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# Rough sets of multi-granulation with topological approaches based kinds of neighborhood systems

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**Abstract** Recently, much attention has been given to multi-granulation rough sets (briefly, MGRS) and different kinds of multi-granulation rough set models have been developed from various viewpoints. In this paper, we propose two types of MGRS models under neighborhood systems from the topological view, where a target concept is approximated by employing the $j$-neighborhoods and $j$-adhesion neighborhoods of objects in a given universe set. Therefore, we investigate some of basic properties of the two types of MGRS models, and discuss the relationships and differences among the classical MGRS model and some other new models. Also, for each new MGRS model, an algorithm will be presented.

**Keywords** Rough sets · Multi-granulation Rough Sets (MGRS) · Neighborhood systems (NS) · Topology · Multi-granulation topological rough space (MGTRS)

## 1 Introduction

Rough set theory, proposed by Pawlak [16,17], is a mathematical tool for characterizing the uncertainty by the difference between the lower and upper approximations. Presently, Presently, rough set has been demonstrated to be useful in various areas such as pattern recognition, image processing, feature selection, neural computing, conflict analysis, decision support, data mining and knowledge discovery process from large data sets [16-20]. A fundamental concept in Pawlak's rough set model is the indiscernibility relation, which is an equivalence relation. By such relation, the equivalence classes are then regarded as the basic knowledge for the construction of the lower and upper approximations. The computation of approximations is a necessary step for attribute reduction and knowledge discovery. As one of the most important research topics along with the fast development of rough set theory, attribute reduction has aroused wide concern and study, and many attribute reduction techniques have been developed in last twenty years. In the information age, complex data is often represented by a multi-source information system [8] in which data come from different sources. How to fuse such data has become a challenging task in the community of granular computing (GrC) [33]. Information granulation is one of three basic issues: information granulation, organization, and causation in granular computing. Information granulation involves decomposition of whole data into parts called granules. Then, these granules are organized into a granular structure (or a granular space). In granular computing, the granules induced by an equivalence relation (or a tolerance relation) form a set of equivalence classes (or tolerance classes), in which each equivalence class (or tolerance class) can be regarded as a Pawlak information granule (or a tolerance information granule). In the view point of Granular Computing [32,34],Pawlak's rough set model and most of its extensions are constructed based on only one granular structure,which is induced by a binary relation(a partition or a covering).Thus,one may call those models the single-granulation rough sets.

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As Qian et al.[25], have mentioned that,in many cases,a target concept is needed to describe concurrently from some independent environments,that is,multi-granulation spaces are needed.Therefore, Qian and Liang [24,25] introduced the concept of multi-granulation rough sets(MGRS),where the approximations of a set of objects are defined by using multi-equivalence relations.The main difference between single-granulation rough sets and multi-granulation ones lies in that the approximations of a target concept in multi-granulation rough sets are constructed by using multi-distinct sets of information granules.When two attribute subsets in an information system contradict each other or possess an inconsistent relationship, MGRS will show its advantages for knowledge discovery [25]. Many scholars have extended the classical MGRS by using various generalized binary relations. For instance, Qian et al. [26] presented a multi-granulation rough set based on multiple tolerance relations in incomplete information systems. Lin et al. [14] proposed a covering-based pessimistic multi-granulation rough set, Xu et al. [29] proposed another generalized version, called variable precision multi-granulation rough set, and Yang et al. [30] proposed a multi-granulation rough set based on a fuzzy binary relation. In fact, the basic idea of multi-granulation has been also discussed by Khan et al. in [8].
From discussions above, our objective is to develop the multi-granulation rough set theory via topology theory by using two kinds of neighborhood systems.

## 2 Preliminaries

In this section, we introduce some fundamental key concepts of topology, rough set and neighborhood systems [5,16]. Throughout this paper, we suppose that the universe $\Omega$ is a non-empty finite set.
We present a brief overview of topological space, a closure operator, an interior operator, and a topology based on a set. They are all important concepts in topology theory and they were used to study rough sets [11,23,31]. In this paper, these topological tools are also employed to investigate multi-granulation rough sets.

**Definition 1** [5] A topological space is a pair $(\Omega, \tau)$ consisting of a set $\Omega$ and a family $\tau$ of subset of $\Omega$ satisfying the following conditions:
1) $\Omega, \phi \in \tau$,
2) $\tau$ is closed under arbitrary union,
3) $\tau$ is closed under finite intersection.

The pair $(\Omega, \tau)$ is called a topological space. The elements of $\Omega$ are called the points of the space and the subsets of $\Omega$ belonging to $\tau$ are called open sets. The complement of the subsets of $\Omega$ belonging to $\tau$ are called closed sets. The family $\tau$ of all open subsets of $\Omega$ is called a topology for $\Omega$.

**Definition 2** [31] Let $R$ be any binary relation on a non empty set $\Omega$, for any set $A \subseteq \Omega$. The interior of $A$ according to $R$ defined as: $int(A)$=$\{x \in \Omega : xR \subseteq A\}$.
If it satisfies the following conditions, then we call it a interior operator $int : 2^{\Omega} \to 2^{\Omega}$ on $\Omega$. $\forall X, Y \subseteq \Omega$,
(1) $int(\Omega) = \Omega$,
(2) $int(X) \subseteq X$,
(3) $int(int(X)) = int(X)$,
(4) $int(X \cap Y) = int(X) \cap int(Y)$.

**Definition 3** [31] Let $R$ be any binary relation on a non empty set $\Omega$, for any set $A \subseteq \Omega$. The interior of $A$ according to $R$ defined as: $cl(A)$=$\{x \in \Omega : xR \cap A \neq \phi\}$.
If it satisfies the following conditions, then we call it a closure operator $cl : 2^{\Omega} \to 2^{\Omega}$ on $\Omega$. $\forall X, Y \subseteq \Omega$,
(1) $cl(\phi) = \phi$,
(2) $X \subseteq cl(X)$,
(3) $cl(cl(X)) = cl(X)$,
(4) $cl(X \cup Y) = cl(X) \cup cl(Y)$.

Obviously, if $R$ is an equivalence relation $xR$=$[x]_R$ and these definitions are equivalent to the original Pawlak$\acute{s}$ definitions.
In [24], Qian analyzed some restrictions of Pawlak classical rough set in practice and proposed a new extension of rough set i.e., multi-granulation rough sets, in which a target concept can be approximated by multiple equivalence relations according to a users different requirements. In other words, a target concept can be approximated by multiple granulation spaces in the view of granular computing [25]. Assume that $\Omega$ is a finite non-empty universe of discourse. Let $R$ be an equivalence relation on $\Omega$, $\Omega/R$ is a corresponding partition of $\Omega$, denoted by $\Omega/R$=$\{[x]_R : x \in \Omega\}$ in which $[x]_R$=$\{y : y \in \Omega, xRy\}$ is an equivalence class consisting $x.\Omega/R$ can generate a topological space, denoted as $(\Omega, \tau_R)$, and $\Omega/R$ is a topology base of $\tau_R$, each subset of $\tau_R$ is both open and close [4].

**Definition 4** [24] Let $(\Omega, AT, f)$ be an information system. Suppose that $X \subseteq \Omega$, $R_1, R_2, ...R_n$ be $n$ equivalence relations on $\Omega$, the lower approximation $\underline{\sum_{i=1}^{n} R_i}(X)$ and the upper approximation $\overline{\sum_{i=1}^{n} R_i}(X)$ of $X$ with respect to $R_1, R_2, ...R_n$ are defined as follows, respectively,
(1) $\underline{\sum_{i=1}^{n} R_i}(X)$=$\{x \in \Omega : \vee([x]_{R_i} \subseteq X), i \leq n\}$.

(2) $\overline{\sum_{i=1}^{n} R_i}(X)$=$\{x \in \Omega : \wedge([x]_{R_i} \cap X \neq \phi), i \leq n\}$.

From the above expressions, the operator '$\vee$' is a disjunctive operator which here indicates that in multiple independent granular structures, one needs only at least one granular structure to satisfy with the inclusion condition between an equivalence class and a target concept. The expression (2) is the upper approximation of the optimistic multi-granulation rough set that can be also defined by the complement of the lower approximation, which has been proved in [12]. the operator '$\wedge$' in expression (2) is a conjunctive operator whose meaning is that in multiple independent granular structures, one needs all granular structures to satisfy with non-empty for joint operator between an equivalence class and a target concept. And $\underline{\sum_{i=1}^{n} R_i}(X) \subseteq X \subseteq \overline{\sum_{i=1}^{n} R_i}(X)$. So we can label multi-granulation rough set $X$= $(\underline{\sum_{i=1}^{n} R_i}(X), \overline{\sum_{i=1}^{n} R_i}(X))$, accordingly, we call $(\Omega, R_1, R_2, ..., R_n)$ a multigranulation approximation space in the view of granular computing. And in [13], is defined interior and closure operators on an equivalence relation as follows $\forall X \in \Omega$ and $\Gamma$=$\{\tau_1, \tau_2, ..., \tau_i\}$:
$mint(X)$= $\bigcup\{A \in \tau_i : \vee(A \subseteq X), i \in n\}$,
$mcl(X)$= $\bigcup\{A \in \tau_i : \wedge(A \cap X \neq \phi), i \in n\}$.

**Definition 5** [31] Let $\Omega$ be a finite nonempty universe. A function $m : 2^{\Omega} \to \mathcal{R}$is called a measure of the granularity of a set if it satisfies the following conditions: for all $A, B \in 2^{\Omega}$,
(M1) $m(A) \geq 0$
(M2) If $A \subset B$, then $m(A) < m(B)$,
(M3) $A \sim_s B \iff m(A) = m(B)$.

Where $A \sim_s B \iff (\sim (A <_s B), \sim (B <_s A))$, " $<_s$ " is the weak order that is an extension of " $\subset$ ".

**Definition 6** [9] Let $T = \Gamma$ be a family of multigranulation topological rough spaces on $\Omega$. A function $G : \Gamma \to \mathcal{R}$ is called a measure of granularity of a partition if it satisfies the following conditions for all $\Gamma_1, \Gamma_2 \in T$,
(G1) $G(\Gamma) \geq 0$.
(G2) If $\Gamma_1 \subset \Gamma_2$, then $G(\Gamma_1) < G(\Gamma_2)$,
(G3) $\Gamma_1 = \Gamma_2 \iff \Gamma_1 = \Gamma_2$.

## 3 Multi-granulation rough sets approximations by using neighborhood systems

In this section, we use the definitions of $j$-neighborhood systems which are seen the first time in [1,31,3]. So, we use it to redefine the MGRS approximations and generalize it.

### 3.1 The first type of neighborhood systems

**Definition 7** [1,31] Let $R$ be an arbitrary binary relation on a nonempty finite set $U$. The j-neighborhoods of $x \in U$ is denoted by $(\mathcal{N}_j(x)), \forall j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$, which are defined as:
(1) $\mathcal{N}_{j_1}(x) = \{y \in U : xRy\}$, is used in [13] as an equivalence relation.
(2) $\mathcal{N}_{j_2}(x) = \{y \in U : yRx\}$.
(3) $\mathcal{N}_{j_3}(x) = \left\{y \in U : \bigcap_{x \in \mathcal{N}_{j_1}(y)} \mathcal{N}_{j_1}(y)\right\}$.
(4) $\mathcal{N}_{j_4}(x) = \left\{y \in U : \bigcap_{x \in \mathcal{N}_{j_2}(y)} \mathcal{N}_{j_2}(y)\right\}$.
(5) $\mathcal{N}_{j_5}(x) = \mathcal{N}_{j_1}(x) \cup \mathcal{N}_{j_2}(x)$.
(6) $\mathcal{N}_{j_6}(x) = \mathcal{N}_{j_1}(x) \cap \mathcal{N}_{j_2}(x)$.
(7) $\mathcal{N}_{j_7}(x) = \mathcal{N}_{j_3}(x) \cup \mathcal{N}_{j_4}(x)$.
(8) $\mathcal{N}_{j_8}(x) = \mathcal{N}_{j_3}(x) \cap \mathcal{N}_{j_4}(x)$.

**Definition 8** Let $(\Omega, R)$ be an approximation space. Suppose that $X \subseteq \Omega$, $R_1, R_2, \cdots, R_n$ be $n$ binary relations on $\Omega$, the lower approximation $\underline{\sum_{i=1}^{n} {}_{\mathcal{N}} R_i}(X)$ and the upper approximation $\overline{\sum_{i=1}^{n} {}^{\mathcal{N}} R_i}(X)$ of $X$ with respect to $R_1, R_2, \cdots, R_n$ are defined as follows, respectively,
(1) $\underline{\sum_{i=1}^{n} {}_{\mathcal{N}} R_i}(X) = \{x \in \Omega : \bigvee(\mathcal{N}_{ij}(x) \subseteq X), i \leq n\}$.
(2) $\overline{\sum_{i=1}^{n} {}^{\mathcal{N}} R_i}(X) = \{x \in \Omega : \bigwedge(\mathcal{N}_{ij}(x) \cap X \neq \phi), i \leq n\}$.

*Example 1* Let $U=\{a,b,c,d,e\}$ and we have three binary relations
$R_1=\{(a,a),(a,b),(b,b),(b,c),(c,c),(d,b),(d,d),(d,e),(e,b),(e,d),(e,e)\}$, $R_2=\{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c),(d,d),(e,e)\}$ and $R_3=\{(a,a),(a,d),(b,a),(b,b),(b,d),(c,c),(d,a),(d,d),(e,a),(e,c),(e,d),(e,e)\}$. If $X=\{b,d\}$, then $\underline{\sum_{i=1}^{3}{}_{\mathscr{N}}R_i(X)}=\{d\}$ and $\overline{\sum_{i=1}^{3}{}^{\mathscr{N}}R_i(X)}=\{a,b,d\}$

**Theorem 1** *Let $(\Omega,R)$ be an approximation space. Suppose that $X\subseteq\Omega$, $R_1,R_2,\cdots,R_n$ be $n$ binary relations on $\Omega$. Then we have the following properties are hold:*
*(1)* $\overline{\sum_{i=1}^{n}{}^{\mathscr{N}}R_i(\phi)}=\phi$.
*(2)* $X\subseteq Y$ *then* $\overline{\sum_{i=1}^{n}{}^{\mathscr{N}}R_i(X)}\subseteq\overline{\sum_{i=1}^{n}{}^{\mathscr{N}}R_i(Y)}$.
*(3)* $\overline{\sum_{i=1}^{n}{}^{\mathscr{N}}R_i}(\cap_{j\in I}X_j)\subseteq\cap_{j\in I}\overline{\sum_{i=1}^{n}{}^{\mathscr{N}}R_i}(X_j)$.
*(4)* $\overline{\sum_{i=1}^{n}{}^{\mathscr{N}}R_i}(\cup_{j\in I}X_j)\supseteq\cup_{j\in I}\overline{\sum_{i=1}^{n}{}^{\mathscr{N}}R_i}(X_j)$.
*(5)* $\overline{\sum_{i=1}^{n}{}^{\mathscr{N}}R_i(X^c)}=(\underline{\sum_{i=1}^{n}{}^{\mathscr{N}}R_i(X)})^c$.

