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RESEARCH ARTICLE

First Integrals and Closed-form Solutions of Some Singular Optimal Control Problems

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This article analyze singular optimal control problems (SOCP) from different areas of engineering and applied mathematics. We use the notion of partial Hamiltonian and we show that every singular optimal control problem can be written in the form of current value or standard Hamiltonian. The partial Hamiltonian approach is used to compute the partial Hamiltonian operators and first integrals. Then these first integrals are utilized to construct the closed-form solutions of hybrid vehicle optimal energy management model, optimal harvesting mathematical model and model of membrane filtration system. We explain how one can use partial Hamiltonian approach for both finite horizon and infinite horizon systems. This study provides a new way of solving singular optimal control problems.

KEYWORDS:
Partial Hamiltonian, Optimal control, Gauge terms, First integrals, Exact solutions.

1 | INTRODUCTION

Differential equations play a major role in modeling almost all physical, natural or biological process regardless of difficulty level. An analytical solution shows what variables are vital and importance of a specific variable relative to the other variables or parameters in the solution. This enable mathematicians or scientists who formulated the problem in their model to track the impact of the inputs on the outputs (their effect on the output, and the degree of that impact), with solid mathematical backing. Exact solutions are generally faster and more reliable. However, to compute exact solutions has thus been a challenge for researchers as it is difficult or some time impossible for non-linear differential equations using existing techniques.

First integrals are particular studied in the theory of Hamiltonian system. First integral plays a vital role and have widespread applications in studies of dynamical system. Their values can be estimated as constants with specific physical significance. The symmetry properties of a physical system are well connected to the first integral characterizing that system.

William Hamilton [1] utilized the notion of Legendre transformation to develop the theory of Hamiltonian mechanics. Dorodnitsyn and Kozlov [2] rewritten the Noether theorem in terms of Hamiltonian functions and symmetry operators. The partial Hamiltonian version of Noether's theorem was discussed by Naz et al. [5] to construct the first integrals and closed-form solutions of some current value Hamiltonian systems from economic growth. Later, this approach was illustrated through various models of mechanics by Naz (see [6]). Afterwards, the concepts of Lagrangian and Hamiltonian gained worth in other fields as well, few of which include continuum and fluid mechanics, mathematical biology, optimal control theory, engineering, quantum mechanics and some other fields dealing with optimization problems. The major techniques to deal with the optimization problems include calculus of variations, optimal control theory and dynamic programming. The calculus of variation approach is based on the existence of standard Lagrangian which in turn provides the Euler-Lagrange set of equations. The optimal control theory deals with finding the control laws of continuous time problems. The dynamic programming is an algorithmic approach for solving optimization problem by breaking it down into sub-problems.

Optimal control has been widely used in modeling physical systems and most of these models can be expressed as canonical Hamiltonian of two types: standard Hamiltonian and current value Hamiltonian. If a system admits Hamiltonian, then the Pontryagin maximum principle gives the necessary condition for the solution of finite and infinite horizon optimal control problems[3]. There have been numerous approaches to deal with the optimization models but most of the models were solved by using numerical techniques. The singular optimal control exhibit lacking of a general method to find the analytical solutions. The local stability of certain systems ([13]-[17]) has been discussed by numerical methods.

In this article, we investigate new approach which include first integrals and closed form solutions of singular optimal control problems. The method introduced here is applicable to any system with arbitrary number of state and control variables. The partial Hamiltonian approach[5] is used to compute partial Hamiltonian operators and first integrals. Using the partial Hamiltonian approach we find the first integrals and closed-form solutions of hybrid vehicle model, bio-economics growth model and water filtration system.

2 | PRELIMINARIES

In this section, we present the basic operators and definitions adapted from literature[5]-[8].
Let $(q^i, p_i) = (q^1, ..., q^n, p_1, ..., p_n)$ be the phase space coordinates and t the independent variable. Suppose $U(t, q^i, p_i)$ and $V(t, q^i, p_i)$ are differentiable and integrable functions and let

$$\dot{q}^i = U(t, q^i, p_i),$$
$$\dot{p}_i = V(t, q^i, p_i), \qquad i = 1, 2, ..., n, \tag{1}$$

be a system of $2n$ first order ordinary differential equations.
The operator

$$D = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i}, \quad i = 1, 2, ..., n.$$

is known as the total derivative operator with respect to time t. The summation convention applies for repeated indices. The relation between variables t, p and q is

$$\dot{p}_i = D(p_i), \quad \dot{q}^i = D(q^i).$$

The Euler operator and variational operator are given by

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - D\frac{\partial}{\partial \dot{q}^i},$$

and

$$\frac{\delta}{\delta p_i} = \frac{\partial}{\partial p_i} - D\frac{\partial}{\partial \dot{p}_i}.$$

The canonical or standard Hamiltonian $H(t, q^i, p_i)$ satisfies the canonical Hamiltonian system

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{2}$$

The generators of symmetry in phase space co-ordinates and time is defined as

$$X = \xi(t, q^i, p_i)\frac{\partial}{\partial t} + \eta^i(t, q^i, p_i)\frac{\partial}{\partial q^i} + \zeta_i(t, q^i, p_i)\frac{\partial}{\partial p_i}. \tag{3}$$

The operator in (3) is a Hamiltonian operator related to the Hamiltonian function $H(t, q^i, p_i)$, if there exists a gauge term $B(t, q^i, p_i)$ such that

$$\zeta_i \frac{\partial H}{\partial p_i} + p_i D(\eta^i) - X(H) - H D(\xi) = D(B). \tag{4}$$

Solving equation (4) for unknowns ξ, η^i and B, the first integrals for system (1) can be constructed from (see[5])

$$I = p_i \eta^i - \xi H - B. \tag{5}$$

The system of equations (1) can be expressed in canonical system (2) if there exist a canonical Hamiltonian. However, for nonholonomic system of equations it is not possible to find a standard Hamiltonian, so there was a need of some alternative technique that allow us to formulate the given problem in the form of canonical co-ordinates. Luckily, there is a road-map to this problem motivated by the partial Hamiltonian approach. The partial Hamiltonian technique[6] is an analogy of partial Noether's approach. In the absence of canonical Hamiltonian, the partial Hamiltonian function, also known as the current value Hamiltonian, satisfy the canonical Hamiltonian system.

