})^{n-1} \otimes (s_{i1}, h_{j1}), n \in N$;
(3) $d((s_{i1}, h_{j1})') \le (d((s_{i1}, h_{j1})))'$;
(4) if $d((s_g, h_t)) = (s_g, h_t)$, then d is an identity derivation.

Proof. (1) Let d be a derivation on $S\times H$, then $d((s_{i1},h_{j1}))=d((s_{i1},h_{j1})\otimes(s_g,h_t))=(d((s_{i1},h_{j1}))\otimes(s_g,h_t))\vee((s_{i1},h_{j1})\otimes d((s_g,h_t)))=d((s_{i1},h_{j1}))\vee((s_{i1},h_{j1})\otimes d((s_g,h_t)))$, thus $(s_{i1},h_{j1})\otimes d((s_g,h_t))\le d((s_{i1},h_{j1}))$.

On the other hand, we have $d((s_{i1},h_{j1}))\le(s_{i1},h_{j1})$ by Proposition 6 (4), and $d((s_{i1},h_{j1}))\le d((s_g,h_t))$ by Proposition 6 (2), thus $d((s_{i1},h_{j1}))\le(s_{i1},h_{j1})\wedge d((s_g,h_t))$. Because $(s_{i1},h_{j1})\wedge d((s_g,h_t))=(s_{i1},h_{j1})\otimes d((s_g,h_t))$ by Proposition 6 (3) and Proposition 3 (1), then we have $d((s_{i1},h_{j1}))=(s_{i1},h_{j1})\wedge d((s_g,h_t))=(s_{i1},h_{j1})\otimes d((s_g,h_t))$.

(2) Let d be a derivation on $S\times H$, then we have $d((s_{i1},h_{j1})^2)=d((s_{i1},h_{j1})\otimes(s_{i1},h_{j1}))=(d((s_{i1},h_{j1}))\otimes(s_{i1},h_{j1}))\vee((s_{i1},h_{j1})\otimes d((s_{i1},h_{j1}))=(s_{i1},h_{j1})\otimes d((s_{i1},h_{j1}))$. Then we can obtain $d((s_{i1},h_{j1})^n)=(s_{i1},h_{j1})^{n-1}\otimes d((s_{i1},h_{j1}))$ by induction for all $n\ge 2$.

(3) Let d be a derivation on $S\times H$, then $d((s_{i1},h_{j1})')\le(s_{i1},h_{j1})'$ by Proposition 6 (4). Thus $d((s_{i1},h_{j1})')\le(s_{i1},h_{j1})'\le(s_{i1},h_{j1})'\vee d((s_g,h_t))'=((s_{i1},h_{j1})\wedge d((s_g,h_t)))'$. Because $d((s_{i1},h_{j1}))=(s_{i1},h_{j1})\wedge d((s_g,h_t))$ by (1), then we have $((s_{i1},h_{j1})\wedge d((s_g,h_t)))'\le(d((s_{i1},h_{j1})))'$. Hence $d((s_{i1},h_{j1})')\le(d((s_{i1},h_{j1})))'$.

(4) If $d((s_g,h_t))=(s_g,h_t)$, we have $(s_{i1},h_{j1})=(s_{i1},h_{j1})\otimes d((s_g,h_t))=d((s_{i1},h_{j1}))$ by (1), which implies d is an identity derivation.

Proposition 8. Let d be a derivation on $S\times H$, then for all $(s_{i1},h_{j1}),(s_{i2},h_{j2})\in S\times H$:

(1) $d((s_{i1},h_{j1})\wedge(s_{i2},h_{j2}))=d((s_{i1},h_{j1}))\wedge d((s_{i2},h_{j2}))$;

(2) $d((s_{i1},h_{j1})\vee(s_{i2},h_{j2}))=d((s_{i1},h_{j1}))\vee d((s_{i2},h_{j2}))$;

(3) $d((s_{i1},h_{j1})\otimes(s_{i2},h_{j2}))=d((s_{i1},h_{j1}))\otimes d((s_{i2},h_{j2}))$.