*Proof* (1) Follows from Definition 8.

(2) By Definition 8 and if $X_1\subseteq X_2$, then

$$\begin{aligned}\overline{\sum_{i=1}^{n}{}^{\mathscr{N}}R_i(X_1)}&=\{x\in\Omega\mid\bigwedge(\mathscr{N}_{ij}(x)\cap X_1\neq\phi),i\leq n\}\\&\subseteq\{x\in\Omega\mid\bigwedge(\mathscr{N}_{ij}(x)\cap X_2\neq\phi),i\leq n\}\\&=\overline{\sum_{i=1}^{n}{}^{\mathscr{N}}R_i(X_2)}.\end{aligned}$$

(3) From (2) and by Definition 8, if $X_1\subseteq X_2\subseteq\cdots\subseteq X_j$, then

$$\begin{aligned}\overline{\sum_{i=1}^{n}{}^{\mathscr{N}}R_i(X_1\cap X_2\cap\cdots\cap X_j)}&=\overline{\sum_{i=1}^{n}{}^{\mathscr{N}}R_i(X_1)}\\&=\{x\in\Omega\mid\bigwedge(\mathscr{N}_{ij}(x)\cap X_1\neq\phi),i\leq n\}\\&\subseteq\{x\in\Omega\mid\bigwedge(\mathscr{N}_{ij}(x)\cap X_2\neq\phi),i\leq n\}\\&\vdots\\&\subseteq\{x\in\Omega\mid\bigwedge(\mathscr{N}_{ij}(x)\cap X_j\neq\phi),i\leq n\}\\&=\cap_{j\in I}\overline{\sum_{i=1}^{n}{}^{\mathscr{N}}R_i(X_j)}.\end{aligned}$$

(4) Analogue to (3) above.

(5) The complement of the $j$-lower approximation of $X$ is

$$\begin{aligned}(\underline{\sum_{i=1}^{n}{}_{\mathscr{N}}R_i}(X))^c&=\{x\in\Omega\mid\bigvee(\mathscr{N}_{ij}(x)\subseteq X),i\leq n\}^c\\&=\{x\in\Omega\mid\bigwedge(\mathscr{N}_{ij}(x))^c\supseteq(X^c)\}\\&=\{x\in\Omega:\bigwedge(\mathscr{N}_{ij}(x)\cap X\neq\phi),i\leq n\}\\&=\overline{\sum_{i=1}^{n}{}^{\mathscr{N}}R_i(X^c)}.\end{aligned}$$

The equality of (2) and (3) in Theorem 1 does not hold, in general, as shown in Example 2.

*Example 2* In Example 1,
(1) Take $X_1=\{b\},X_2=\{c\}$ and $X_3=\{d\}$. Thus $\overline{\sum_{i=1}^{3}{}^{\mathscr{N}}R_i}\ (X_1\cup X_2\cup X_3)=\{a,b,c,d\}$ and $\overline{\sum_{i=1}^{3}{}^{\mathscr{N}}R_i}(X_1)\cup\overline{\sum_{i=1}^{3}{}^{\mathscr{N}}R_i}(X_2)\cup\overline{\sum_{i=1}^{3}{}^{\mathscr{N}}R_i}(X_3)=\{b,c,d\}$. Therefore, $\overline{\sum_{i=1}^{3}{}^{\mathscr{N}}R_i}(\cup_{j\in I}X_j)\neq\cup_{j\in I}\overline{\sum_{i=1}^{3}{}^{\mathscr{N}}R_i}(X_j)\ \forall j\in\{1,2,3\}$.
(2) Take $X_1=\{b,d\}$, $X_2=\{b,cc\}$ and $X_3=\{a,c,d,e\}$. Then $\overline{\sum_{i=1}^{3}{}^{\mathscr{N}}R_i}(X_1\cap X_2\cap X_3)=\{d\}$ and $\overline{\sum_{i=1}^{3}{}^{\mathscr{N}}R_i}(X_1)\cup\overline{\sum_{i=1}^{3}{}^{\mathscr{N}}R_i}(X_2)\cup\overline{\sum_{i=1}^{3}{}^{\mathscr{N}}R_i}(X_3)=\{a,b,d\}$. Therefore, $\overline{\sum_{i=1}^{3}{}^{\mathscr{N}}R_i}(\cap_{j\in I}X_j)\neq\cap_{j\in I}\overline{\sum_{i=1}^{3}{}^{\mathscr{N}}R_i}(X_j)\ \forall j\in\{1,2,3\}$.

**Theorem 2** *Let $(\Omega,R)$ be an approximation space. Suppose that $X\subseteq\Omega$, $R_1,R_2,\cdots,R_n$ be $n$ binary relations on $\Omega$. Then we have the following properties are hold:*
*(1)* $\underline{\sum_{i=1}^{n}{}_{\mathscr{N}}R_i}(\Omega)=\Omega$.
*(2)* $\underline{\sum_{i=1}^{n}{}_{\mathscr{N}}R_i}(\cap_{j\in I}X_j)\subseteq\cap_{j\in I}\underline{\sum_{i=1}^{n}{}_{\mathscr{N}}R_i}(X_j)$.
*(3)* $\underline{\sum_{i=1}^{n}{}_{\mathscr{N}}R_i}(\cup_{j\in I}X_j)\supseteq\cup_{j\in I}\underline{\sum_{i=1}^{n}{}_{\mathscr{N}}R_i}(X_j)$.
*(4)* $X\subseteq Y$ *then* $\underline{\sum_{i=1}^{n}{}_{\mathscr{N}}R_i(X)}\subseteq\underline{\sum_{i=1}^{n}{}_{\mathscr{N}}R_i(Y)}$.
*(5)* $\underline{\sum_{i=1}^{n}{}_{\mathscr{N}}R_i(X^c)}=(\overline{\sum_{i=1}^{n}{}_{\mathscr{N}}R_i(X)})^c$.

*Proof* Analogue to Theorem 1.

The equality of (2) and (3) in Theorem 2 does not hold, in general, as shown in Example 3.

*Example 3* In Example 1,
(1) Put $X_1 = \{a,b\}$, $X_2 = \{a,b,c\}$ and $X_3 = \{a,c,d,e\}$. Thus $\underline{\sum_{i=1}^{3} R_i}(X_1 \cap X_2 \cap X_3) = \phi$ and $\underline{\sum_{i=1}^{3} {}_{\mathscr{N}} R_i}(X_1) \cap \underline{\sum_{i=1}^{3} {}_{\mathscr{N}} R_i}(X_2) \cap \underline{\sum_{i=1}^{3} {}_{\mathscr{N}} R_i}(X_3) = \{a\}$. Therefore $\underline{\sum_{i=1}^{3} {}_{\mathscr{N}} R_i}(\cap_{j\in I} X_j) \neq \cap_{j\in I} \underline{\sum_{i=1}^{3} {}_{\mathscr{N}} R_i}(X_j)$ $\forall j \in \{1,2,3\}$.
(2) Put $X_1 = \{a\}$, $X_2 = \{b\}$ and $X_3 = \{c\}$. Thus $\underline{\sum_{i=1}^{3} {}_{\mathscr{N}} R_i}(X_1 \cup X_2 \cup X_3) = \{a,b,c\}$ and $\underline{\sum_{i=1}^{3} {}_{\mathscr{N}} R_i}(X_1) \cup \underline{\sum_{i=1}^{3} {}_{\mathscr{N}} R_i}(X_2) \cup \underline{\sum_{i=1}^{3} {}_{\mathscr{N}} R_i}(X_3) = \{c\}$. Therefore $\underline{\sum_{i=1}^{3} {}_{\mathscr{N}} R_i}(\cup_{j\in I} X_j) \neq \cup_{j\in I} \underline{\sum_{i=1}^{3} {}_{\mathscr{N}} R_i}(X_j)$ $\forall j \in \{1,2,3\}$.

By using Definition 8, we can define the accuracy measure as the following.

**Definition 9** Let $(\Omega, R_n)$ be an approximation space. Suppose that $A \subseteq \Omega$, $R_1, R_2, \cdots, R_n$ be $n$ binary relations on $\Omega$, the accuracy measure of $A$ is defined by

$$\alpha_{\mathscr{N}}(A) = \frac{|\underline{\sum_{i=1}^{n} {}_{\mathscr{N}} R_i}(A)|}{|\overline{\sum_{i=1}^{n} {}^{\mathscr{N}} R_i}(A)|}$$

Where $|.|$ denotes the cardinality of the set.

*Remark 1* If $R_i$, for $i \in \{1,2,\cdots,n\}$ are binary relations on a nonempty set $\Omega$. The properties does not hold, in general.
(1) $\underline{\sum_{i=1}^{n} {}_{\mathscr{N}} R_i}(\phi) = \phi$.
(2) $\underline{\sum_{i=1}^{n} {}_{\mathscr{N}} R_i}(X) \subseteq X$.
(3) $\underline{\underline{\sum_{i=1}^{n} {}_{\mathscr{N}} R_i}}(\underline{\sum_{i=1}^{n} {}_{\mathscr{N}} R_i}(X)) = \underline{\sum_{i=1}^{n} {}_{\mathscr{N}} R_i}(X)$.
(4) $\overline{\sum_{i=1}^{n} {}_{\mathscr{N}} R_i}(\Omega) = \Omega$‘.
(5) $X \subseteq \overline{\sum_{i=1}^{n} {}_{\mathscr{N}} R_i}(X)$.
(6) $\overline{\sum_{i=1}^{n} {}_{\mathscr{N}} R_i}(\overline{\sum_{i=1}^{n} {}_{\mathscr{N}} R_i}(X)) = \overline{\sum_{i=1}^{n} {}_{\mathscr{N}} R_i}(X)$.
$\forall X \in \Omega$ as in Example 4.

*Example 4* Let $R_1 = \{(a,b),(b,c)\}$ and $R_2 = \{(a,c),(b.c)\}$ are any two binary relations on non-empty set $\Omega = \{a,b,c\}$. Then we have, $\underline{\sum_{i=1}^{2} {}_{\mathscr{N}} R_i}(\phi) = \{c\}$ and $\overline{\sum_{i=1}^{2} {}_{\mathscr{N}} R_i}(\Omega) = \{a,b\}$. Hence (1) and (4) does not hold.
If $X = \{b\}$, then $\underline{\sum_{i=1}^{2} {}_{\mathscr{N}} R_i}(X) = \{a,c\}$ and $\overline{\sum_{i=1}^{2} {}_{\mathscr{N}} R_i}(X) = \{a\}$. Consequently $\underline{\sum_{i=1}^{2} {}_{\mathscr{N}} R_i}(X) \nsubseteq X \nsubseteq \overline{\sum_{i=1}^{2} {}_{\mathscr{N}} R_i}(X)$, i.e., (2) and (5) does not hold.
If $X = \{b\}$, hence $\underline{\sum_{i=1}^{2} {}_{\mathscr{N}} R_i}(\underline{\sum_{i=1}^{2} {}_{\mathscr{N}} R_i}(X)) = \{a,b,c\}$ and $\underline{\sum_{i=1}^{2} {}_{\mathscr{N}} R_i}(X) = \{b,c\}$. If $X = \{a,b,c\}$ $\overline{\sum_{i=1}^{2} {}_{\mathscr{N}} R_i}(X) = \{a,b\}$ and $\overline{\sum_{i=1}^{2} {}_{\mathscr{N}} R_i}(\overline{\sum_{i=1}^{2} {}_{\mathscr{N}} R_i}(X)) = \phi$. Therefore (3) and (6) does not hold.

### 3.2 The second type of neighborhood systems

In the following, $j$-neighborhood systems for Zhu [35] will be used to define another MGRS model. In this model, some properties in Remark 1 will be satisfied.

**Definition 10** Let $R$ be an arbitrary binary relation on a nonempty finite set $U$. The j-adhesion neighborhoods of $x \in U$ is denoted by $(\mathscr{P}_j(x))$, $\forall j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$, which are defined as:
(1) $\mathscr{P}_{j_1}(x)$= $\{y \in U : \mathscr{N}_{j_1}(x) = \mathscr{N}_{j_1}(y)\}$.
(2) $\mathscr{P}_{j_2}(x)$= $\{y \in U : \mathscr{N}_{j_2}(x) = \mathscr{N}_{j_2}(y)\}$.
(3) $\mathscr{P}_{j_3}(x)$= $\left\{ y \in U : \bigcap_{x \in \mathscr{N}_{j_1}} \mathscr{N}_{j_1}(y) = \bigcap_{y \in \mathscr{N}_{j_1}} \mathscr{N}_{j_1}(x) \right\}$.
(4) $\mathscr{P}_{j_4}(x)$= $\left\{ y \in U : \bigcap_{x \in \mathscr{N}_{j_2}} \mathscr{N}_{j_2}(y) = \bigcap_{x \in \mathscr{N}_{j_2}(y)} \mathscr{N}_{j_2}(x) \right\}$.
(5) $\mathscr{P}_{j_5}(x)$= $\mathscr{P}_{j_1}(x) \cup \mathscr{P}_{j_2}(x)$.
(6) $\mathscr{P}_{j_6}(x)$= $\mathscr{P}_{j_1}(x) \cap \mathscr{P}_{j_2}(x)$.
(7) $\mathscr{P}_{j_7}(x)$= $\mathscr{P}_{j_3}(x) \cup \mathscr{P}_{j_4}(x)$.
(8) $\mathscr{P}_{j_8}(x)$= $\mathscr{P}_{j_3}(x) \cap \mathscr{P}_{j_4}(x)$.