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} + \Gamma_i(t, q^i, p_i). \tag{6}$$

The partial Hamiltonian determining equation in this case attains the form[5]

$$\zeta_i \frac{\partial H}{\partial p_i} + p_i D(\eta^i) - X(H) - H D(\xi) = D(B) - (\eta^i - \xi \frac{\partial H}{\partial p_i})(\Gamma_i). \tag{7}$$

Once the partial Hamiltonian operator and gauge terms are computed from (7), then the first integrals can be constructed from

$$I = p_i \eta^i - \xi H - B. \tag{8}$$

3 | OPTIMAL PATH OF SOME PHYSICAL PROBLEMS

In this section, we discuss some important physical problems. We express the given functional in the form of canonical/partial Hamiltonian system and then using Hamiltonian version of Noether's theorem[5] compute first integrals. These first integrals will be utilized along with the Hamiltonian system in order to find the closed form solutions.

3.1 | Hybrid vehicle optimal energy management

We study the following series hybrid energy management[13], where the power split between the different sources allows optimizing fuel consumption, the energy storage level is the state variable. Fuel consumption is the functional

$$\min_{u(t) \in \Phi(t)} J(u) = \int_0^T l(t, x, u) dt, \tag{9}$$

to be minimized subject to the energy storage level and specified limits

$$\frac{dx}{dt} = f(t, x, u), \tag{10}$$

with

$$f(t, x, u) = -\frac{a'[I_m(t) - u(t)]^2 + b'[I_m(t) - u(t)] + c'}{Q}, \tag{11}$$

where a', b' and c' are the constants of battery current model and Q is the energy storage capacity. $I_m(t)$ is the motor current and can be easily computed by using the proposed technique.
The proposed initial and final state of charge values of the batteries (Yuan et al., 2013)

$$x(0) = x_0, \ x(T) = x_T \tag{12}$$

must be controlled between two limits of the state of charge.
In (9) $l(t, x, u) = au(t)^2 + bu(t) + c$, is the fuel mass rate with constants of auxiliary power unit fuel consumption a, b and c. The current provided by the auxiliary power unit is denoted by $I_{apu}(t) = u(t)$, and is limited by the physical constraint: $I_{apu}(t) \in [0, \ \overline{I_{apu}}]$.
The Hamiltonian function associated with (9) is

$$H(t, x, u, \lambda) = au^2 + bu + c - \frac{\lambda}{Q}[a'(I_m - u)^2 + b'(I_m - u) + c'], \tag{13}$$

where $\lambda(t)$ is the co-state variable. The necessary first order conditions for optimal control for finite horizon are

$$\begin{aligned}\frac{\partial H}{\partial u} &= 0,\\ \frac{dx}{dt} &= \frac{\partial H}{\partial \lambda},\\ \frac{d\lambda}{dt} &= -\frac{\partial H}{\partial x}.\end{aligned} \tag{14}$$

System (14) along with Hamiltonian H in (13) yields:

$$\lambda = \frac{Q(2au+b)}{-2a'(I_m-u)-b'}, \tag{15a}$$

$$\frac{dx}{dt} = -\frac{1}{Q}[a'(I_m-u)^2+b'(I_m-u)+c'], \tag{15b}$$

$$\frac{d\lambda}{dt} = 0. \tag{15c}$$

We solve system of equations (15) by using the Hamiltonian version of Noether-type theorem. The canonical Hamiltonian determining Eq. (4) with the aid of (13) and (15) takes the form

$$\lambda\left[\eta_t+\frac{dx}{dt}\eta_x\right]-\left[au^2+bu+c+\lambda\frac{dx}{dt}\right]\left[\xi_t+\frac{dx}{dt}\xi_x\right]=B_t+\frac{dx}{dt}B_x, \tag{16}$$

where $\xi=\xi(t,x)$, $\eta=\eta(t,x)$ and $B=B(t,x)$. One can also choose ξ, η and B as function of $(t,\ x,\ \lambda)$ but one has to face difficulty in solving the determining equations. Separating Eq. (16) with respect to u and its powers, we obtain a system of partial differential equations in unknowns ξ , η and B:

$$u^4:\ \frac{aa'\xi_x}{Q}-\frac{\lambda a'^2\xi_x}{Q^2}=0, \tag{17}$$

$$u^3:\ \frac{ba'\xi_x}{Q}-\frac{ab'\xi_x}{Q}+\frac{4\lambda a'^2 I_m\xi_x}{Q^2}+\frac{2\lambda a'b'\xi_x}{Q^2}-\frac{2aa'I_m\xi_x}{Q}=0, \tag{18}$$

$$\begin{aligned}u^2:\ &\frac{a'B_x}{Q}+\frac{ac'\xi_x}{Q}+\frac{ca'\xi_x}{Q}+\frac{\lambda a'\xi_t}{Q}-\frac{bb'\xi_x}{Q}-\frac{\lambda a'\eta_x}{Q}-\frac{\lambda b'^2\xi_x}{Q^2}-\frac{6\lambda a'b'I_m\xi_x}{Q^2}\\ &+\frac{aa'I_m^2\xi_x}{Q}+\frac{ab'I_m\xi_x}{Q}-\frac{6\lambda a'^2I_m^2\xi_x}{Q^2}-\frac{2\lambda a'c'\xi_x}{Q^2}-\frac{2ba'I_m\xi_x}{Q}-a'\xi_t=0,\end{aligned} \tag{19}$$

$$\begin{aligned}u:\ &-\frac{b'B_x}{Q}+\frac{6\lambda a'b'I_m^2\xi_x}{Q^2}+\frac{4\lambda a'c'I_m\xi_x}{Q^2}+\frac{bc'\xi_x}{Q}-\frac{2a'I_mB_x}{Q}-\frac{cb'\xi_x}{Q}\\ &+\frac{\lambda b'\eta_x}{Q}+\frac{2\lambda a'I_m\eta_x}{Q}+\frac{bb'I_m\xi_x}{Q}-\frac{\lambda b'\xi_t}{Q}+\frac{ba'I_m^2\xi_x}{Q}-\frac{2ca'I_m\xi_x}{Q}\\ &-b\xi_t+\frac{4\lambda a'^2I_m^3\xi_x}{Q^2}-\frac{2\lambda a'I_m\xi_t}{Q}+\frac{2\lambda b'^2I_m\xi_x}{Q^2}+\frac{2\lambda b'c'\xi_x}{Q^2}=0,\end{aligned} \tag{20}$$