Proof. (1) Suppose d be a derivation on $S\times H$, then we have $d((s_{i1},h_{j1})\wedge(s_{i2},h_{j2}))=((s_{i1},h_{j1})\wedge(s_{i2},h_{j2}))\otimes d((s_g,h_t))$ by Proposition 7 (1). Since $d((s_g,h_t))\in B(S\times H)$ by Proposition 6 (3), we get $((s_{i1},h_{j1})\wedge(s_{i2},h_{j2}))\otimes d((s_g,h_t))=((s_{i1},h_{j1})\otimes d((s_g,h_t)))\wedge((s_{i2},h_{j2})\otimes d((s_g,h_t)))=d((s_{i1},h_{j1}))\wedge d((s_{i2},h_{j2}))$ by Proposition 7 (1), that is $d((s_{i1},h_{j1})\wedge(s_{i2},h_{j2}))=d((s_{i1},h_{j1}))\wedge d((s_{i2},h_{j2}))$.

(2) and (3) can be proved analogously.

Proposition 9. Let d be a derivation on $S\times H$, then for all $(s_{i1},h_{j1}),(s_{i2},h_{j2})\in S\times H$:

(1) $d((s_{i1},h_{j1})\to(s_{i2},h_{j2}))\le d((s_{i1},h_{j1}))\to d((s_{i2},h_{j2}))$;

(2) $d((s_{i1},h_{j1})\to(s_{i2},h_{j2}))\le(s_{i1},h_{j1})\to d((s_{i2},h_{j2}))$;

(3) $(s_{i1},h_{j1})\to(s_{i2},h_{j2})\le d((s_{i1},h_{j1}))\to d((s_{i2},h_{j2}))$.

Proof. (1) Let d be a derivation on $S\times H$, then we have $d((s_{i1},h_{j1})\to(s_{i2},h_{j2}))=d((s_{i1},h_{j1})'\oplus(s_{i2},h_{j2}))=d((s_{i1},h_{j1})')\oplus d((s_{i2},h_{j2}))$ by Definition 7 (1). Since $d((s_{i1},h_{j1})')\oplus d((s_{i2},h_{j2}))\le d((s_{i1},h_{j1}))'\oplus d((s_{i2},h_{j2}))=d((s_{i1},h_{j1}))\to d((s_{i2},h_{j2}))$ by Proposition 7 (3), then we have $d((s_{i1},h_{j1})\to(s_{i2},h_{j2}))\le d((s_{i1},h_{j1}))\to d((s_{i2},h_{j2}))$.

(2) Let d be a derivation on $S\times H$, since $(s_{i1},h_{j1})\otimes((s_{i1},h_{j1})\to(s_{i2},h_{j2}))\le(s_{i2},h_{j2})$, we have $d((s_{i1},h_{j1})\otimes((s_{i1},h_{j1})\to(s_{i2},h_{j2})))\le d((s_{i2},h_{j2}))$ by Proposition 6 (2). Then $d((s_{i1},h_{j1})\otimes((s_{i1},h_{j1})\to(s_{i2},h_{j2})))=(d((s_{i1},h_{j1}))\otimes((s_{i1},h_{j1})\to(s_{i2},h_{j2})))\vee((s_{i1},h_{j1})\otimes d((s_{i1},h_{j1})\to(s_{i2},h_{j2})))$ by Definition 7 (2), that is $(s_{i1},h_{j1})\otimes d((s_{i1},h_{j1})\to(s_{i2},h_{j2}))\le d((s_{i2},h_{j2}))$ and $d((s_{i1},h_{j1}))\otimes((s_{i1},h_{j1})\to(s_{i2},h_{j2}))\le d((s_{i2},h_{j2}))$. Hence $d((s_{i1},h_{j1})\to(s_{i2},h_{j2}))\le(s_{i1},h_{j1})\to d((s_{i2},h_{j2}))$. And $(s_{i1},h_{j1})\to(s_{i2},h_{j2})\le d((s_{i1},h_{j1}))\to d((s_{i2},h_{j2}))$, which implies (3) holds.

The following theorem shows the relationships between derivations and lattice implication homomorphisms in 2DLIAs.