**Definition 11** Let $(\Omega, R)$ be an approximation space. Suppose that $X \subseteq \Omega$, $R_1, R_2, \cdots, R_n$ be $n$ binary relations on $\Omega$, the lower approximation and the upper approximation with respect to $R_1, R_2, \cdots, R_n$ are
(1) $\underline{\sum_{i=1}^{n} {}_{\mathscr{P}} R_i}(X) = \{x \in \Omega : \bigvee(\mathscr{P}_{ij}(x) \subseteq X), i \leq n\}$.

(2) $\overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i}(X) = \{x \in \Omega : \bigwedge(\mathscr{P}_{ij}(x) \cap X \neq \phi), i \leq n\}$, respectively.

*Example 5* From Example 1, if we have $X = \{b, d\}$, then $\underline{\sum_{i=1}^{3} {}_{\mathscr{P}}R_i}(X) = \{b, d\}$ and $\overline{\sum_{i=1}^{3} {}^{\mathscr{P}}R_i}(X) = \{b, d\}$

From Examples 1 and 5, we note that
$\underline{\sum_{i=1}^{n} {}_{\mathscr{N}}R_i}$ and $\underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i}$ are independent. Also, $\overline{\sum_{i=1}^{n} {}^{\mathscr{N}}R_i}$ and $\overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i}$ are independent.

**Theorem 3** *Let $(\Omega, R)$ be an approximation space. Suppose that $X \subseteq \Omega$, $R_1, R_2, \cdots, R_n$ be $n$ binary relations on $\Omega$. Then the following properties are hold:*
*(1)* $\overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i}(\Omega) = \Omega$.
*(2)* $\overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i}(\phi) = \phi$.
*(3)* $X \subseteq \overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i}(X)$.
*(4)* $X \subseteq Y$ *then* $\overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i(X)} \subseteq \overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i(Y)}$.
*(5)* $\overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i}(\cap_{j\in I} X_j) \subseteq \cap_{j\in I} \overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i}(X_j)$.
*(6)* $\overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i}(\cup_{j\in I} X_j) = \cup_{j\in I} \overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i}(X_j)$.
*(7)* $\overline{\sum_{i=1}^{n} R_i}(\overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i(X)}) = \overline{\sum_{i=1}^{n} {}^{\mathscr{P}}}(X)$.
*(8)* $\overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i}(X^c) = (\underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i(X)})^c$.

*Proof* (1) By Definition 11,

$$\overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i(\Omega)} = \{x \in \Omega \mid \wedge(\mathscr{P}_{ij}(x) \cap \Omega \neq \phi), i \leq n\} = \Omega. \text{ Thus, } \overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i(\Omega)} = \Omega.$$

(2) Similarly, from (1) $\overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i(\phi)} = \phi$.

(3) Follows from Definition 11.

(4) By Definition 11 and if $X_1 \subseteq X_2$, then

$$\begin{aligned}\overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i(X_1)} &= \{x \in \Omega \mid \wedge(\mathscr{P}_{ij}(x) \cap X_1 \neq \phi), i \leq n\} \\ &\subseteq \{x \in \Omega \mid \wedge(\mathscr{P}_{ij}(x) \cap X_2 \neq \phi), i \leq n\} \\ &= \overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i(X_2)}.\end{aligned}$$

(5) From (4) and Definition 11, if $X_1 \subseteq X_2 \subseteq \ldots \subseteq X_j$, then

$$\begin{aligned}\overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i(X_1 \cap X_2 \cap \ldots \cap X_j)} &= \overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i(X_1)} \\ &= \{x \in \Omega \mid \wedge(\mathscr{P}_{ij}(x) \cap X_1 \neq \phi), i \leq n\} \\ &\subseteq \{x \in \Omega \mid \wedge(\mathscr{P}_{ij}(x) \cap X_2 \neq \phi), i \leq n\} \\ &\vdots \\ &\subseteq \{x \in \Omega \mid \wedge(\mathscr{P}_{ij}(x) \cap X_j \neq \phi), i \leq n\} \\ &= \cap_{j\in I} \overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i(X_j)}.\end{aligned}$$

(6) Analogue to (5) above.

(7) By Definition 11, we have

$$\begin{aligned}&\overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i(X)} = \{x \in \Omega : \wedge(\mathscr{P}_{ij}(x) \cap X \neq \phi), i \leq n\} \\ \Longleftrightarrow \quad &\subseteq \{x \in \Omega : \wedge(\mathscr{P}_{ij}(x) \cap \overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i(X)} \neq \phi), i \leq n\} \\ \Longleftrightarrow \quad &= \overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i}(\overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i(X)})\end{aligned}$$

(8) The complement of the $j$-lower approximation of $X$ is

$$\begin{aligned}(\underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i}(X))^c &= \{x \in \Omega \mid \vee(\mathscr{P}_{ij}(x) \subseteq X), i \leq n\}^c \\ &= \{x \in \Omega \mid \wedge(\mathscr{P}_{ij}(x))^c \supseteq (X^c)\} \\ &= \{x \in \Omega : \wedge(\mathscr{P}_{ij}(x) \cap X \neq \phi), i \leq n\} \\ &= \overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i(X^c)}.\end{aligned}$$

The equality of (5) in Theorem 3 does not hold, in general, as in Example 6.

*Example 6* Let $U = \{a, b, c, d\}$ and we have two binary relations
$R_1 = \{(a, a), (b, b), (c, b), (c, c), (d, a)\}$ and $R_2 = \{(a, b), (b, a), (a, c), (c, a), (c, d), (d, c), (d, b)\}$. Then if we have $X_1 = \{a, b\}$ and $X_2 = \{b, d\}$, then $\cap_{j\in I}\overline{\sum_{i=1}^{2} {}^{\mathscr{P}}R_i}(X_j) = \{a, b, d\}$ and $\overline{\sum_{i=1}^{2} {}^{\mathscr{P}}R_i}(\cap_{j\in I}X_j) = \{b\}$. Thus $\overline{\sum_{i=1}^{2} {}^{\mathscr{P}}R_i}(\cap_{j\in I}X_j) \neq \cap_{j\in I}\overline{\sum_{i=1}^{2} {}^{\mathscr{P}}R_i}(X_j)$.

**Theorem 4** *Let $(\Omega, R)$ be an approximation space. Suppose that $X \subseteq \Omega$, $R_1, R_2, \cdots, R_n$ be $n$ binary relations on $\Omega$. Then the following properties are hold:*
*(1)* $\underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i(\Omega)} = \Omega$.
*(2)* $\underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i(\phi)} = \phi$.
*(3)* $\underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i}(X) \subseteq X$.
*(4)* $\underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i}(\cap_{j\in I}X_j) = \cap_{j\in I}\underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i}(X_j)$.
*(5)* $\underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i}(\cup_{j\in I}X_j) \supseteq \cup_{j\in I}\underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i}(X_j)$.
*(6)* $\underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i}(\underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i(X)}) = \underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i(X)}$.
*(7)* $X \subseteq Y$ *then* $\underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i(X)} \subseteq \underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i(Y)}$.
*(8)* $\underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i(X^c)} = (\overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i}(X))^c$.

*Proof* Analogue to Theorem 3.

The equality of (5) in Theorem does not hold, in general, as in Example 7.

*Example 7* From Example 5, we have $X_1 = \{a\}$ and $X_2 = \{d\}$, then $\underline{\sum_{i=1}^{2} {}_{\mathscr{P}}R_i}(\cup_{j\in I}X_j) = \{a, d\}$ and $\cup_{j\in I}\underline{\sum_{i=1}^{2} {}_{\mathscr{P}}R_i}(X_j) = \phi$. Thus $\underline{\sum_{i=1}^{2} {}_{\mathscr{P}}R_i}(\cup_{j\in I}X_j) \neq \cup_{j\in I}\underline{\sum_{i=1}^{2} {}_{\mathscr{P}}R_i}(X_j)$.

By using Definition 11, we can define the accuracy measure as the following.

**Definition 12** Let $(\Omega, R_n)$ be an approximation space. Suppose that $A \subseteq \Omega$, $R_1, R_2, \cdots, R_n$ be $n$ binary relations on $\Omega$, the accuracy measure of $A$ is defined by

$$\alpha_{\mathscr{P}}(A) = \frac{|\underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i}(A)|}{|\overline{\sum_{i=1}^{n} {}^{\mathscr{P}}R_i}(A)|}$$

Where $|.|$ denotes the cardinality of the set.

Example 8 illustrates a comparison between the accuracy measure for new MGRS models.

*Example 8* (Chemical application) Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ be five amino acids (AAs) which described in terms of five attributes: $a_1 = PIE$, $a_2 = SAC =$ surface area, $a_3 = MR =$ molecular refractivity, $a_4 = LAM =$ the side chain polarity and $a_5 = Vol =$ molecular volume [28]. Table 1 shows all quantitative attributes of five AAs.

**Table 1** Quantitative attributes of five amino acids.

| $X$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---|---|---|---|---|---|
| $x_1$ | 0.23 | 254.2 | 2.126 | 0.02 | 82.2 |
| $x_2$ | 0.48 | 303.6 | 2.994 | 1.24 | 112.3 |
| $x_3$ | 0.61 | 287.9 | 2.994 | 1.08 | 103.7 |
| $x_4$ | 0.45 | 282.9 | 2.933 | 0.11 | 99.1 |
| $x_5$ | 0.11 | 335.0 | 3.458 | 0.19 | 127.5 |

Consider five reflexive relations on $X$, $R_k = \{(x_i, x_j) \in X \times X : x_i(a_k) - x_j(a_k) < \frac{\sigma_k}{2}, i, j, k \in \{1, 2, \cdots, 5\}\}$, where $\sigma_k$ represents the standard deviation of the quantitative attributes $a_k$, $k = 1, 2, 3, 4, 5$. The right neighborhood for each element with respect to the relation $R_k$, for $k = 1, 2, 3, 4, 5$. We have binary relations
$R_1 = \{(x_1, x_1), (x_1, x_4), (x_2, x_1), (x_2, x_2), (x_2, x_3), (x_2, x_4), (x_2, x_5), (x_3, x_1), (x_3, x_2), (x_3, x_3), (x_3, x_4), (x_3, x_5), (x_4, x_4), (x_5, x_1), (x_5, x_4), (x_5, x_5)\}$.
$R_2 = \{(x_1, x_1), (x_1, x_2), (x_1, x_3), (x_1, x_4), (x_1, x_5), (x_2, x_2), (x_2, x_5), (x_3, x_2), (x_3, x_3), (x_3, x_4), (x_3, x_5), (x_4, x_2), (x_4, x_3), (x_4, x_4), (x_4, x_5), (x_5, x_5)\}$.
$R_3 = \{(x_1, x_1), (x_1, x_2), (x_1, x_3), (x_1, x_4), (x_1, x_5), (x_2, x_2), (x_2, x_3), (x_2, x_4), (x_2, x_5), (x_3, x_2), (x_3, x_3), (x_3, x_4), (x_3, x_5), (x_4, x_2), (x_4, x_3), (x_4, x_4), (x_4, x_5), (x_5, x_5)\}$.

$R_4 = \{(x_1,x_1),(x_1,x_4),(x_2,x_1),(x_2,x_2),(x_2,x_3),(x_2,x_4)\ (x_2,x_5),(x_3,x_1),(x_3,x_3),(x_3,x_4),(x_3,x_5),(x_4,x_1),(x_4,x_4),(x_4,x_5),(x_5,x_1),(x_5,x_4)\}$.
$R_5 = \{(x_1,x_1),(x_1,x_2),(x_1,x_3),(x_1,x_4),(x_1,x_5),\ (x_2,x_2),(x_2,x_5),(x_3,x_2),(x_3,x_3),(x_3,x_4),(x_3,x_5),(x_4,x_2),(x_4,x_3),(x_4,x_4),(x_4,x_5),(x_5,x_5)\}$.

Tables 2, 3, 4, 5 show a comparison between different kinds of $j$.