$$\begin{aligned}u^0:\ &-\frac{\lambda c'\eta_x}{Q}+\frac{cc'\xi_x}{Q}+\frac{\lambda c'\xi_t}{Q}-\frac{\lambda c'^2\xi_x}{Q^2}+\frac{a'I_m^2B_x}{Q}+\frac{b'I_mB_x}{Q}+\frac{c'B_x}{Q}+\lambda\eta_t\\ &-c\xi_t-\frac{\lambda a'I_m^2\eta_x}{Q}-\frac{\lambda b'I_m\eta_x}{Q}+\frac{ca'I_m^2\xi_x}{Q}+\frac{cb'I_m\xi_x}{Q}+\frac{\lambda a'I_m^2\xi_t}{Q}-\frac{\lambda a'^2I_m^4\xi_x}{Q^2}\\ &+\frac{\lambda b'I_m\xi_t}{Q}-\frac{\lambda b'^2I_m^2\xi_x}{Q^2}-B_t-\frac{2\lambda a'b'I_m^3\xi_x}{Q^2}-\frac{2\lambda a'c'I_m^2\xi_x}{Q^2}-\frac{2\lambda c'b'I_m\xi_x}{Q^2}=0.\end{aligned} \tag{21}$$

Solving Eqs. (17 - 19), we arrive at

$$\begin{aligned}\xi &= c_1t+c_2,\\ \eta &= c_1x+c_3,\\ B &= \frac{Qc_1ax}{a'}+r(t),\end{aligned} \tag{22}$$

where $r(t)$ is the constant of integration as t and x are acting as independent variables. In order to solve Eqs. (20 - 21), the following cases need to be considered.

Case 1: $c_1 = 0$

For this case, we find

$$\begin{aligned} \xi &= c_2, \\ \eta &= c_3, \\ B &= c_4. \end{aligned} \tag{23}$$

Making the specific choice for constants (taking one constant equal to unity and rest to zero) the Hamiltonian operators and the gauge terms are

$$X_1 = \frac{\partial}{\partial t},\ B = 0, \tag{24}$$
$$X_2 = \frac{\partial}{\partial x},\ B = 0.$$

Using (5), we evaluate the following first integrals with the aid of Hamiltonian operators and the gauge terms

$$I_1 = -au^2 - bu - c + \frac{\lambda}{Q}[a'(I_m - u)^2 + b'(I_m - u) + c'], \tag{25}$$

$$I_2 = \lambda. \tag{26}$$

We use the first integrals obtained in Case 1 to construct the closed-form solution. Setting $I_1 = A_1$, $I_2 = A_2$ and using (15a) - (15b), we obtain

$$x(t) = \frac{x_0 T + (x_T - x_0)t}{T}, \tag{27}$$

$$I_m(t) = \frac{-Qab'T - Qba'T + (A_1 a' - Qa)\sqrt{4Qa'x_0T - 4Qa'x_TT - 4a'c'T^2 + b'^2T^2}}{2Qaa'}, \tag{28}$$

$$u(t) = \frac{-QbT + A_1\sqrt{4Qa'x_0T - 4Qa'x_TT - 4a'c'T^2 + b'^2T^2}}{2Qa}. \tag{29}$$

Case 2: $c_1 \neq 0$, $I_m = -\frac{ab'+ba'}{2aa'}$.

Some simple but lenghty manipulations yield

$$\begin{aligned} \xi &= c_1 t + c_2, \\ \eta &= c_1 x + c_3, \\ B &= \frac{4Qa^2a'x + 4a^2a'c't - a^2b'^2t - 4aa'^2ct + a'^2b^2t}{4aa'^2}c_1 + c_4. \end{aligned} \tag{30}$$

We obtain the following operator and the gauge term

$$X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x},\ B = \frac{4Qa^2a'x + 4a^2a'c't - a^2b'^2t - 4aa'ct + a'^2b^2t}{4aa'^2}.$$

In this case, following first integral corresponding to the Hamiltonian operator and gauge term is computed using Eq. (5)

$$I_3 = \lambda x - tH - \frac{4Qa^2a'x + 4a^2a'c't - a^2b'^2t - 4aa'ct + a'^2b^2t}{4aa'^2}. \tag{31}$$

Writting $I_3 = A_3$ and solving for $x(t)$, we obtain

$$x(t) = -\frac{(4u(t)^2a^2a'^2 + 4u(t)aba'^2 + 4a^2a'c' - a^2b'^2 + b^2a'^2)t}{4a^2a'Q} + x_0, \tag{32}$$

Substituting the value of $x(t)$ from Eq. (32) and initial and boundary conditions from Eq. (12) in Eq. (15b), we obtain

$$u(t) = \frac{-ba'T \pm a\sqrt{-4a'x_TQT + 4a'x_0QT - 4a'c'T^2 + b'^2T^2}}{2aa'T}, \tag{33}$$

which represents the closed form solution of (9). The hybrid vehicle uses the engine as primary source and to save fuel it can be switched on and off. So the optimal control can be obtained for a constant supply of current.

3.2 | Optimal Harvesting Problem

Allee effect plays an important role in the population dynamics. There are several process which generate allee effect. In this section, we discuss an important depensation growth model[14]

$$\max_{E\in[E_{min},\ E_{max}]} I(E) = \int_0^{\infty} e^{-\delta t}(pE(t)x(t) - cE(t))dt,$$

subject to:

$$\frac{dx}{dt} = x(x-k)(1-x) - Ex, \quad x(0) = x_0, \tag{34}$$

where E (effort) and x (renewable resource) are control and state variables. Moreover, $c,\ p$ and δ denote cost per unit effort, price per unit harvest and discount factor, respectively.
The current value Hamiltonian function for depensation growth model is

$$H(t, x, E, \lambda) = pEx - cE + \lambda[x(x-k)(1-x) - Ex], \tag{35}$$

where λ is the co-state variable.
The necessary first order conditions for infinite horizon continuous optimal control yield

$$\begin{aligned} \frac{\partial H}{\partial E} &= 0, \\ \frac{dx}{dt} &= \frac{\partial H}{\partial \lambda}, \\ \frac{d\lambda}{dt} &= -\frac{\partial H}{\partial x} + r\lambda. \end{aligned} \tag{36}$$