Theorem 6. Let d be a derivation on $S\times H$, if for all $(s_i,h_j)\in S\times H$, $(d((s_i,h_j)))'\le d((s_i,h_j)')$, then d is a lattice implication homomorphism.

Proof. Suppose $\forall(s_i,h_j)\in S\times H$, $(d((s_i,h_j)))'\le d((s_i,h_j)')$, then we have $(d((s_i,h_j)))'=d((s_i,h_j)')$ by Proposition 7 (3). Let d be a derivation on $S\times H$, then by Definition 7 (1), we have $d((s_{i1},h_{j1})\to(s_{i2},h_{j2}))=d((s_{i1},h_{j1})'\oplus(s_{i2},h_{j2}))$ $=d((s_{i1},h_{j1})')\oplus d((s_{i2},h_{j2}))=(d((s_{i1},h_{j1})))'\oplus d((s_{i2},h_{j2}))=d((s_{i1},h_{j1}))\to d((s_{i2},h_{j2}))$, that is $d((s_{i1},h_{j1})\to(s_{i2},h_{j2}))=d((s_{i1},h_{j1}))\to d((s_{i2},h_{j2}))$. Combining with Proposition 8 (1) and (2), we get that d is a lattice implication homomorphism.

Now, some special mappings are defined by Boolean elements, which can be verified to be derivations on 2DL-LIAs.

Let $S\times H$ be a 2DL-LIA, $(s_i,h_j)\in B(S\times H)$, define $f_1:S\times H\to S\times H$ and $f_2:S\times H\to S\times H$ be the mappings such that $f_1:x\to x\wedge(s_i,h_j)$, $f_2:x\to x\otimes(s_i,h_j)$.

Proposition 10. Let f_1, f_2 be defined as above, then $f_1=f_2$.

Proof. The conclusions are obvious by Proposition 3 (1).

Then we investigate some properties of the mapping f_1 on 2DL-LIAs.

Proposition 11. Let f_1 be defined as above, then for all $(s_{i1}, h_{j1}), (s_{i2}, h_{j2}) \in S \times H$, we have:
(1) $f_1((s_{i1}, h_{j1}) \vee (s_{i2}, h_{j2})) = f_1((s_{i1}, h_{j1})) \vee f_1((s_{i2}, h_{j2}))$;
(2) $f_1((s_{i1}, h_{j1}) \wedge (s_{i2}, h_{j2})) = f_1((s_{i1}, h_{j1})) \wedge f_1((s_{i2}, h_{j2}))$;
(3) $f_1((s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2})) = f_1((s_{i1}, h_{j1})) \oplus f_1((s_{i2}, h_{j2}))$;
(4) $f_1((s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2})) = f_1((s_{i1}, h_{j1})) \otimes f_1((s_{i2}, h_{j2}))$;
(5) $f_1((s_{i1}, h_{j1})') = f_1(f_1((s_{i1}, h_{j1}))')$.

Proof. (1) Let $f_1 : x \to x \wedge (s_i, h_j)$, then we have $f_1((s_{i1}, h_{j1}) \vee (s_{i2}, h_{j2})) = ((s_{i1}, h_{j1}) \vee (s_{i2}, h_{j2})) \wedge (s_i, h_j) = ((s_{i1}, h_{j1}) \wedge (s_i, h_j)) \vee ((s_{i2}, h_{j2})) \wedge (s_i, h_j)) = f_1((s_{i1}, h_{j1})) \vee f_1((s_{i2}, h_{j2}))$. Analogously for (2).

(3) Let $f_1 : x \to x \wedge (s_i, h_j)$, then we have $f_1((s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2})) = ((s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2})) \wedge (s_i, h_j) = ((s_{i1}, h_{j1}) \wedge (s_i, h_j)) \oplus ((s_{i2}, h_{j2})) \wedge (s_i, h_j)) = f_1((s_{i1}, h_{j1})) \oplus f_1((s_{i2}, h_{j2}))$ by Proposition 4 (1). Analogously for (4).