**Table 2** A comparison between accuracies when $j = j1 = j3$.

| P(X) | $\underline{\sum_{i=1}^{n} \mathscr{N} R_i}(X)$ | $\overline{\sum_{i=1}^{n} \mathscr{N} R_i}(X)$ | $\alpha_{\mathscr{N}}(X)$ | $\underline{\sum_{i=1}^{n} \mathscr{P} R_i}(X)$ | $\overline{\sum_{i=1}^{n} \mathscr{P} R_i}(X)$ | $\alpha_{\mathscr{N}}(X)$ |
|---|---|---|---|---|---|---|
| $\{a\}$ | $\phi$ | $\{a\}$ | $0$ | $\{a\}$ | $\{a\}$ | $1$ |
| $\{b\}$ | $\phi$ | $\{b,c\}$ | $0$ | $\{b\}$ | $\{b\}$ | $1$ |
| $\{c\}$ | $\phi$ | $\{c\}$ | $0$ | $\phi$ | $\{c\}$ | $1$ |
| $\{d\}$ | $\phi$ | $\{a,c,d\}$ | $0$ | $\{d\}$ | $\{d\}$ | $1$ |
| $\{e\}$ | $\{e\}$ | $X$ | $\frac{1}{5}$ | $\{e\}$ | $\{e\}$ | $1$ |
| $\{a,b\}$ | $\phi$ | $\{a,b,c,d\}$ | $0$ | $\{a,b\}$ | $\{a,b\}$ | $1$ |
| $\{a,c\}$ | $\phi$ | $\{a,c,d\}$ | $0$ | $\{a\}$ | $\{a,c\}$ | $\frac{1}{2}$ |
| $\{a,d\}$ | $\phi$ | $\{a,c,d\}$ | $0$ | $\{a,d\}$ | $\{a,d\}$ | $1$ |
| $\{a,e\}$ | $\{e\}$ | $X$ | $\frac{1}{5}$ | $\{a,e\}$ | $\{a,e\}$ | $1$ |
| $\{b,c\}$ | $\phi$ | $\{b,c\}$ | $0$ | $\{b,c\}$ | $\{b,c\}$ | $1$ |
| $\{b,d\}$ | $\phi$ | $\{a,b,c,d\}$ | $0$ | $\{b,d\}$ | $\{b,c,d\}$ | $\frac{2}{3}$ |
| $\{b,e\}$ | $\{b,e\}$ | $X$ | $\frac{2}{5}$ | $\{b,e\}$ | $\{b,e\}$ | $1$ |
| $\{c,d\}$ | $\phi$ | $\{a,c,d\}$ | $0$ | $\{c,d\}$ | $\{c,d\}$ | $1$ |
| $\{c,e\}$ | $\{e\}$ | $X$ | $\frac{1}{5}$ | $\{e\}$ | $\{c,e\}$ | $\frac{1}{2}$ |
| $\{d,e\}$ | $\{e\}$ | $X$ | $\frac{1}{5}$ | $\{d,e\}$ | $\{d,e\}$ | $1$ |
| $\{a,b,c\}$ | $\phi$ | $\{a,b,c,d\}$ | $0$ | $\{a,b,c\}$ | $\{a,b,c\}$ | $1$ |
| $\{a,b,d\}$ | $\phi$ | $\{a,b,c,d\}$ | $0$ | $\{a,b,d\}$ | $\{a,b,c,d\}$ | $\frac{3}{4}$ |
| $\{a,b,e\}$ | $\{b,e\}$ | $X$ | $\frac{2}{5}$ | $\{a,b,e\}$ | $\{a,b,e\}$ | $1$ |
| $\{a,c,d\}$ | $\phi$ | $\{a,c,d\}$ | $0$ | $\{a,c,d\}$ | $\{a,c,d\}$ | $1$ |
| $\{a,c,e\}$ | $\{e\}$ | $X$ | $\frac{1}{5}$ | $\{a,e\}$ | $\{a,c,e\}$ | $\frac{2}{3}$ |
| $\{a,d,e\}$ | $\{a,d,e\}$ | $X$ | $\frac{3}{5}$ | $\{a,d,e\}$ | $\{a,d,e\}$ | $1$ |
| $\{b,c,d\}$ | $\phi$ | $\{a,b,c,d\}$ | $0$ | $\{b,c,d\}$ | $\{b,c,d\}$ | $1$ |
| $\{b,c,e\}$ | $\{b,e\}$ | $X$ | $\frac{2}{5}$ | $\{b,c,e\}$ | $\{b,c,e\}$ | $1$ |
| $\{b,d,e\}$ | $\{b,e\}$ | $X$ | $\frac{2}{5}$ | $\{b,c,e\}$ | $\{b,c,d,e\}$ | $\frac{3}{4}$ |
| $\{c,d,e\}$ | $\{e\}$ | $X$ | $\frac{1}{5}$ | $\{c,d,e\}$ | $\{c,d,e\}$ | $1$ |
| $\{a,b,c,d\}$ | $\phi$ | $\{a,b,c,d\}$ | $0$ | $\{a,b,c,d\}$ | $\{a,b,c,d\}$ | $1$ |
| $\{a,b,c,e\}$ | $\{b,e\}$ | $X$ | $\frac{2}{5}$ | $\{a,b,c,e\}$ | $\{a,b,c,e\}$ | $1$ |
| $\{a,b,d,e\}$ | $\{a,b,d,e\}$ | $X$ | $\frac{4}{5}$ | $\{a,b,d,e\}$ | $X$ | $\frac{4}{5}$ |
| $\{a,c,d,e\}$ | $\{a,d,e\}$ | $X$ | $\frac{3}{5}$ | $\{a,c,d,e\}$ | $\{a,c,d,e\}$ | $1$ |
| $\{b,c,d,e\}$ | $\{b,c,d,e\}$ | $X$ | $\frac{4}{5}$ | $\{b,c,d,e\}$ | $\{b,c,d,e\}$ | $1$ |
| $X$ | $X$ | $X$ | $1$ | $X$ | $X$ | $1$ |

## 4 Topological approach to MGRS by neighborhood systems

In this section, we give some topological approaches for new MGRS models using definitions 7 and 10.

### 4.1 The first type of topology by NS and its algorithms

**Definition 13** [2] The topology which is generated by $j$-neighborhood systems is

$$\mathscr{T}_j = \bigcup \{A \in \Omega : \forall x \in A, \mathscr{N}_j(x) \subseteq A\}$$

$\forall j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ is called the topology generated by $j$-neighborhoods, denoted by $\mathscr{T}_j$.

*Example 9* Let $X = \{a,b,c,d\}$ and $R$ be a binary relation defined by

$$R = \{(a,a),(b,b),(c,b),(c,c),(d,a)\}$$

Then, we compute $\mathscr{T}_j$ as follows

**Table 3** A comparison between accuracies when $j = j2 = j4$.

| P(X) | $\underline{\sum_{i=1}^{n} \mathscr{N} R_i}(X)$ | $\overline{\sum_{i=1}^{n} \mathscr{N} R_i}(X)$ | $\alpha_{\mathscr{N}}(X)$ | $\underline{\sum_{i=1}^{n} \mathscr{P} R_i}(X)$ | $\overline{\sum_{i=1}^{n} \mathscr{P} R_i}(X)$ | $\alpha_{\mathscr{N}}(X)$ |
|---|---|---|---|---|---|---|
| $\{a\}$ | $\{a\}$ | $\{a,d\}$ | $\frac{1}{2}$ | $\{a\}$ | $\{a\}$ | 1 |
| $\{b\}$ | $\phi$ | $\{b,e\}$ | 0 | $\{b\}$ | $\{b\}$ | 1 |
| $\{c\}$ | $\phi$ | $\{b,c,d,e\}\}$ | 0 | $\phi$ | $\{c\}$ | 0 |
| $\{d\}$ | $\phi$ | $\{d,e\}$ | 0 | $\phi$ | $\{d\}$ | 0 |
| $\{e\}$ | $\phi$ | $\{e\}$ | 0 | $\{e\}$ | $\{e\}$ | 1 |
| $\{a,b\}$ | $\{a\}$ | $X$ | $\frac{1}{5}$ | $\{a,b\}$ | $\{a,b\}$ | 1 |
| $\{a,c\}$ | $\{a\}$ | $X$ | $\frac{1}{5}$ | $\{a\}$ | $\{a,c,d\}$ | $\frac{1}{3}$ |
| $\{a,d\}$ | $\{a\}$ | $\{a,d,e\}$ | $\frac{1}{3}$ | $\{a,d\}$ | $\{a,d\}$ | 1 |
| $\{a,e\}$ | $\{a\}$ | $\{a,d,e\}$ | $\frac{1}{3}$ | $\{a,e\}$ | $\{a,e\}$ | 1 |
| $\{b,c\}$ | $\{b,c\}$ | $\{b,c,d,e\}$ | $\frac{1}{2}$ | $\{b,c\}$ | $\{b,c\}$ | 1 |
| $\{b,d\}$ | $\phi$ | $\{b,c,d,e\}$ | 0 | $\{b\}$ | $\{b,c,d\}$ | $\frac{1}{3}$ |
| $\{b,e\}$ | $\phi$ | $\{b,e\}$ | 0 | $\{b,e\}$ | $\{b,e\}$ | 1 |
| $\{c,d\}$ | $\phi$ | $\{b,c,d,e\}$ | 0 | $\{c,d\}$ | $\{c,d\}$ | 1 |
| $\{c,e\}$ | $\phi$ | $\{b,c,d,e\}$ | 0 | $\{e\}$ | $\{c,e\}$ | $\frac{1}{2}$ |
| $\{d,e\}$ | $\phi$ | $\{d,e\}$ | 0 | $\{e\}$ | $\{d,e\}$ | $\frac{1}{2}$ |
| $\{a,b,c\}$ | $\{a,b,c\}$ | $X$ | $\frac{3}{5}$ | $\{a,b,c\}$ | $\{a,b,c,d\}$ | $\frac{3}{4}$ |
| $\{a,b,d\}$ | $\{a\}$ | $X$ | $\frac{1}{5}$ | $\{a,b,d\}$ | $\{a,b,c,d\}$ | $\frac{3}{4}$ |
| $\{a,b,e\}$ | $\{a\}$ | $X$ | $\frac{1}{5}$ | $\{a,b,e\}$ | $\{a,b,e\}$ | 1 |
| $\{a,c,d\}$ | $\{a,c,d\}$ | $X$ | $\frac{3}{5}$ | $\{a,c,d\}$ | $\{a,c,d\}$ | 1 |
| $\{a,c,e\}$ | $\{a\}$ | $X$ | $\frac{1}{5}$ | $\{a,e\}$ | $\{a,c,d,e\}$ | $\frac{1}{2}$ |
| $\{a,d,e\}$ | $\{a\}$ | $\{a,d,e\}$ | $\frac{1}{3}$ | $\{a,d,e\}$ | $\{a,d,e\}$ | 1 |
| $\{b,c,d\}$ | $\{b,c\}$ | $\{b,c,d,e\}$ | $\frac{1}{2}$ | $\{b,c,d\}$ | $\{b,c,d\}$ | 1 |
| $\{b,c,e\}$ | $\{b,c\}$ | $\{b,c,d,e\}$ | $\frac{1}{2}$ | $\{b,c,e\}$ | $\{b,c,e\}$ | 1 |
| $\{b,d,e\}$ | $\phi$ | $\{b,c,d,e\}$ | 0 | $\{b,e\}$ | $\{b,c,d,e\}$ | $\frac{1}{2}$ |
| $\{c,d,e\}$ | $\phi$ | $\{b,c,d,e\}$ | 0 | $\{c,d,e\}$ | $\{c,d,e\}$ | 1 |
| $\{a,b,c,d\}$ | $\{a,b,c,d\}$ | $X$ | $\frac{4}{5}$ | $\{a,b,c,d\}$ | $\{a,b,c,d\}$ | 1 |
| $\{a,b,c,e\}$ | $\{a,b,c\}$ | $X$ | $\frac{3}{5}$ | $\{a,b,c,e\}$ | $X$ | $\frac{4}{5}$ |
| $\{a,b,d,e\}$ | $\{a\}$ | $X$ | $\frac{1}{5}$ | $\{a,b,d,e\}$ | $X$ | $\frac{4}{5}$ |
| $\{a,c,d,e\}$ | $\{a,c,d\}$ | $X$ | $\frac{3}{5}$ | $\{a,c,d,e\}$ | $\{a,c,d,e\}$ | 1 |
| $\{b,c,d,e\}$ | $\{b,c,e\}$ | $\{b,c,d,e\}$ | $\frac{3}{4}$ | $\{b,c,d,e\}$ | $\{b,c,d,e\}$ | 1 |
| $X$ | $X$ | $X$ | 1 | $X$ | $X$ | 1 |

(1) As $j = j_1$, the topology $\mathscr{T}_{j_1}$ is

$$\mathscr{T}_{j_1} = \{X, \phi, \{a\}, \{b\}, \{a,b\}, \{a,d\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}\}.$$

(2) As $j = j_2$, the topology $\mathscr{T}_{j_2}$ is

$$\mathscr{T}_{j_2} = \{X, \phi, \{c\}, \{d\}, \{a,d\}, \{c,d\}, \{b,c\}, \{a,c,d\}, \{b,c,d\}\}.$$

(3) As $j = j_3$, the topology $\mathscr{T}_{j_3}$ is

$$\mathscr{T}_{j_3} = \{X, \phi, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,d\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}\}$$

(4) As $j = j_4$, the topology $\mathscr{T}_{j_4}$ is

$$\mathscr{T}_{j_4} = \{X, \phi, \{c\}, \{d\}, \{c,d\}, \{a,d\}, \{b,c\}, \{a,c,d\}, \{b,c,d\}\}$$

(5) As $j = j_5$, the topology $\mathscr{T}_{j_5}$ is

$$\mathscr{T}_{j_5} = \{X, \phi, \{a,d\}, \{b,c\}\}$$

(6) As $j = j_6$, the topology $\mathscr{T}_{j_6}$ is

$$\mathscr{T}_{j_6} = \{X, \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$$