System (36) with the help of (35) gives

$$\lambda = p - \frac{c}{x}, \tag{37a}$$

$$\frac{dx}{dt} = x(x-k)(1-x) - Ex, \tag{37b}$$

$$\frac{d\lambda}{dt} = -pE + \lambda(-2x + 3x^2 + k - 2kx + E) + \lambda\delta. \tag{37c}$$

Note that the Hamiltonian is linear in control variable E, hence, it is difficult to find the optimal solution. The Generalized Legendre-Clebsch condition is used as singular control condition $-\frac{\partial}{\partial E}(\frac{\partial^2}{\partial t^2}H_E) \geq 0$, which gives rise to

$$k = \frac{3px(t)^2 + 2cx(t) - 2px(t) + \delta p - c}{4px(t) - c - p}. \tag{38}$$

In addition, if $x^*(t)$ is the optimal singular solution, then the admissible singular effort will be

$$E^*(t) = x^*(t)^2 + kx^*(t) + x^*(t) - k, \tag{39}$$

hence, the optimal harvesting can be defined as

$$E(t) = \begin{cases} E_{min} & \text{if } x(t) < x^*, \\ E_{max} & \text{if } x(t) > x^*, \\ E^* & \text{if } x(t) = x^*, \end{cases} \tag{40}$$

where $E(t) = E_{min} = 0$ when $x = 1$ and $E(t) = E_{max} = \frac{(k-1)^2}{4}$ when $x = \frac{k+1}{2}$, which implies $1 < x^* < \frac{k+1}{2}$.
In order to find the value of x^*, we use the partial Hamiltonian approach. Assuming that $\xi = \xi(t, x),\ \eta = \eta(t, x),\ B = B(t, x)$, we can write the partial Hamiltonian determining equation (7) as

$$\zeta\frac{\partial H}{\partial \lambda} + \lambda(\eta_t + \dot{x}\eta_x) - \eta\frac{\partial H}{\partial x} - H(\xi_t + \dot{x}\xi_x) = B_t + \dot{x}B_x - (\eta - \dot{x}\xi)\Gamma. \tag{41}$$

Substituting the values of H and $\dot{x}$ in Eq. (41) to obtain

$$\begin{aligned}&\lambda[\eta_t + (x - x^2 - k + kx - E)x\eta_x] - \eta[pE + \lambda(2x - 3x^2 - k + 2kx - E)]-\\&[(px - c)E + \lambda x(x - x^2 - k + kx - E)][\xi_t + (x - x^2 - k + kx - E)x\xi_x]\\&= B_t + (x - x^2 - k + kx - E)xB_x - \delta\lambda[\eta - (x - x^2 - k + kx - E)x\xi].\end{aligned} \tag{42}$$

Separation of Eq. (42) with respect to powers of E yields

$$E^2 : \; -\xi_x\lambda x^2 + \xi_x px^2 - \xi_x cx = 0, \tag{43}$$

$$\begin{aligned}&E : \; x(x-1)(k-x)(2\lambda x - px + c)\xi_x + (\lambda x - px + c)\xi_t\\&- \lambda x\eta_x + xB_x + (\lambda - p)\eta + \delta\lambda x\xi = 0,\end{aligned} \tag{44}$$

$$\begin{aligned}&rest : \; [\delta\lambda x^3 - (\delta k + \delta)\lambda x^2 + k\delta\lambda x]\xi - \lambda x^2(x-1)^2(-x+k)^2\xi_x\\&+ [3\lambda x^2 - (2k+2)\lambda x + (k+\delta)\lambda]\eta + \lambda(x^2 - x)(-x+k)\eta_x\\&- \lambda x(x^2 - x)(k - x)\xi_t - (x^2 - x)(k - x)B_x + \lambda\eta_t - B_t = 0.\end{aligned} \tag{45}$$

Eqs. (43)-(44) give rise to

$$\xi(t,x) = F_1(t), \tag{46}$$

$$\eta(t,x) = \delta x lnx F_1(t) + x lnx F_1^{\dot{}}(t) + xF_2(t), \tag{47}$$

$$B(t,x) = \delta px(lnx - 1)F_1(t) + lnx(px - c)F_1^{\dot{}}(t) + pxF_2(t) + F_3(t). \tag{48}$$

Replcing the values of ξ, η and B in Eq. (45) finally provides

$$F_1(t) = c_1, \; F_2(t) = -c_1\delta \ln x, \; F_3(t) = c_2 \; .$$

Which in turn results in

$$\begin{aligned}&\xi(t,x) = c_1,\\&\eta(t,x) = 0,\\&B(t,x) = -c_1\delta px + c_2.\end{aligned} \tag{49}$$

For this case, the operator and the guage term are

$$X = \frac{\partial}{\partial t}, \quad B = -\delta px. \tag{50}$$

Using formula (5), the first integral corresponding to the operator and gauge term is

$$I_1 = cE - pEx - \lambda[x^2 - x^3 - kx + kx^2 - Ex] + \delta px. \tag{51}$$

Taking in account $DI_1 = 0$, we have $I_1 = A_1$. Eq. (51) can be expressed as

$$cE - pEx - \lambda[x^2 - x^3 - kx + kx^2 - Ex] + \delta px = A_1.$$

Eqs. (37a-37c) with the help of (38) results in

$$\begin{aligned}&x^*(t) = \beta,\\&\lambda(t) = \frac{px_0 - c}{x_0},\end{aligned} \tag{52}$$

where β is an arbitrary constant along with the terminal condition

$$\lim_{t\to\infty} x(t) = \begin{cases} 0 & \text{if } x(0) < \frac{(k+1)}{2}, \\ \frac{(k+1)}{2} & \text{if } x(0) \geqslant \frac{(k+1)}{2}. \end{cases} \tag{53}$$

Using the terminal condition, we find

$$x^*(t) = \begin{cases} 0 & \text{if } x(0) < k, \\ -1 + \sqrt{(2-\delta)} & \text{if } x(0) \geqslant k. \end{cases} \tag{54}$$

The partial Hamiltonian approach provides an alternative way to obtain exact solutions of singular optimal control problems. Observe that, when the threshold value k is less than the initial stock value x_0, a stable optimal solution exist. The closed-form solutions are interpreted graphically. We consider the following three sets of parameter values and initial states $x_1 = 0.3,\ x_2 = 0.8$.