(5) Let $f_1 : x \to x \wedge (s_i, h_j)$, then we have $f_1(f_1((s_{i1}, h_{j1}))') = f_1(((s_{i1}, h_{j1}) \wedge (s_i, h_j))') = ((s_{i1}, h_{j1}) \wedge (s_i, h_j))' \wedge (s_i, h_j) = ((s_{i1}, h_{j1})' \vee (s_i, h_j)') \wedge (s_i, h_j)$. Since $(s_i, h_j) \in B(S \times H)$, we have $((s_{i1}, h_{j1})' \vee (s_i, h_j)') \wedge (s_i, h_j) = (s_{i1}, h_{j1})' \wedge (s_i, h_j) = f_1((s_{i1}, h_{j1})')$, that is $f_1((s_{i1}, h_{j1})') = f_1(f_1((s_{i1}, h_{j1}))')$.

Finally, we prove that the mapping f_1 is a derivation on $S \times H$.

Theorem 7. Let f_1 be defined as above, then f_1 is a derivation on $S \times H$.

Proof. Let $f_1 : x \to x \wedge (s_i, h_j)$, then we have $f_1((s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2})) = ((s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2})) \wedge (s_i, h_j)$, $f_1((s_{i1}, h_{j1})) \otimes (s_{i2}, h_{j2}) = ((s_{i1}, h_{j1}) \wedge (s_i, h_j)) \otimes (s_{i2}, h_{j2})$, and $(s_{i1}, h_{j1}) \otimes f_1((s_{i2}, h_{j2})) = (s_{i1}, h_{j1}) \otimes ((s_{i2}, h_{j2}) \wedge (s_i, h_j))$. Since $(s_i, h_j) \in B(S \times H)$, we get $((s_{i1}, h_{j1}) \wedge (s_i, h_j)) \otimes (s_{i2}, h_{j2}) = (((s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2})) \otimes (s_i, h_j) = (((s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2})) \wedge (s_i, h_j)$ and $(s_{i1}, h_{j1}) \otimes ((s_{i2}, h_{j2}) \wedge (s_i, h_j)) = (((s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2})) \wedge (s_i, h_j)$ by Proposition 3 (1), which implies that $f_1(s_{i1}, h_{j1}) \otimes (s_{i2}, h_{j2})) = (f_1((s_{i1}, h_{j1})) \otimes (s_{i2}, h_{j2})) \vee ((s_{i1}, h_{j1}) \otimes f_1((s_{i2}, h_{j2})))$.

By Proposition 11 (3), we have $f_1((s_{i1}, h_{j1}) \oplus (s_{i2}, h_{j2})) = f_1((s_{i1}, h_{j1})) \oplus f_1((s_{i2}, h_{j2}))$. Hence f_1 is a derivation on $S \times H$ by Definition 7.

5 | CONCLUSIONS

The theory of derivations plays an important role for investigating algebraic structures and properties of logical algebras. In this paper, derivations on rings and derivations on residuated lattices are extended to 2DL-LIAs. Concretely, a Boolean element is proposed in a 2DL-LIA along with its some properties. Then logical operator $\oplus$ is applied to aggregate 2-dimension fuzzy linguistic information, which can build a bridge for 2-dimension fuzzy linguistic information aggregations and logical algebras. Next, the concept of a derivation is introduced in a 2DL-LIA and some properties of derivations are studied. Finally, some special mappings defined by Boolean elements are proved to be derivations on 2DL-LIAs.

The above work not only enriches algebraic structures and properties of 2DL-LIAs, but also provides theoretical foundations for lattice-valued logic systems based on 2DL-LIAs. Further, it hopes that the results of this manuscript can supply theoretical supports for 2-dimension linguistic information application. In future, we will consider derivations on substructures of 2DL-LIAs including LI-ideals, filters, sl-ideals etc and apply the theory of derivations on 2DL-LIAs to deal with 2-dimension fuzzy linguistic information.

6 | ACKNOWLEDGMENTS

This work is partially supported by the Natural Science Foundation of China (Grant No.11501523 and 61673320). The authors also gratefully acknowledge the helpful comments and suggestions of the reviewers.

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