(7) As $j = j_7$, the topology $\mathscr{T}_{j_7}$ is

$$\mathscr{T}_{j_7} = \{X, \phi, \{d\}, \{a,d\}, \{b,c\}, \{b,c,d\}\}$$

**Table 4** A comparison between accuracies when $j = j5 = j7$.

| P(X) | $\underline{\sum_{i=1}^{n} {}_{\mathscr{N}}R_i}(X)$ | $\overline{\sum_{i=1}^{n} {}_{\mathscr{N}}R_i}(X)$ | $\alpha_{\mathscr{N}}(X)$ | $\underline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i}(X)$ | $\overline{\sum_{i=1}^{n} {}_{\mathscr{P}}R_i}(X)$ | $\alpha_{\mathscr{N}}(X)$ |
|---|---|---|---|---|---|---|
| $\{a\}$ | $\phi$ | $X$ | 0 | $\{a\}$ | $\{a\}$ | 1 |
| $\{b\}$ | $\phi$ | $X$ | 0 | $\{b\}$ | $\{b\}$ | 1 |
| $\{c\}$ | $\phi$ | $X$ | 0 | $\phi$ | $\{c\}$ | 0 |
| $\{d\}$ | $\phi$ | $X$ | 0 | $\phi$ | $\{d\}$ | 0 |
| $\{e\}$ | $\phi$ | $X$ | 0 | $\{e\}$ | $\{e\}$ | 1 |
| $\{a,b\}$ | $\phi$ | $X$ | 0 | $\{a,b\}$ | $\{a,b\}$ | 1 |
| $\{a,c\}$ | $\phi$ | $X$ | 0 | $\{a\}$ | $\{a,c,d\}$ | $\frac{1}{3}$ |
| $\{a,d\}$ | $\phi$ | $X$ | 0 | $\{a\}$ | $\{a,d\}$ | $\frac{1}{2}$ |
| $\{a,e\}$ | $\phi$ | $X$ | 0 | $\{a,e\}$ | $\{a,e\}$ | 1 |
| $\{b,c\}$ | $\phi$ | $\{b,c\}$ | 0 | $\{b,c\}$ | $\{b,c\}$ | 1 |
| $\{b,d\}$ | $\phi$ | $X$ | 0 | $\{b\}$ | $\{b,c,d\}$ | $\frac{1}{3}$ |
| $\{b,e\}$ | $\phi$ | $X$ | 0 | $\{b,e\}$ | $\{b,e\}$ | 1 |
| $\{c,d\}$ | $\phi$ | $X$ | 0 | $\{c,d\}$ | $\{c,d\}$ | 1 |
| $\{c,e\}$ | $\phi$ | $X$ | 0 | $\{e\}$ | $\{c,d,e\}$ | $\frac{1}{3}$ |
| $\{d,e\}$ | $\phi$ | $X$ | 0 | $\{e\}$ | $\{d,e\}$ | $\frac{1}{2}$ |
| $\{a,b,c\}$ | $\phi$ | $X$ | 0 | $\{a,b,c\}$ | $\{a,b,c,d\}$ | $\frac{3}{4}$ |
| $\{a,b,d\}$ | $\phi$ | $X$ | 0 | $\{a,b\}$ | $\{a,b,c,d\}$ | $\frac{1}{2}$ |
| $\{a,b,e\}$ | $\phi$ | $X$ | 0 | $\{a,b,e\}$ | $\{a,b,e\}$ | 1 |
| $\{a,c,d\}$ | $\phi$ | $X$ | 0 | $\{a,c,d\}$ | $\{a,c,d\}$ | 1 |
| $\{a,c,e\}$ | $\phi$ | $X$ | 0 | $\{a,e\}$ | $\{a,c,d,e\}$ | $\frac{1}{2}$ |
| $\{a,d,e\}$ | $\phi$ | $X$ | 0 | $\{a,d,e\}$ | $\{a,d,e\}$ | 1 |
| $\{b,c,d\}$ | $\phi$ | $X$ | 0 | $\{b,c,d\}$ | $\{b,c,d\}$ | 1 |
| $\{b,c,e\}$ | $\phi$ | $X$ | 0 | $\{b,c,e\}$ | $\{b,c,d,e\}$ | $\frac{3}{4}$ |
| $\{b,d,e\}$ | $\phi$ | $X$ | 0 | $\{b,e\}$ | $\{b,c,d,e\}$ | $\frac{1}{2}$ |
| $\{c,d,e\}$ | $\phi$ | $X$ | 0 | $\{c,d,e\}$ | $\{c,d,e\}$ | 1 |
| $\{a,b,c,d\}$ | $\phi$ | $X$ | 0 | $\{a,b,c,d\}$ | $\{a,b,c,d\}$ | 1 |
| $\{a,b,c,e\}$ | $\phi$ | $X$ | 0 | $\{a,b,c,e\}$ | $X$ | $\frac{4}{5}$ |
| $\{a,b,d,e\}$ | $\phi$ | $X$ | 0 | $\{a,b,d,e\}$ | $X$ | $\frac{4}{5}$ |
| $\{a,c,d,e\}$ | $\phi$ | $X$ | 0 | $\{a,c,d,e\}$ | $\{a,c,d,e\}$ | 1 |
| $\{b,c,d,e\}$ | $\phi$ | $X$ | 0 | $\{b,c,d,e\}$ | $\{b,c,d,e\}$ | 1 |
| $X$ | $X$ | $X$ | 1 | $X$ | $X$ | 1 |

(8) As $j = j_8$, the topology $\mathscr{T}_{j_8}$ is

$$\mathscr{T}_{j_8} = \{X, \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$$

**Definition 14** For every $j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$, we call $\mathcal{G}$ is a $j$-open set if $\mathcal{G} \in \mathscr{T}_j$ and the complement $\mathcal{G}^c = \Omega - \mathcal{G}$ of $j$-open set is called a $j$-closed set. The set of all $j$-closed sets denoted by $\mathscr{F}_j$.

*Example 10* From Example 9, by the complement. Then we obtain the $j$-closed set for each $\mathscr{T}_j$. For instance at $j = j_1$, the $j_1$-closed set is

$$\mathscr{F}_{j_1} = \{X, \phi, \{b,c,d\}, \{a,c,d\}, \{c,d\}, \{b,c\}, \{a,d\}, \{d\}, \{c\}\}.$$

**Definition 15** Let $(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively, and $X \subseteq \Omega$. Then we define ${}^m\mathcal{I}_j$ and ${}^m\mathcal{C}_j$ operators of $X$ with respect to $\Gamma$, where $\Gamma$=$\{\mathscr{T}_1, \mathscr{T}_2, \cdots, \mathscr{T}_n\}$, respectively, $\forall j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$, as follows:
(1) ${}^m\mathcal{I}_j(X) = \{\mathcal{G} \in \mathscr{T}_i : \bigvee(\mathcal{G} \subseteq X), i \in n\}$,
(2) ${}^m\mathcal{C}_j(X) = \{\mathcal{F} \in \mathscr{F}_i : \bigwedge(X \subseteq \mathcal{F}), i \in n\}$.

*Example 11* Consider $X$ and $R_1$ are given in Example 9, and we have another binary relation $R_2 = \{(a,a), (a,b), (b,c), (c,c), (d,b)\}$. Take $j = j_1$ (and also $j \in \{j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). The topology is determined by $R_2$ is

$$\mathscr{T}_{j_1} = \{X, \phi, \{c\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}\}$$

If $X = \{b,d\}$. Thus ${}^2\mathcal{I}_{j_1}(X) = \{b\}$ and ${}^2\mathcal{C}_{j_1}(X) = \{b,d\}$.

**Definition 16** Let $(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively, and $X \subseteq \Omega$. Then the $m$-boundary, $m$-positive and $m$-negative regions of $H$ using $j$-neighborhoods are denoted by ${}^m\bigtriangledown_j$, ${}^m\triangleright_j$ and ${}^m\triangleleft_j$, respectively, $\forall j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$, and defined as with respect to $\Gamma$, where $\Gamma$=$\{\mathscr{T}_1, \mathscr{T}_2, \cdots, \mathscr{T}_n\}$, respectively, as follows:
(1) ${}^m\bigtriangledown_j(X) = {}^m\mathcal{C}_j(X) - {}^m\mathcal{I}_j(X)$,

**Table 5** A comparison between accuracies when $j = j6 = j8$.

| P(X) | $\underline{\sum_{i=1}^{n} {}_{\mathscr{N}} R_i}(X)$ | $\overline{\sum_{i=1}^{n} {}_{\mathscr{N}} R_i}(X)$ | $\alpha_{\mathscr{N}}(X)$ | $\underline{\sum_{i=1}^{n} {}_{\mathscr{P}} R_i}(X)$ | $\overline{\sum_{i=1}^{n} {}_{\mathscr{P}} R_i}(X)$ | $\alpha_{\mathscr{N}}(X)$ |
|---|---|---|---|---|---|---|
| $\{a\}$ | $\{a\}$ | $\{a\}$ | 1 | $\{a\}$ | $\{a\}$ | 1 |
| $\{b\}$ | $\{b\}$ | $\{b\}$ | 1 | $\{b\}$ | $\{b\}$ | 1 |
| $\{c\}$ | $\phi$ | $\{c\}$ | 0 | $\{c\}$ | $\{c\}$ | 1 |
| $\{d\}$ | $\phi$ | $\{d\}$ | 0 | $\{d\}$ | $\{d\}$ | 1 |
| $\{e\}$ | $\{e\}$ | $\{e\}$ | 1 | $\{e\}$ | $\{e\}$ | 1 |
| $\{a,b\}$ | $\{a,b\}$ | $\{a,b\}$ | 1 | $\{a,b\}$ | $\{a,b\}$ | 1 |
| $\{a,c\}$ | $\{a\}$ | $\{a,c,d\}$ | $\frac{1}{3}$ | $\{a,c\}$ | $\{a,c\}$ | 1 |
| $\{a,d\}$ | $\{a\}$ | $\{a,d\}$ | $\frac{1}{2}$ | $\{a,d\}$ | $\{a,d\}$ | 1 |
| $\{a,e\}$ | $\{a,e\}$ | $\{a,e\}$ | 1 | $\{a,e\}$ | $\{a,e\}$ | 1 |
| $\{b,c\}$ | $\{b,c\}$ | $\{b,c\}$ | 1 | $\{b,c\}$ | $\{b,c\}$ | 1 |
| $\{b,d\}$ | $\{b\}$ | $\{b,c,d\}$ | $\frac{1}{3}$ | $\{b,d\}$ | $\{b,d\}$ | 1 |
| $\{b,e\}$ | $\{b,e\}$ | $\{b,e\}$ | 1 | $\{b,e\}$ | $\{b,e\}$ | 1 |
| $\{c,d\}$ | $\{c,d\}$ | $\{c,d\}$ | 1 | $\{c,d\}$ | $\{c,d\}$ | 1 |
| $\{c,e\}$ | $\{e\}$ | $\{c,d,e\}$ | $\frac{1}{3}$ | $\{c,e\}$ | $\{c,e\}$ | 1 |
| $\{d,e\}$ | $\{e\}$ | $\{d,e\}$ | $\frac{1}{2}$ | $\{d,e\}$ | $\{d,e\}$ | 1 |
| $\{a,b,c\}$ | $\{a,b,c\}$ | $\{a,b,c,d\}$ | $\frac{3}{4}$ | $\{a,b,c\}$ | $\{a,b,c\}$ | 1 |
| $\{a,b,d\}$ | $\{a,b\}$ | $\{a,b,c,d\}$ | $\frac{1}{2}$ | $\{a,b,d\}$ | $\{a,b,d\}$ | 1 |
| $\{a,b,e\}$ | $\{a,b,e\}$ | $\{a,b,e\}$ | 1 | $\{a,b,e\}$ | $\{a,b,e\}$ | 1 |
| $\{a,c,d\}$ | $\{a,c,d\}$ | $\{a,c,d\}$ | 1 | $\{a,c,d\}$ | $\{a,c,d\}$ | 1 |
| $\{a,c,e\}$ | $\{a,e\}$ | $\{a,c,d,e\}$ | $\frac{1}{2}$ | $\{a,c,e\}$ | $\{a,c,e\}$ | 1 |
| $\{a,d,e\}$ | $\{a,d,e\}$ | $\{a,d,e\}$ | 1 | $\{a,d,e\}$ | $\{a,d,e\}$ | 1 |
| $\{b,c,d\}$ | $\{b,c,d\}$ | $\{b,c,d\}$ | 1 | $\{b,c,d\}$ | $\{b,c,d\}$ | 1 |
| $\{b,c,e\}$ | $\{b,c,e\}$ | $\{b,c,d,e\}$ | $\frac{3}{4}$ | $\{b,c,e\}$ | $\{b,c,e\}$ | 1 |
| $\{b,d,e\}$ | $\{b,e\}$ | $\{b,c,d,e\}$ | $\frac{1}{2}$ | $\{b,d,e\}$ | $\{b,d,e\}$ | 1 |
| $\{c,d,e\}$ | $\{c,d,e\}$ | $\{c,d,e\}$ | 1 | $\{c,d,e\}$ | $\{c,d,e\}$ | 1 |
| $\{a,b,c,d\}$ | $\{a,b,c,d\}$ | $\{a,b,c,d\}$ | 1 | $\{a,b,c,d\}$ | $\{a,b,c,d\}$ | 1 |
| $\{a,b,c,e\}$ | $\{a,b,c,e\}$ | $X$ | $\frac{4}{5}$ | $\{a,b,c,e\}$ | $\{a,b,c,e\}$ | 1 |
| $\{a,b,d,e\}$ | $\{a,b,d,e\}$ | $X$ | $\frac{4}{5}$ | $\{a,b,d,e\}$ | $\{a,b,d,e\}$ | 1 |
| $\{a,c,d,e\}$ | $\{a,c,d,e\}$ | $\{a,c,d,e\}$ | 1 | $\{a,c,d,e\}$ | $\{a,c,d,e\}$ | 1 |
| $\{b,c,d,e\}$ | $\{b,c,d,e\}$ | $\{b,c,d,e\}$ | 1 | $\{b,c,d,e\}$ | $\{b,c,d,e\}$ | 1 |
| $X$ | $X$ | $X$ | 1 | $X$ | $X$ | 1 |

(2) ${}^m \rhd_j (X) = {}^m \mathcal{I}_j(X)$,
(3) ${}^m \lhd_j (X) = \Omega - {}^m \mathcal{C}_j(X)$.

*Example 12* From Examples 9 and 11. Take $j = j_1$ (and also $j \in \{j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). So, we have
(1) ${}^2\bigtriangledown_{j_1}(X) = \{d\}$.
(2) ${}^2 \rhd_{j_1} (X) = \{b\}$.
(3) ${}^2 \lhd_{j_1} (X) = \{a, c\}$.

**Theorem 5** *Let $(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively, and $X \subseteq \Omega$. We have ${}^m\mathcal{I}_j(X) = \underline{\sum_{i=1}^{n} {}_{\mathscr{N}} R_i(X)}$, ${}^m\mathcal{C}_j(X) = \overline{\sum_{i=1}^{n} {}^{\mathscr{N}} R_i(X)}$.*

*Proof* Follows from Definition 8 and 15.

According to above results of multi-granulation rough sets via neighborhood systems, we have Theorems 6, 7 and 8.

**Theorem 6** *Let $(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively, and $X \subseteq \Omega$. Then, with respect to the operators ${}^m\mathcal{I}_j$, we have*
*(i)* ${}^m\mathcal{I}_j(\phi) = \phi$,
*(ii)* ${}^m\mathcal{I}_j(\Omega) = \Omega$,
*(iii)* ${}^m\mathcal{I}_j(X) \subseteq X$,
*(iv)* *if* $X \subseteq Y \implies {}^m\mathcal{I}_j(X) \subseteq {}^m \mathcal{I}_j(Y)$,
*(v)* ${}^m\mathcal{I}_j({}^m\mathcal{I}_j(X)) = {}^m \mathcal{I}_j(X)$.

*Proof* (1) Since $\phi \in \mathscr{T}_i$ and by Definition 15,

$${}^m\mathcal{I}_j(\phi) = \{A \in \mathscr{T}_i : \bigvee(\mathscr{N}_j(A) \subseteq \phi), i \in n\} = \phi.$$

Thus, ${}^m\mathcal{I}_j(\phi) = \phi$.

(2) Follows from Definition 15 and from (1).

(3) Follows from Definition 15.

(4) By Definition 15 and if $X \subseteq Y$, then

$$\begin{aligned} {}^m\mathcal{I}_j(\mathcal{X}) &= \{A \in \mathscr{T}_i : \bigvee(\mathscr{N}_j(A) \subseteq X), i \in n\} \\ &\subseteq \{A \in \mathscr{T}_i : \bigvee(\mathscr{N}_j(A) \subseteq Y), i \in n\} \\ &=^m \mathcal{I}_j(Y). \end{aligned}$$

(5) Follows from (3) and Definition 15.