Set 1: When $p = 1.6,\ c = 0.04,\ k = 0.08,\ \delta = 0.27,\ E_{min} = 0,\ E_{max} = 0.23$.
Taking set 1 in account, the singular optimal solution is only possible if the initial state is greater than the Allee threshold value. Since $x_1 > k$, we have a stable optimal solution. Moreover, $x^* > x_1$ leads to $E = E_{min}$ and the corresponding singular effort is $E^* = 0.1611$.

We compare the optimal control effort policies with the work of Sirinvasu et al. [14], using the same parameter values and initial conditions. For the parameter set 1, $x^* = 0.60741$ is the optimal singular solution and $E_2^* = 0.2071$ is the corresponding singular effort.

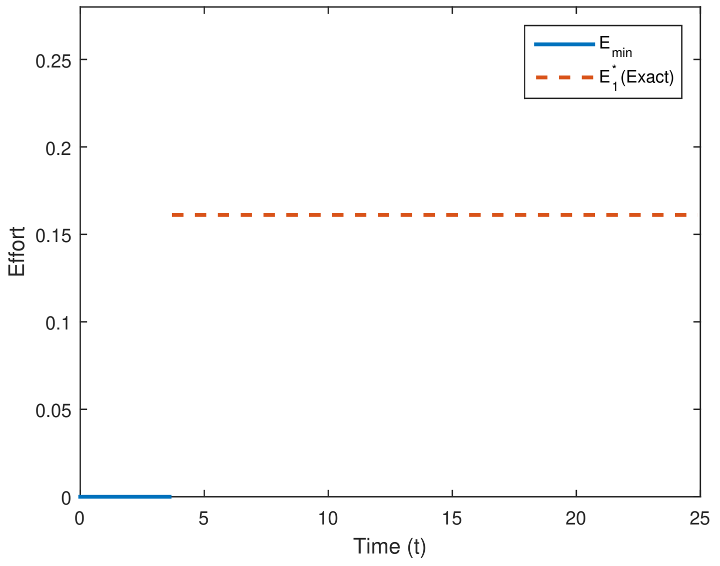

FIGURE 1 The optimal effort policy when $k < x_1 < x^*$.

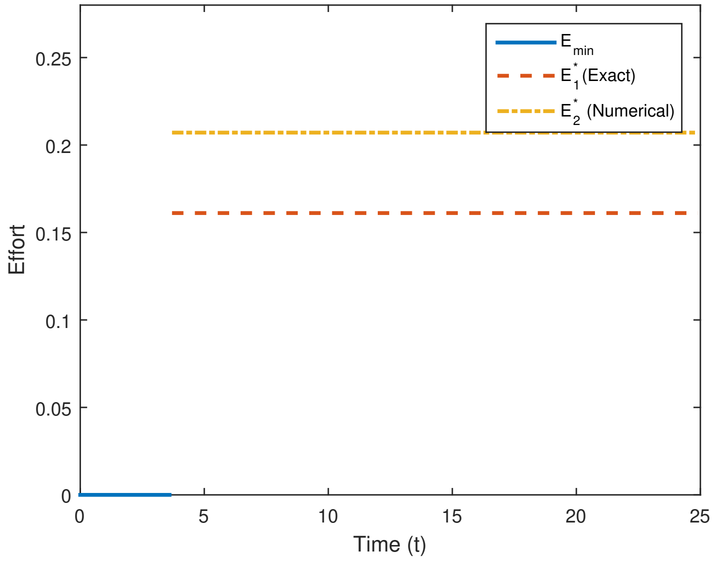

FIGURE 2 Comparison of exact and numerical solutions: when $k < x_1 < x^*$.

The optimal strategy for the optimal control problem (34) with initial state value $x_2 > x^* > k$ is given by the effort $E = E_{max}$, before switching to the singular optimal effort $E^* = 0.1611$ (Fig. 3). **Set 2:** When $p = 14.56$, $c = 2.3$, $k = 0.146$, $\delta = 0.355$, $E_{min} = 0$, $E_{max} = 0.25$.
In this case, $k < x_1 \leq x^* < x_2$ and the singular effort is $E^* = 0.0980$. The optimal effort policies corresponding to the initial states are presented in (Fig. 5 -8).

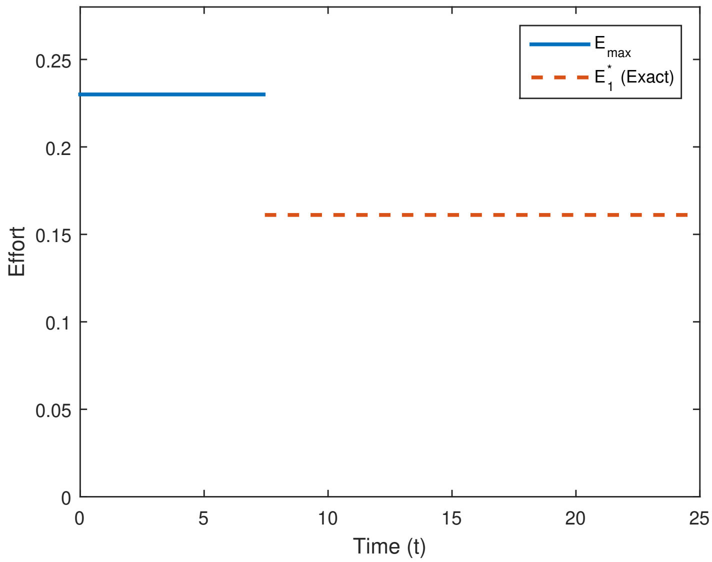

FIGURE 3 The optimal effort policy when $k < x^* < x_2$.

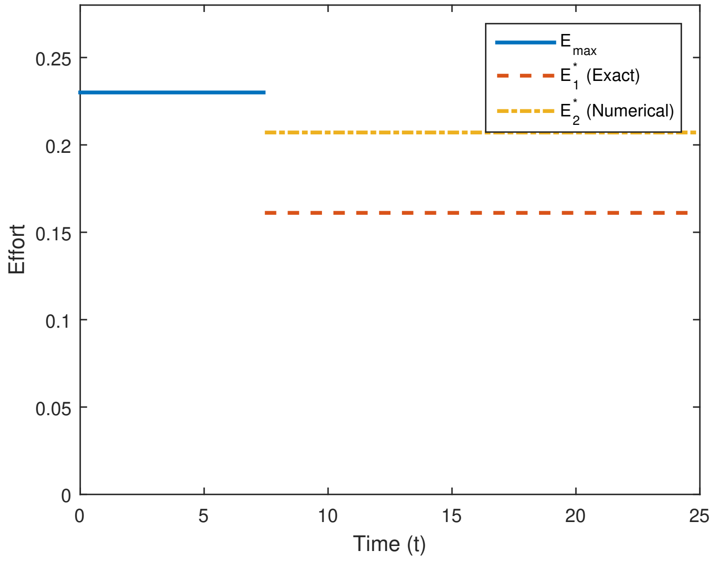

FIGURE 4 Comparison of exact and numerical solutions: when $k < x^* < x_2$.

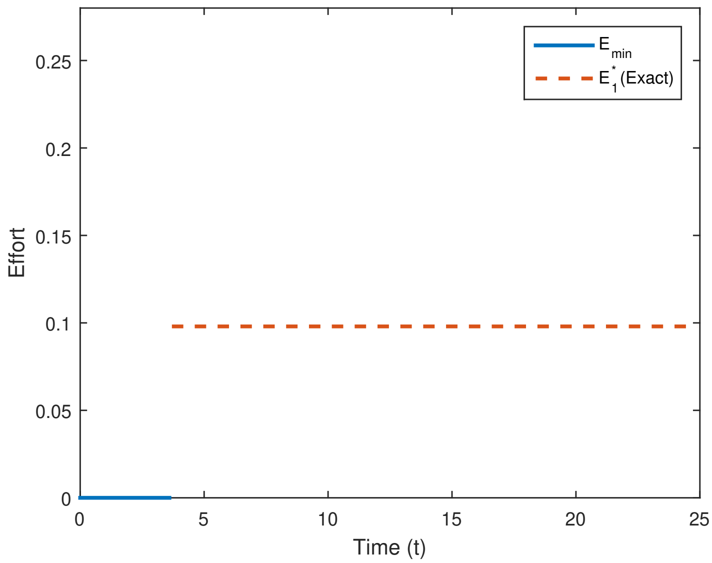

FIGURE 5 The optimal effort policy with initial state x_1 satisfying $k < x_1 \leq x^*$.

The comparison of exact and numerical optimal control strategies when initial state value is less than the optimal value is given in (Fig. 6) and it can be seen that the exact value requires less effort in all the cases.

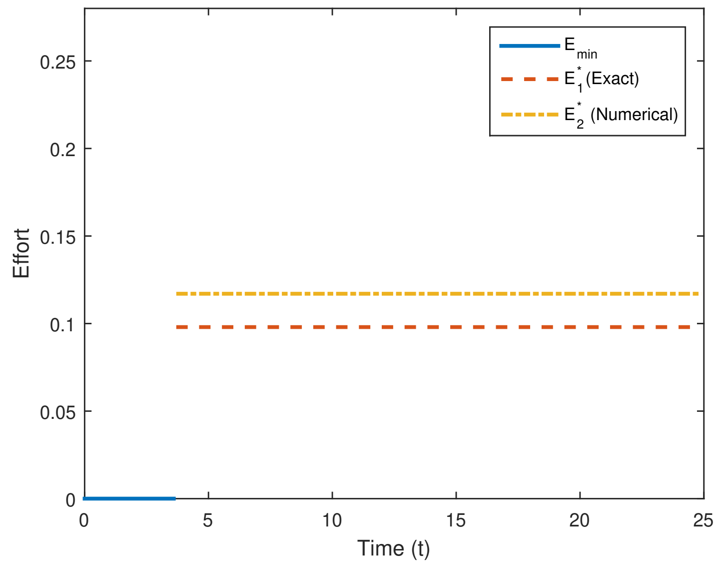

FIGURE 6 Comparison of exact and numerical solutions when $k < x_1 \leq x^*$.

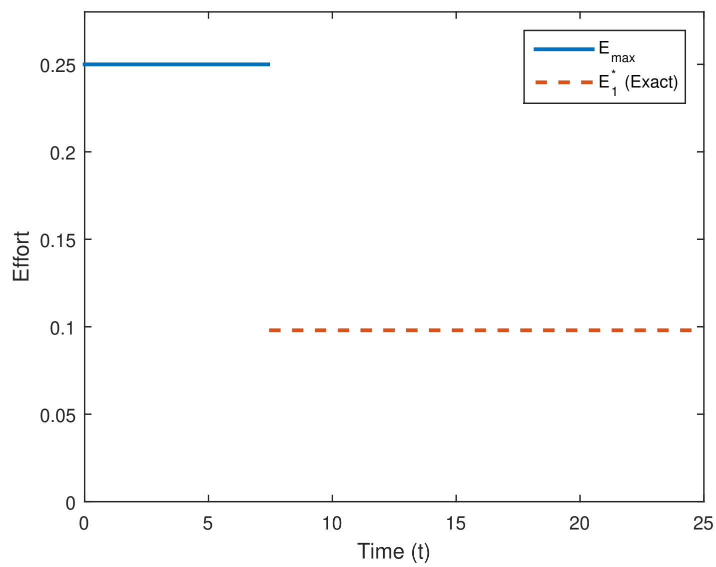

FIGURE 7 The optimal effort policy with initial state x_2 satisfying $k < x^* < x_2$.

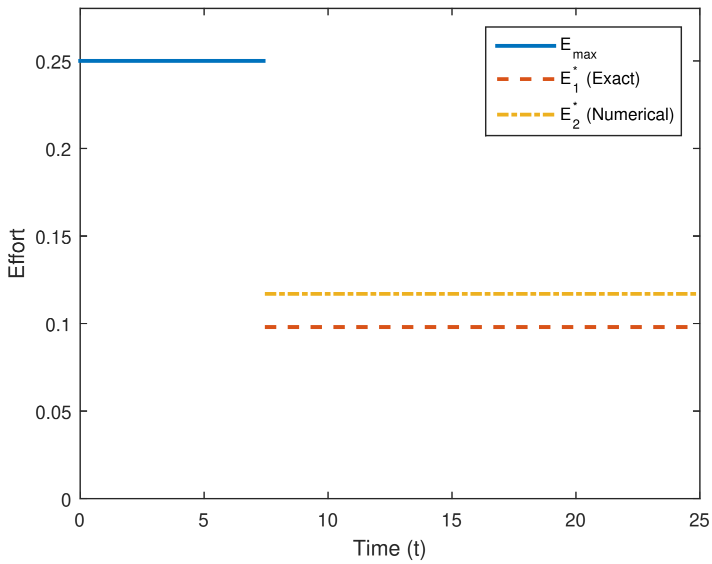

FIGURE 8 Comparison of exact and numerical solutions when $k < x^* < x_2$.