**Theorem 7** *Let $(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively, and $X \subseteq \Omega$. Then, with respect to the operators ${}^m\mathcal{C}_j$, we have*
*(i)* ${}^m\mathcal{C}_j(\Omega) = \Omega$,
*(ii)* ${}^m\mathcal{C}_j(\phi) = \phi$,
*(iii)* $X \subseteq^m \mathcal{C}_j(X)$,
*(iv)* *if* $X \subseteq Y \implies {}^m\mathcal{C}_j(X) \subseteq^m \mathcal{C}_j(Y)$,
*(v)* ${}^m\mathcal{C}_j({}^m\mathcal{C}_j(X)) =^m \mathcal{C}_j(X)$.

*Proof* Analogue to Theorem 6.

**Theorem 8** *Let $(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively, and $X, Y \subseteq \Omega$. Then*
*(1)* ${}^m\mathcal{I}_j(X \cap Y) =^m \mathcal{I}_j(X) \cap^m \mathcal{I}_j(Y)$,
*(2)* ${}^m\mathcal{I}_j(X \cup Y) \supseteq^m \mathcal{I}_j(X) \cup^m \mathcal{I}_j(Y)$.

*Proof*
(1) It is sufficient to show
$$^m\mathcal{I}_j(X \cap Y) =^m \mathcal{I}_j(X) \cap^m \mathcal{I}_j(Y).$$
By Definition 15, we have
$$^m\mathcal{I}_j(X \cap Y) = \{A \in \mathscr{T}_i : \bigvee(\mathscr{N}_j(A) \subseteq X \cap Y), i \in n\},$$
since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$ implies that $\mathscr{N}_j(A) \subseteq X$ and $\mathscr{N}_j(A) \subseteq Y$. Thus, ${}^m\mathcal{I}_j(X \cap Y) \subseteq^m \mathcal{I}_j(X)$ and ${}^m\mathcal{I}_j(X \cap Y) \subseteq^m \mathcal{I}_j(Y)$ by (3). Therefore,
$$\begin{aligned} &{}^m\mathcal{I}_j(X) \cap^m \mathcal{I}_j(Y) \\ &= \{A \in \mathscr{T}_i : \bigvee(\mathscr{N}_j(A) \subseteq X), i \in n\} \end{aligned}$$
and
$$\begin{aligned} &\{A \in \mathscr{T}_i : \bigvee(\mathscr{N}_j(A) \subseteq Y), i \in n\} \\ &= \{A \in \mathscr{T}_i : \bigvee(\mathscr{N}_j(X) \cap \mathscr{N}_j(Y) \subseteq \Omega), i \in n\} \\ &=^m \mathcal{I}_j(X \cap Y) =^m \mathcal{I}_j(X) \cap^m \mathcal{I}_j(Y). \end{aligned}$$
(2) Analogue to (1).

*Example 13* From Examples 9 and 11, take $j = j_1$ (and also $j \in \{j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly).
(1) If $X = \{a, b\}$ and $Y = \{a\}$. Then, we have, ${}^2\mathcal{I}_{j_1}(X \cap Y) = \{a\}$ and ${}^2\mathcal{I}_{j_1}(X) \cap^2 \mathcal{I}_{j_1}(Y) = \{a, b\} \cap \{a\} = \{a\}$. Thus ${}^m\mathcal{I}_j(X \cap Y) =^m \mathcal{I}_j(X) \cap^m \mathcal{I}_j(Y)$
(2) If $X = \{b, d\}$ and $Y = \{a\}$. Then we have, ${}^2\mathcal{I}_{j_1}(X \cup Y) = \{a, b, d\}$ and ${}^2\mathcal{I}_{j_1}(X) \cup^2 \mathcal{I}_{j_1}(Y) = \{a, b\}$. Therefore, ${}^m\mathcal{I}_j(X \cup Y) \supseteq^m \mathcal{I}_j(X) \cup^m \mathcal{I}_j(Y)$.

**Theorem 9** *Let $(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively, and $X, Y \subseteq \Omega$. Then*
*(1)* ${}^m\mathcal{C}_j(X \cup Y) =^m \mathcal{C}_j(X) \cup^m \mathcal{C}_j(Y)$,
*(2)* ${}^m\mathcal{C}_j(X \cap Y) \subseteq^m \mathcal{C}_j(X) \cap^m \mathcal{C}_j(Y)$.

*Proof* Analogue to Theorem 8.

*Example 14* From Examples 9 and 11, take $j = j_1$ (and also $j \in \{j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly).
(1) If $X = \{b, d\}$ and $Y = \{a\}$. Then we have, ${}^2\mathcal{C}_{j_1}(X \cup Y) = \{a, b, d\}$ and ${}^2\mathcal{C}_{j_1}(X) \cup^2 \mathcal{C}_{j_1}(Y) = \{a, b, d\}$. Therefore ${}^m\mathcal{C}_j(X \cup Y) =^m \mathcal{C}_j(X) \cup^m \mathcal{C}_j(Y)$.
(2) If $X = \{a, b\}$ and $Y = \{b, c\}$. Then, we have, ${}^2\mathcal{C}_{j_1}(X \cap Y) = \{b\}$ and ${}^2\mathcal{C}_{j_1}(X) \cap^2 \mathcal{C}_{j_1}(Y) = \{b, d\}$. Thus ${}^m\mathcal{C}_j(X \cap Y) \subseteq^m \mathcal{C}_j(X) \cap^m \mathcal{C}_j(Y)$.

**Theorem 10** *Let $(\Omega,\mathscr{T}_1),(\Omega,\mathscr{T}_2),\cdots,(\Omega,\mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1,R_2,\cdots,R_n$, respectively, and $X,Y\subseteq\Omega$. Then, ${}^m\mathcal{I}_j$ and ${}^m\mathcal{C}_j$ are interior and closure operators, respectively.*

*Proof* Obvious.

**Theorem 11** *Let $(\Omega,\mathscr{T}_1),(\Omega,\mathscr{T}_2),\cdots,(\Omega,\mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1,R_2,\cdots,R_n$, respectively, and $X\subseteq\Omega$. Then, with respect to ${}^m\bigtriangledown_j(X)$, we have*
*(1) ${}^m\bigtriangledown_j(X)={}^m\mathcal{C}_j(X)\cap{}^m\mathcal{C}_j(X^c)$,*
*(2) ${}^m\bigtriangledown_j(X)={}^m\bigtriangledown_j(X^c)$,*
*(3) ${}^m\mathcal{C}_j(X)=X\cup{}^m\bigtriangledown_j(X)$,*
*(4) ${}^m\mathcal{I}_j(X)=X\setminus{}^m\bigtriangledown_j(X)$,*
*(5) ${}^m\bigtriangledown_j(X)\cap{}^m\mathcal{I}_j(X)=\phi$,*
*(6) ${}^m\bigtriangledown_j(X_1\cup X_2)\subseteq{}^m\bigtriangledown_j(X_1)\cup{}^m\bigtriangledown_j(X_2)$,*
*(7) ${}^m\bigtriangledown_j(X_1\cap X_2)\subseteq{}^m\bigtriangledown_j(X_1)\cup{}^m\bigtriangledown_j(X_2)$,*
*(8) ${}^m\bigtriangledown_j({}^m\mathcal{C}_j(X))\subseteq{}^m\bigtriangledown_j(X)$,*
*(9) ${}^m\bigtriangledown_j({}^m\mathcal{I}_j(X))\subseteq{}^m\bigtriangledown_j(X)$.*

*Proof* (1) By Theorem 6 and Definition 16, we have
$$^m\bigtriangledown_j(X)={}^m\mathcal{C}_j(X)\cap({}^m\mathcal{I}_j(X))^c={}^m\mathcal{C}_j(X)\cap{}^m\mathcal{C}_j(X^c).$$

(2) Follows from (1), where
$$\begin{aligned}{}^m\bigtriangledown_j(X^c)&={}^m\mathcal{C}_j(X^c)\cap({}^m\mathcal{C}_j(X^c)^c)\\&={}^m\mathcal{C}_j(X^c)\cap{}^m\mathcal{C}_j(X)={}^m\bigtriangledown_j(X).\end{aligned}$$

(3) By (1) and Theorem 6, we have
$$\begin{aligned}X\cup{}^m\bigtriangledown_j(X)&=X\cup({}^m\mathcal{C}_j(X)\cap{}^m\mathcal{C}_j(X^c))\\&=(X\cup{}^m\mathcal{C}_j(X))\cap(X\cup{}^m\mathcal{C}_j(X^c))\\&={}^m\mathcal{C}_j(X)\cap[X\cup({}^m\mathcal{I}_j(X))^c]\\&={}^m\mathcal{C}_j(X)\cap\Omega\\&={}^m\mathcal{C}_j(X).\end{aligned}$$

(4)
$$\begin{aligned}X\setminus{}^m\bigtriangledown_j(X)&=X\setminus({}^m\mathcal{C}_j(X)\cap{}^m\mathcal{C}_j(X^c))\\&=X\wedge([{}^m\mathcal{C}_j(X)\cap{}^m\mathcal{C}_j(X^c)]^c)\\&=X\wedge([{}^m\mathcal{C}_j(X)]^c\cup[{}^m\mathcal{C}_j(X^c)]^c)\\&=[X\cap{}^m\mathcal{I}_j(X^c)]\cup[X\cap{}^m\mathcal{I}_j(X)]\\&=\phi\cap{}^m\mathcal{I}_j(X)\\&={}^m\mathcal{I}_j(X).\end{aligned}$$

(5) It is clear.

(6) By Theorem 7 and Definition 16, we have
$$\begin{aligned}{}^m\bigtriangledown_j(X_1\cup X_2)&={}^m\mathcal{C}_j(X_1\cup X_2)\cap({}^m\mathcal{C}_j(X_1\cup X_2)^c)\\&\subseteq[{}^m\mathcal{C}_j(X_1)\cup{}^m\mathcal{C}_j(X_2)]\cap[{}^m\mathcal{C}_j(X_1^c)\cap{}^m\mathcal{C}_j(X_2^c)]\\&=[{}^m\mathcal{C}_j(X_1)\cup{}^m\mathcal{C}_j(X_2)\cap{}^m\mathcal{C}_j(X_1^c)\cap{}^m\mathcal{C}_j(X_2^c)]\\&=[{}^m\mathcal{C}_j(X_1)\cap{}^m\mathcal{C}_j(X_1^c)]\cup[{}^m\mathcal{C}_j(X_2)\cap{}^m\mathcal{C}_j(X_2^c)]\\&={}^m\bigtriangledown_j(X_1)\cup{}^m\bigtriangledown_j(X_2).\end{aligned}$$

(7) Analogue to (6) above.

(8)
$$\begin{aligned}{}^m\bigtriangledown_j({}^m\mathcal{C}_j(X))&={}^m\mathcal{C}_j({}^m\mathcal{C}_j(X)\cap{}^m\mathcal{C}_j^c(X)\\&\subseteq{}^m\mathcal{C}_j(X)\cap{}^m\mathcal{C}_j({}^m\mathcal{C}_j^c(X))\\&\subseteq{}^m\mathcal{C}_j(X)\cap{}^m\mathcal{C}_j(X^c)\\&={}^m\bigtriangledown_j(X).\end{aligned}$$

(9) Analogue to (8).

**Definition 17** Let $R_1$ and $R_2$ be any binary relations on a finite universe $\Omega$, $f_1$ and $f_2$ are the subbase of $R_1$ and $R_2$ respectively. Then we define a intersection mapping $\mathcal{F}_{\mathscr{N}\cap}:\Omega\to2^\Omega$ satisfies
$$\mathcal{F}_{\mathscr{N}\cap}(x)=f_1(x)\cap f_2(x).$$
Where $f_1(x)=\{(\mathscr{N}_j(x))_{R_1}:x\in\Omega\}$ and $f_2(x)=\{(\mathscr{N}_j(x))_{R_2}:x\in\Omega\}$.

**Definition 18** Let $\mathscr{T}_1, \mathscr{T}_2$ are two topologies induced by $R_1$ and $R_2$. Then we can define $\sqcap$ between two topologies which is defined as follows
$$\mathscr{T}_1 \sqcap \mathscr{T}_2 = \bigcup\{\bigcap \mathcal{F}_{\mathscr{N}\cap}(x) : x \in \Omega\}.$$

**Theorem 12** *If $\mathscr{T}_1, \mathscr{T}_2$ are two topologies induced by $R_1$ and $R_2$, then $\mathscr{T}_1 \sqcap \mathscr{T}_2$ is a topology.*

*Proof* Suppose $Y$ be a finite universe, $R_1$ and $R_2$ two binary relations, and
$\mathscr{T}_1$=$\{\Omega, \phi, (\mathscr{N}_j(x_{i1}))_{R_1}, (\mathscr{N}_j(x_{i2}))_{R_1}, \cdots, (\mathscr{N}_j(x_{ik}))_{R_1},$
$\bigcup_{i=1}^{k}(\mathscr{N}_j(x_{ik}))_{R_1}, \bigcap_{i=1}^{k}(\bigcup_{i=1}^{k} \mathscr{N}_j(x_{ik}))_{R_1}\}$,
$\mathscr{T}_2$=$\{\Omega, \phi, (\mathscr{N}_j(x_{j1}))_{R_2}, (\mathscr{N}_j(x_{j2}))_{R_2}, \cdots, (\mathscr{N}_j(x_{jl}))_{R_2},$
$\bigcup_{i=1}^{l}(\mathscr{N}_j(x_{il}))_{R_2}, \bigcap_{i=1}^{l}(\bigcup_{i=1}^{l} \mathscr{N}_j(x_{il}))_{R_2}\}$,
induced by $R_1$ and $R_2$, $k, l \leq |\Omega|$, where $|.|$ is cardinality of $\Omega$.
(i) According to the definition of $\mathscr{T}_1 \sqcap \mathscr{T}_2$, obviously, $\phi \in \mathscr{T}_1 \sqcap \mathscr{T}_2$, $\Omega \in \mathscr{T}_1 \sqcap \mathscr{T}_2$.
(ii) Assume that $X, Y \in \mathscr{T}_1 \sqcap \mathscr{T}_2$, then there exists two classes $(\mathscr{N}_j(x_1))_{R_1} \in \mathscr{T}_1$, $(\mathscr{N}_j(x_2))_{R_2} \in \mathscr{T}_2$ such that $X \subseteq (\mathscr{N}_j(x_1))_{R_1}, Y \subseteq (\mathscr{N}_j(x_2))_{R_2}$.
Hence $X \cap Y \in (\mathscr{N}_j(x_1))_{R_1} \cap (\mathscr{N}_j(x_2))_{R_2} \in \mathscr{T}_1 \sqcap \mathscr{T}_2$.
(iii) Let $\mathscr{T} \in \mathscr{T}_1 \sqcap \mathscr{T}_2$, suppose that $\bigcup_{X\in\mathscr{T}} X \notin \mathscr{T}_1 \sqcap \mathscr{T}_2$. Then there at least exists an element $x \in X \in \mathscr{T}$, we have an class $\mathscr{N}_j(x)$ consisting $x$ in $\mathscr{T}_1 \sqcap \mathscr{T}_2$ such that $\mathscr{N}_j(x) \notin \mathscr{T}_1 \sqcap \mathscr{T}_2$.
Thus $\mathscr{N}_j(x)$=$((\mathscr{N}_j(x))_{R_1} \cap (\mathscr{N}_j(x))_{R_2})$ holds, a contradiction. Therefore, $\mathscr{T}_1 \sqcap \mathscr{T}_2$ is still a topology.