Set 3: When $p = 12.6,\ c = 1.5,\ k = 0.81,\ \delta = 0.23,\ E_{min} = 0,\ E_{max} = 0.02$.
For set 3, all the initial states are less than the threshold value i.e $k = 0.81$, making all the solutions inadmissible.

3.3 | Membrane filtration system

In this section, we discuss the membrane filtration system which are widely used as physical separation techniques in different industrial fields e.g. water desalination, waste-water treatment, food, medicine and biotechnology etc. The membrane provides a barrier that separates substances. Sequences of filtration and relaxation (cleaning) are performed to limit membrane fouling. The following system is operated by alternating two functional modes: Filtration and Relaxation. For this reason, we consider a control that only takes value 1 during filtration period and 0 during membrane relaxation period. The dynamics of fouling layer formed by attachment of mass m onto the membrane surface is governed by

$$\frac{dm}{dt} = u(t)f_1(m(t)) - \{1 - u(t)\}f_2(m(t)). \tag{55}$$

Our aim is to determine the optimal switching between two functioning nodes that maximize the water production of water filtration process during the time interval [0,T]. The associated objective function is

$$\max_{u(t)\in U} J(u) = \int_0^T u(t)g(m(t))dt. \tag{56}$$

Let us consider an experimentally validated functions[18]

$$f_1(m) = \frac{b}{(e+m)},\ f_2(m) = am,\ g(m) = \frac{1}{e+m},$$

where $a,\ b$ and e are positive numbers, f_1 and g are decreasing functions whereas f_2 is an increasing function. The following functional represents the aforesaid optimal membrane filtration problem:

$$\max_{u(t)\in U} J(u) = \int_0^T u(t)\frac{1}{e+m}dt, \tag{57}$$

which we need to maximize with respect to the constraint,

$$\frac{dm}{dt} = \frac{bu(t)}{(e+m)} - \{1 - u(t)\}am, \quad m(0) = 0. \tag{58}$$

The Hamiltonian H associated with the problem is

$$H(m, \lambda, u) = \frac{u}{e+m} + \lambda(t)\left[\frac{ub}{e+m} - (1-u)am\right],$$

where $\lambda(t)$ is an costate variable. Pontryagin's maximum principle provides the following set of equations which are regarded as an essential criteria for optimality

$$\lambda(t) = -\frac{1}{b + (e+m)am}, \tag{59a}$$

$$\frac{dm}{dt} = \frac{ub}{e+m} - (1-u)am, \tag{59b}$$

$$\frac{d\lambda}{dt} = \frac{u}{(e+m)^2} - \lambda\left[-\frac{ub}{(e+m)^2} - a + ua\right], \tag{59c}$$

together with terminal condition $\lambda(T) = 0$.
Singular control strategy shows that

$$u^*(t) = \begin{cases} u_{min} & \text{if } m(t) > m^* \\ u_{max} & \text{if } m(t) < m^* \\ \bar{u} & \text{if } m(t) = m^*, \end{cases} \tag{60}$$

where m^* can be determined using the singular control condition $-\frac{\partial}{\partial u}(\frac{\partial^2}{\partial t^2}H_u) = 0$, which gives $m^* = \sqrt{\frac{b}{a}}$. To determine the value of $\bar{u}$, we use the partial Hamiltonian approach[5].

Assuming $\xi=\xi(t,m)$, $\eta=\eta(t,m)$ and $B=B(t,m)$, the partial Hamiltonian determining equation (4) gives rise to

$$\begin{aligned}&-\lambda\eta_m am-\lambda\eta au-\xi_m\frac{bu^2}{(e+m)^2}+\lambda\xi_t am-\lambda\xi_m a^2m^2-B_m\frac{bu}{e+m}-B_m amu\\&-B_t+2\lambda\xi_m a^2m^2u-\lambda\xi_m a^2m^2u^2-\lambda\xi_m\frac{b^2u^2}{(e+m)^2}-\lambda\xi_t amu+\xi_m\frac{amu}{e+m}\\&-\lambda\xi_t\frac{bu}{e+m}+\lambda\eta_m amu+\lambda\eta\frac{bu}{(e+m)^2}+\lambda\eta_t-\xi_m\frac{amu^2}{e+m}+\lambda\eta_m\frac{bu}{e+m}\\&-\xi_t\frac{u}{e+m}+\lambda\eta a+\eta\frac{u}{(e+m)^2}+B_m am-2\lambda\frac{abmu^2}{e+m}+2\lambda\xi_m\frac{abmu}{e+m}=0.\end{aligned}\tag{61}$$

The separation of Eq. (61) with respect to powers of λ yields a system of linear partial differential equations

$$\begin{aligned}\lambda:\ &\xi_t(am-amu)-\xi_m(a^2m^2+a^2m^2u^2-2a^2m^2)+\eta_t+\eta a-\eta au\\&+am\eta_m u-am\eta_m+\eta\frac{bu}{(e+m)^2}-\xi_m\frac{b^2u^2}{(e+m)^2}-\xi_m\frac{2abmu}{e+m}\\&-\xi_t\frac{bu}{e+m}+\eta_m\frac{bu}{e+m}-2\frac{abmu^2}{e+m}=0,\end{aligned}\tag{62}$$

$$\begin{aligned}\lambda^0:\ &-\frac{\xi_m bu^2}{(e+m)^2}-\frac{B_m bu}{e+m}+\frac{\xi_m amu}{e+m}-\frac{\xi_m amu^2}{e+m}-\frac{\xi_t u}{e+m}+\frac{\eta u}{(e+m)^2}\\&+amB_m-amuB_m-B_t=0.\end{aligned}\tag{63}$$

Separating Eq. (62) with respect to the powers of u yields

$$u^2:\ -\xi_m a^2m^2-2\xi_m\frac{abm}{e+m}-\xi_m\frac{b^2}{(e+m)^2}=0,\tag{64}$$

$$\begin{aligned}u:\ &\eta_m am-\eta a+2a^2m^2\xi_m-\xi_t am-\xi_t\frac{b}{e+m}+\eta_m\frac{b}{e+m}\\&+2\xi_m\frac{abm}{e+m}+\eta\frac{b}{(e+m)^2}=0,\end{aligned}\tag{65}$$

$$u^0:\ \xi_t am-\xi_m a^2m^2-\eta_m am+\eta_t+\eta a=0.\tag{66}$$

From Eq. (64) and (65), we obtain

$$\xi=F_1(t),$$

$$\eta=\left[\frac{\dot F_1\ln(aem+am^2+b)}{2a}-\frac{\dot F_1 e\arctan\left(\frac{ae+2am}{\sqrt{a^2e^2-4ab}}\right)}{2\sqrt{a^2e^2-4ab}}+F_2\right]\left[\frac{aem+am^2+b}{e+m}\right].$$

Substituting the values of ξ and η in Eq. (66), we find

$$F_1(t)=c_1t+c_2,\tag{67}$$

$$F_2(t)=c_3.\tag{68}$$

Similarly, separating Eq. (63) with respect to the powers of u and substituting the values of ξ and η from (67) and (68), the solution of determining equation yields,
$\xi(t,m)=c_2$, $\eta(t,m)=0$, $B(t,m)=c_4$.
Using formula (5), we obtain the following first integral of (56)

$$I=-\frac{u(t)}{e+m(t)}-\lambda(t)\left[\frac{u(t)b}{e+m(t)}-am(t)+u(t)am(t)\right].\tag{69}$$

Since this first integral is a solution along the constant, hence, writing $I=A_1$, we have

$$-\frac{u(t)}{e+m(t)}-\lambda(t)\left[\frac{u(t)b}{e+m(t)}-am(t)+u(t)am(t)\right]=A_1.\tag{70}$$

The solution of (59a-59c) along with (70) yields

$$\bar{u} = \frac{e\sqrt{ab} + b}{e\sqrt{ab} + 2b}, \tag{71}$$

$$\lambda = -\frac{1}{e\sqrt{ab} + 2b}. \tag{72}$$

We have plotted the closed-form solution curves for optimal control $u(t)$ and the dynamics of mass $m(t)$ onto membrane surface in Fig. 9 . All calculations were performed for specific parameter values, which has been experimentally validated by Benyahia et al.[18]. In Fig. 9 , as $m_0 \leq \bar{m}$, the process operates in filtration ($u = 1$) until $m(t) = \bar{m}$. Then the singular control is applied before switching back to filtration until total time.

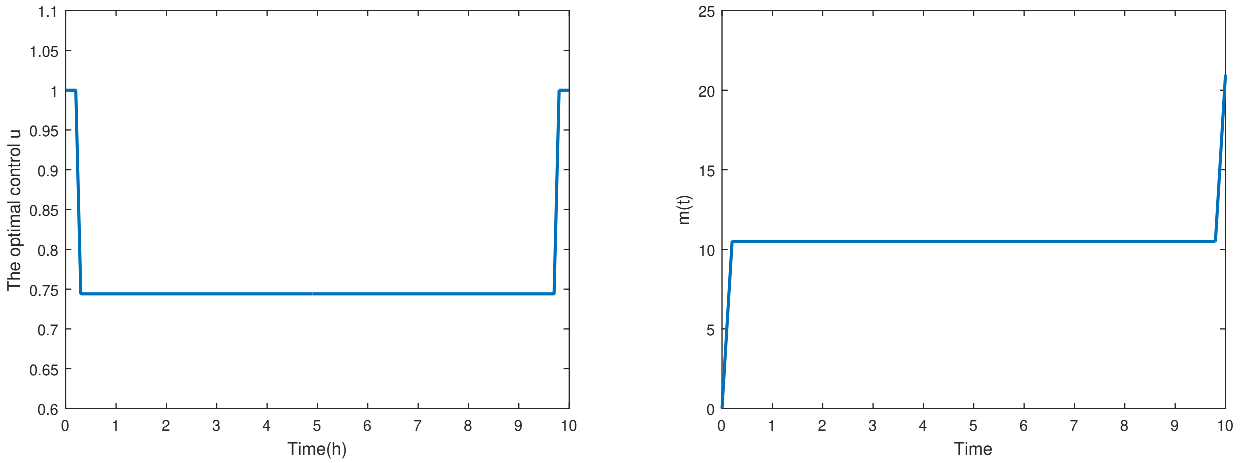

FIGURE 9 The dynamics of $u(t)$ and $m(t)$ for $a = 25,\ b = 0.00275,\ e = 20$

4 | CONCLUDING REMARKS

The optimal control problems exist in variety of areas of applied sciences. The partial Hamiltonian approach is a systemic technique for the reductions of Hamiltonian/partial Hamiltonian systems using the first integrals. We have utilized partial Hamiltonian approach along with Pontryagin's maximum principle to construct the closed-form solutions of optimal control problems. This elegant approach has been used to solve singular optimal control problems of varying complexity level. In this article, we have restricted our work to systems having one control and one state variable. A hybrid vehicle optimal energy management model from engineering has been studied using partial Hamiltonian approach to obtain three first integrals. Using these first integrals with ODEs obtained through Pontryagin's principle, we evaluate the closed-form solutions for two distinct cases that arise during the solution process of the model. In section 3.2, we have discussed the bio-economic growth model for the well known harvesting problem. The partial Hamiltonian determining equation (7) has been formulated using the current value Hamiltonian of this singular optimal control problem. We obtain one first integral, using which we evaluate the explicit expressions for optimal values of control and state variables. The last section of this article discusses a membrane filtration system for which we have considered experimentally validated functions for two functioning nodes. Application of partial Hamiltonian approach to the system yields the first integral which is utilized to evaluate the optimal state expressions for the target variables. For physical illustration, we have graphically presented the solutions for singular control problems and provided the comparison of our solutions with those obtained using numerical scheme. It is observed that both solutions are in good agreement. We have shown that our algorithmic approach presents an alternative way for the construction of exact solutions for a wide range of optimal control models belonging to diverse areas.

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