*Example 15* Consider $R_1$ and $R_2$ are defined on Examples 1 and 11 respectively. Then we obtain by a relation $R_1$
$f_1(a) = \{a\}, f_1(b) = \{b\}, f_1(c) = \{b, c\}, f_1(d) = \{a\}$.
And the relation $R_2$
$f_2(a) = \{a, b\}, f_2(b) = \{c\}, f_2(c) = \{c\}$ and $f_2(d) = \{b\}$.
Thus $\mathcal{F}_{\mathscr{N}\cap}(a) = \{a\}, \mathcal{F}_{\mathscr{N}\cap}(b) = \phi, \mathcal{F}_{\mathscr{N}\cap}(c) = \{c\}$ and $\mathcal{F}_{\mathscr{N}\cap}(d) = \phi$. Therefore
$$\mathscr{T}_1 \sqcap \mathscr{T}_2 = \{\Omega, \phi, \{a\}, \{c\}, \{a, c\}\}.$$

Similarly, we can prove that the intersection of the finite topologies is topology, i.e., $\sqcap_{i=1}^{m}\tau_i$ is a topology with respect to $\mathscr{T}_1, \mathscr{T}_2, \cdots, \mathscr{T}_n$, denoted by $\sqcap_{i=1}^{m}\mathscr{T}_i = \Gamma$.

**Definition 19** Let $(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively. An intersection operation $\mathcal{F}_\cap : \Omega \to 2^\Omega$. Then $(\Omega, \sqcap_{i=1}^{n}\mathscr{T}_i)$ is called a multi-granulation topological rough space, denoted as $(\Omega, \sqcap_{i=1}^{n}\mathscr{T}_i)$=$(\Omega, \Gamma)$.

**Corollary 1** *Let $\mathscr{T}_1, \mathscr{T}_2$ be two topologies on $\Omega$, if for any $X \in \mathscr{T}_1$, there exists $Y \in \mathscr{T}_2$ such that $X \subseteq Y$. Then we call $\mathscr{T}_1$ finer than $\mathscr{T}_2$, denoted by $\mathscr{T}_1 \leq^{\mathscr{T}} \mathscr{T}_2$. If $\mathscr{T}_1$ is strictly finer than $\mathscr{T}_2$, denoted by $\mathscr{T}_1 <^{\mathscr{T}} \mathscr{T}_2$. If and only if $X = Y$, then $\mathscr{T}_1 = \mathscr{T}_2$. Similarly, let $\Gamma_1, \Gamma_2$ be two multi-granulation topological rough spaces on X, if for any $\mathscr{T}_1 \in \Gamma_1$, there exists $\mathscr{T}_2 \in \Gamma_2$ such that $\mathscr{T}_1 \leq^{\mathscr{T}} \mathscr{T}_2$, then we call $\Gamma_1$ than $\Gamma_2$, denoted by $\Gamma_1 \leq^{\Gamma} \Gamma_2$. If $\Gamma_1$ is strictly finer than $\Gamma_2$, denoted by $\Gamma_1 <^{\Gamma} \Gamma_2$. If and only if $\mathscr{T}_1 = \mathscr{T}_2$, then $\Gamma_1 = \Gamma_2$.*

**Theorem 13** *Let $\mathscr{T}_1, \mathscr{T}_2, \cdots, \mathscr{T}_n$ be n topologies on $\Omega$ induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively. If $\mathscr{T}_1 <^{\mathscr{T}} \mathscr{T}_2 <^{\mathscr{T}} \cdots <^{\mathscr{T}} \mathscr{T}_n$, then $\Gamma = \mathscr{T}_1$.*

Thus, from the above definition, we know $(\Omega, \Gamma)$ is finer than each topology on $\Omega$.

**Corollary 2** *If $\mathscr{T}_1 <^{\mathscr{T}} \mathscr{T}_2$, and $\mathcal{S}_1^{\mathscr{N}}, \mathcal{S}_2^{\mathscr{N}}$ are the topology subbase of $\mathscr{T}_1, \mathscr{T}_2$, respectively. Then $\mathcal{S}_1^{\mathscr{N}} <^{\mathscr{T}} \mathcal{S}_2^{\mathscr{N}}$.*

**Corollary 3** *If $\Gamma_1 <^{\Gamma} \Gamma_2$, and $\mathcal{S}_{1M}^{\mathscr{N}}, \mathcal{S}_{2M}^{\mathscr{N}}$ are their family of the topology subbase of $\Gamma_1, \Gamma_2$, respectively. Then $\mathcal{S}_{1M}^{\mathscr{N}} <^{\Gamma} \mathcal{S}_{2M}^{\mathscr{N}}$.*

**Theorem 14** *Let $\mathcal{S}_i^{\mathscr{N}}$ be the topology subbase of $\mathscr{T}_i$, then $\sqcap_{i=1}^{n}\mathcal{S}_i^{\mathscr{N}}$ is a topology subbase of multi-granulation topological rough space $\Gamma$.*

*Proof*
(i) For any $x \in X$, there exists $(\mathscr{N}_j(x))_{R_i} \in \beta_i$ such that $x \in (\mathscr{N}_j(x))_{R_i}$. Since $(\mathscr{N}_j(x))_{R_i} \in \sqcap_{i=1}^{n}\mathcal{S}_i^{\mathscr{N}}$. Hence, let $A = (\mathscr{N}_j(x))_{R_i} \in \sqcap_{i=1}^{n}\mathcal{S}_i^{\mathscr{N}}$, we have $x \in A$.
(ii) For any $A_1, A_2 \in \sqcap_{i=1}^{n}\mathcal{S}_i^{\mathscr{N}}$, suppose $x \in A_1 \cap A_2$, but $A_1$ is a set which is intersection of $(\mathscr{N}_j(x))_{R_i}$, $A_2$ is a set which is intersection of $(\mathscr{N}_j(y))_{R_i}$, then $x = y$, otherwise $(\mathscr{N}_j(x))_{R_i} = (\mathscr{N}_j(y))_{R_i}$. Hence $A_1 \cap A_2 = \phi$. There exists $A_3 = \phi$ obviously, $\phi \subset \phi$ holds.

Therefore $\sqcap_{i=1}^{n}\mathcal{S}_i^{\mathscr{N}}$ is a topology subbase of multi-granulation topological rough space (MGTRS) $\Gamma$.

**Definition 20** Let $(\Omega, \Gamma)$ be a MGTRS and $X \subseteq \Omega$. Then the interior operator is defined as
$$\mathcal{I}_j(X) = \bigcup\{\mathcal{G} \in \Gamma : \mathcal{G} \subseteq X\}.$$

**Definition 21** Let $(\Omega, \Gamma)$ be a MGTRS and $X \subseteq \Omega$. Then the closure operator is defined as

$$\mathcal{C}_j(X) = \bigcap\{\mathcal{F} \in \Gamma^c : X \subseteq \mathcal{F}\}.$$

*Example 16* From Example 15, take $j = j_1$ (and also $j \in \{j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). If $X = \{a\}$, then $\mathcal{I}_{j_1}(X) = \{a\}$ and $\mathcal{C}_{j_1}(X) = \{a, c\}$.

**Proposition 1** *Let $(\Omega, \Gamma)$ be a MGTRS and $X, Y \subseteq \Omega$. Then, with respect to the interior operator $\mathcal{I}_j$, we have*
*(i)* $\mathcal{I}_j(\phi) = \phi$,
*(ii)* $\mathcal{I}_j(\Omega) = \Omega$,
*(iii)* $\mathcal{I}_j(X) \subseteq X$,
*(iv) if* $X \subseteq Y \implies \mathcal{I}_j(X) \subseteq \mathcal{I}_j(Y)$,
*(v)* $\mathcal{I}_j(\mathcal{I}_j(X)) = \mathcal{I}_j(X)$.
*(vi)* $\mathcal{I}_j(X \cap Y) = \mathcal{I}_j(X) \cap \mathcal{I}_j(Y)$,
*(vii)* $\mathcal{I}_j(X \cup Y) \supseteq \mathcal{I}_j(X) \cup \mathcal{I}_j(Y)$.

*Proof* By Definition 20 and Theorems 6 and 8.

**Proposition 2** *Let $(\Omega, \Gamma)$ be a MGTRS and $X, Y \subseteq \Omega$. Then, with respect to the closure operator $\mathcal{I}_j$, we have*
*(i)* $\mathcal{C}_j(\phi) = \phi$,
*(ii)* $\mathcal{C}_j(\Omega) = \Omega$,
*(iii)* $X \subseteq \mathcal{C}_j(X)$,
*(iv) if* $X \subseteq Y \implies \mathcal{C}_j(X) \subseteq \mathcal{C}_j(Y)$,
*(v)* $\mathcal{C}_j(\mathcal{C}_j(X)) = \mathcal{C}_j(X)$.
*(vi)* $\mathcal{C}_j(X \cup Y) = \mathcal{C}_j(X) \cup \mathcal{C}_j(Y)$,
*(vii)* $\mathcal{C}_j(X \cap Y) \subseteq \mathcal{C}_j(X) \cap \mathcal{C}_j(Y)$.

*Proof* By Definitions 21 and Theorems 7 and 9.

**Theorem 15** *Let $\Omega_R = \{\mathcal{N}_j(A_1), \mathcal{N}_j(A_2), ..., \mathcal{N}_j(A_k)\}$. Then we call*

$$m_{\mathcal{N}}(\mathcal{N}_j(A)) = \frac{|\Omega|}{k}(1 - \frac{1}{k.|\mathcal{N}_j(A)|}).$$

*A measure of granularity of a set $\mathcal{N}_j(A)$, $k$ is the number of blocks in $\Omega_R$, denoted by $|\Omega_R| = k$.*

*Proof* It is sufficient to show that $m$ meets all the conditions in Definition 5.
(i) Obviously, $m_{\mathcal{N}}(\mathcal{N}_j(A)) = \frac{|\Omega|}{k}(1 - \frac{1}{k.|\mathcal{N}_j(A)|}) \geq 0$.
(ii) If $\mathcal{N}_j(A) \subset \mathcal{N}_j(B)$, then $|N_j(A)| < |N_j(B)|$, then

$$m_{\mathcal{N}}(\mathcal{N}_j(A)) - m(\mathcal{N}_j(B))$$
$$= \frac{|\Omega|}{k}(1 - \frac{1}{k.|\mathcal{N}_j(A)|}) - \frac{|\Omega|}{k}(1 - \frac{1}{k.|\mathcal{N}_j(B)|})$$
$$= \frac{|\Omega|}{k}(1 - \frac{1}{k.|\mathcal{N}_j(A)|} - 1 + \frac{1}{k.|\mathcal{N}_j(B)|})$$
$$= \frac{|\Omega|}{k}(\frac{1}{k.|\mathcal{N}_j(B)|} - \frac{1}{k.|\mathcal{N}_j(A)|}) < 0,$$

i.e., $m(\mathcal{N}_j(A)) < m(\mathcal{N}_j(B))$.
(iii) If $A \sim_s B$, then $|\mathcal{N}_j(A)| \leq |\mathcal{N}_j(B)|$ and $|\mathcal{N}_j(A)| \geq |\mathcal{N}_j(B)|$. Hence $m_{\mathcal{N}}(\mathcal{N}_j(A)) = m_{\mathcal{N}}(\mathcal{N}_j(B))$.

**Proposition 3** *Let $\Omega_R$ be a classes of $\Omega$ induced by any binary relation on $\Omega$ and $X \in \Omega_R$. The maximum granularity measure of $X$ with respect to $R$ is one. This value is achieved if and only if $k = 1$, $max(m_{\mathcal{N}}(\mathcal{N}_j(A))) = |\Omega|(1 - \frac{1}{|\Omega|})$.*

**Proposition 4** *Let $\Omega_R$ be a classes of $\Omega$ induced by any binary relation on $\Omega$ and $X \in \Omega_R$. The minimum granularity measure of $X$ with respect to $R$ is one. This value is achieved if and only if $k = |\Omega|$, $min(m_{\mathcal{N}}(\mathcal{N}_j(A))) = (1 - \frac{1}{|\Omega|})$.*

*Example 17* From Example 15, the first topology have the following subbase $\mathcal{S}_{j_1} = \{\{a\}, \{c\}\}$. Then $m_{\mathcal{N}}(\{a\}) = \frac{4}{2}(1 - \frac{1}{2*1}) = 1$ and $m_{\mathcal{N}}(\{c\}) = \frac{4}{2}(1 - \frac{1}{2*1}) = 1$.

**Theorem 16** *Let $(\Omega, \Gamma)$ be a MGTRS, $m_{\mathcal{N}} : 2^{\Omega} \to \mathcal{R}$ a measure of the granularity of subsets of $\Omega$, and $\mathcal{S}_M^{\mathcal{N}} = \{A_1, A_2, ..., A_n\}$ a topology subbase of $\Gamma_M^{\mathcal{N}}$. Then a measure $G_M^{\mathcal{N}}(\Gamma) = \sum_{i=1}^{n} m^{\mathcal{N}}(A_i).p^{\mathcal{N}}(A_i)$ is a topological granularity of $\Gamma$, where $p^{\mathcal{N}}(A_i) = \frac{|A_i|}{|\Omega|}$.*

*Proof* It is sufficient to show that $G_M^{\mathcal{N}}$ satisfies all the conditions in Definition 6.

(i) Obviously, $G_M^{\mathscr{N}}(\Gamma) \geq 0$ holds.
(ii) Suppose $\Gamma_1 <^{\Gamma} \Gamma_2$, $\mathcal{S}_{1M}^{\mathscr{N}} \subset \mathcal{S}_{2M}^{\mathscr{N}}$ holds. This means that every class of $\mathcal{S}_{2M}^{\mathscr{N}}$ is a union of one or more blocks of $\mathcal{S}_{1M}^{\mathscr{N}}$ and at least one class of $\mathcal{S}_M^{'\mathscr{N}}$ is the union of at least two blocks from $\mathcal{S}_{1M}^{\mathscr{N}}$. By the fact $\Omega$ is a finite universe, there exists a finite sequence of partitions $\mathcal{S}_{1M}^{\mathscr{N}} = \mathcal{S}_{M1}^{\mathscr{N}} \subset \mathcal{S}_{M2}^{\mathscr{N}} \subset ... \subset \mathcal{S}_{Mi}^{\mathscr{N}} = \mathcal{S}_{M2}^{\mathscr{N}}$ such that exactly one block of $\mathcal{S}_{j+1}^{\mathscr{N}}$ is the union of two classes from $\mathcal{S}_j^{\mathscr{N}}$ for $j = 1, 2, ..., n-1$ and $n \geq 2$. We want to show that $G(\mathcal{S}_j^{\mathscr{N}}) < G(\mathcal{S}_{j+1}^{\mathscr{N}})$. Without loss of generality, suppose a class of $\mathcal{S}_{j+1}^{\mathscr{N}}$ is obtained by the union of two classes $A_{j1}$ and $A_{j2}$ of $\mathcal{S}_j^{\mathscr{N}}$, that is,$\mathcal{S}_j^{\mathscr{N}} = \{A_{j1}, A_{j2}, ..., A_{jk}\}$, $k \geq 2$ and $\mathcal{S}_{j+1}^{\mathscr{N}} = \{A_{j1} \cup A_{j2}, ..., \cup A_{jk}\}$. According to the definition of $G_M^{\mathscr{N}}(\Omega)$ and monotonicity of $m^{\mathscr{N}}$, we have:

$$
\begin{aligned}
G_M^{\mathscr{N}}(\mathcal{S}_j^{\mathscr{N}}) &= \textstyle\sum_{i=1}^k m^{\mathscr{N}}(A_{ji}.p^{\mathscr{N}}(A_{ji})) \\
&= m^{\mathscr{N}}(A_{j1}).p^{\mathscr{N}}(A_{j1}) + m^{\mathscr{N}}(A_{j2}).p^{\mathscr{N}}(A_{j2}) + \textstyle\sum_{i=3}^k m^{\mathscr{N}}(A_{ji}.p^{\mathscr{N}}(A_{ji})) \\
&< m^{\mathscr{N}}(A_{j1} \cup A_{j2}).p^{\mathscr{N}}(A_{j1}) + m^{\mathscr{N}}(A_{j2} \cup A_{j1}).p^{\mathscr{N}}(A_{j2}) + \textstyle\sum_{i=3}^k m^{\mathscr{N}}(A_{ji}.p^{\mathscr{N}}(A_{ji})) \\
&= m^{\mathscr{N}}(A_{j1} \cup A_{j2}).p^{\mathscr{N}}(A_{j1}) + p^{\mathscr{N}}(A_{j1}) + \textstyle\sum_{i=3}^k m^{\mathscr{N}}(A_{ji}.p^{\mathscr{N}}(A_{ji})) \\
&= m^{\mathscr{N}}(A_{j1} \cup A_{j2}).p^{\mathscr{N}}(A_{j1} \cup A_{j1}) + \textstyle\sum_{i=3}^k m^{\mathscr{N}}(A_{ji}.p^{\mathscr{N}}(A_{ji})) \\
&= G_M(\mathcal{S}_{j+1}^{\mathscr{N}}).
\end{aligned}
$$

Then $\Gamma_1 <^{\Gamma} \Gamma_2$ holds.
(iii) Assume that $\Gamma_1 = \Gamma_2$, $\mathcal{S}_{1M}^{\mathscr{N}} \subset \mathcal{S}_{2M}^{\mathscr{N}}$ holds. And from Definition 19, $G(\Gamma_1) = G(\Gamma_2)$ holds.

**Proposition 5** *Let $(\Omega, \Gamma)$ be a MGTRS. The maximum topological granularity measure of $\mathscr{T}$ with respect to $\Omega$ is one. This value is achieved if and only if $m^{\mathscr{N}} = 1$, $max(G_M^{\mathscr{N}}(\Gamma)) = |\Omega| - 1$.*

**Proposition 6** *Let $(\Omega, \Gamma)$ be a MGTRS. The minimum topological granularity of s with respect to $\Omega$ is one. This value is achieved if and only if $m^{\mathscr{N}} = |\Omega|$, $min(G_M^{\mathscr{N}}(\Gamma)) = 1 - \frac{1}{|\Omega|}$.*

Thus, $1 - \frac{1}{|\Omega|} \leq G(\Gamma_M^{\mathscr{N}}) \leq |\Omega| - 1$.

**Theorem 17** *Let $\Gamma_1, \Gamma_2$ be two MGTRS. If $\Gamma_1 <^{\mathscr{T}} \Gamma_2$, then $G_M^{\mathscr{N}}(\Gamma_1) < G_M^{\mathscr{N}}(\Gamma_2)$.*

**Definition 22** Let $(\Omega, \Gamma)$ be a MGTRS and $\mathcal{S}_M^{\mathscr{N}} = \{A_1, A_2, \cdots, A_n\}$ is a topology subbase of $\Gamma$. Then the topological entropy of $\Gamma$ is defined as:

$$E_M^{\mathscr{N}}(\Gamma) = 1 - \frac{1}{|\Omega|}\sum_{i=1}^{q} m^{\mathscr{N}}(A_i).p^{\mathscr{N}}(A_i),$$

where $p^{\mathscr{N}}(A_i) = \frac{|A_i|}{|\Omega|}$.

**Definition 23** Let $(\Omega, \Gamma)$ be a MGTRS. Then $\mathscr{T}_k$ is significant in $\Gamma$, if $E_\Gamma^{\mathscr{N}}(\Omega, \sqcap_{i=1}^q \mathscr{T}_i) \neq E_\Gamma^{\mathscr{N}}(\Omega, \sqcap_{i=1, i\neq k}^q \mathscr{T}_i)$. So, $\mathscr{T}_k$ is not significant in $\Gamma$ if $E_\Gamma^{\mathscr{N}}(\Omega, \sqcap_{i=1}^q \mathscr{T}_i) = E_\Gamma^{\mathscr{N}}(\Omega, \sqcap_{i=1, i\neq k}^q \mathscr{T}_i)$.

**Definition 24** Let $(\Omega, \Gamma)$ be a MGTRS. The significance measure of $\mathscr{T}_k$ in $\Gamma$ is defined as

$$S_\Gamma^{\mathscr{N}}(\mathscr{T}_k) = \frac{E_\Gamma^{\mathscr{N}}(\Omega, \sqcap_{i=1}^q \mathscr{T}_i)}{E_\Gamma^{\mathscr{N}}(\Omega, \sqcap_{i=1, i\neq k}^q \mathscr{T}_i)}.$$

**Definition 25** Let $\Gamma = \{\mathscr{T}_1, \mathscr{T}_2, \cdots, \mathscr{T}_q\}$ be $q$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively on $\Omega$. If there exists a subset $\Gamma_i = \{\mathscr{T}_{i1}, \mathscr{T}_{i2}, \cdots, \mathscr{T}_{iq}\} \subseteq \Gamma$, such that $E_\Gamma^{\mathscr{N}}(\Omega, \sqcap_{i=1}^q \mathscr{T}_i) = E_\Gamma^{\mathscr{N}}(\Omega, \sqcap_{i=k}^q \mathscr{T}_{ik})$, but $E_\Gamma^{\mathscr{N}}(\Omega, \mathscr{T}_{i1} \sqcap \mathscr{T}_{i2} \sqcap \cdots \sqcap \mathscr{T}_{ik} \sqcap \mathscr{T}_{i(k+1)}) \neq E_\Gamma^{\mathscr{N}}(\Omega, \mathscr{T}_1 \sqcap \mathscr{T}_2 \sqcap \cdots \sqcap \mathscr{T}_q)$, then we call $\Gamma_i$ is a granularity reduct of $\Gamma$.

Now, we establish Algorithm 1 to produce the intersection of topologies and Algorithm 2 to make reduction for topologies.

**Algorithm 1.** An algorithm for $\sqcap_{i=1}^n \mathscr{T}_i$

**Input**: $x \in \Omega$ and Binary Relations $R_i$.
**Output**: $\sqcap_{i=1}^n \mathscr{T}_i$.
**for** $(i = 1; i <= |R_i|; i++)$
  **for** $(k = 1; k <= |R_k|; k++)$
    $f_i(x) = (\mathscr{N}_j(x))_{R_k}$.
    $\mathcal{F}^i_{\mathscr{N}\cap}(x) = \bigcap_{k=1}^n f_i(x)$.
  **endfor**
**endfor**
**for** $(i = 1; i <= |R_i|; i++)$
    $\sqcap_{i=1} \mathscr{T}_i = \bigcup(\bigcap \mathcal{F}^i_{\mathscr{N}\cap}(x))$.

**endfor**
**end**

---

**Algorithm 2.** An algorithm for a reduction

---

**Input**: $\Omega$, $\Omega_R$ and Multi-source information decision table.
**Output**: make a reduction ( $reduct$ is a set which conserve the selected granularities).
$reduct = \phi$
**for** $(i = 1; i <= n; i++)$
    $m^{\mathscr{N}}(\mathscr{N}_j(A_i)) = \frac{|\Omega|}{k}(1 - \frac{1}{k.|\mathscr{N}_j(A_i)|})$, where $k = |\Omega_R|$
**endfor**
**for** $(i = 1; i <= n; i++)$
    $p^{\mathscr{N}}(A_i) = \frac{|A_i|}{|\Omega|}$
**endfor**
**for** $(i = 1; i <= n; i++)$
    $E_M^{\mathscr{N}}(\Gamma) = 1 - \frac{1}{|\Omega|} \sum_{i=1}^{q} m^{\mathscr{N}}(A_i).p^{\mathscr{N}}(A_i)$
**endfor**
**for** $(i = 1; i <= |R_i|; i++)$
    $S_\Gamma^{\mathscr{N}}(\mathscr{T}_k) = \frac{E_\Gamma^{\mathscr{N}}(\Omega, \sqcap_{i=1}^{q} \mathscr{T}_i)}{E_\Gamma^{\mathscr{N}}(\Omega, \sqcap_{i=1, i\neq k}^{q} \mathscr{T}_i)}$, where $k \leq |R_i|$
    $reduct = \mathscr{T}_k$
      **if**$(E_\Gamma^{\mathscr{N}}(\Omega, reduct) = E_\Gamma^{\mathscr{N}}(\Omega, \sqcap_{i=1}^{q} \mathscr{T}_i))$
        **goto:** end
      **endif**
**endfor**
**end**

---

## 4.2 The second type of topology by $j$-adhesion NS and its algorithms

**Definition 26** The topology which is generated by $j$-adhesion neighborhood systems is

$$\mathscr{T}_j = \bigcup \{A \in \Omega : \forall x \in A, \mathscr{P}_j(x) \subseteq A\}$$

$\forall j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ is called the topology generated by $j$-adhesion neighborhoods, denoted by $\mathscr{T}_j$.

*Example 18* Let $X = \{a, b, c, d\}$ and $R$ be a binary relation defined by

$$R = \{(a, a), (b, b), (c, b), (c, c), (d, a)\}$$

Then, we compute $\mathscr{T}_j$ as follows. For instance at $j = j1$ :

$$\mathscr{T}_{j_1} = \{X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}.$$

**Definition 27** For every $j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$, we call $\mathcal{G}$ is a $j$-open set if $\mathcal{G} \in \mathscr{T}_j$ and the complement $\mathcal{G}^c = \Omega - \mathcal{G}$ of $j$-open set is called a $j$-closed set. The set of all $j$-closed sets denoted by $\mathscr{F}_j$.

*Example 19* From Example 18, by the complement. Then we obtain the $j$-closed set for each $\mathscr{T}_j$. For instance at $j = j_1$, the $j_1$-closed set is

$$\mathscr{F}_{j_1} = \{X, \phi, \{a, c, d\}, \{a, b, d\}, \{a, d\}, \{b, c\}, \{c\}, \{b\}\}.$$

**Definition 28** Let $(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively, and $X \subseteq \Omega$. Then we define ${}^m\mathcal{L}_j$ and ${}^m\mathcal{U}_j$ operators of $X$ with respect to $\Gamma$, where $\Gamma = \{\mathscr{T}_1, \mathscr{T}_2, \cdots, \mathscr{T}_n\}$, respectively, $\forall j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$, as follows:
(1) ${}^m\mathcal{L}_j(X) = \{\mathcal{G} \in \mathscr{T}_i : \bigvee(\mathcal{G} \subseteq X), i \in n\}$,
(2) ${}^m\mathcal{U}_j(X) = \{\mathcal{F} \in \mathscr{F}_i : \bigwedge(A \subseteq \mathcal{F}), i \in n\}$.

*Example 20* Consider $X$ and $R_1$ are given in Example 9, and we have another binary relation $R_2 = \{(a,a),(a,b),(b,c),(c,c),(d,b)\}$. Take $j = j_1$ (and also $j \in \{j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). The topology is determined by $R_2$ is

$$\mathscr{T}_{j_1} = \{U, \phi, \{a\}, \{d\}, \{b,c\}, \{a,d\}, \{a,b,c\}, \{b,c,d\}\}$$

If $X = \{b,d\}$. Thus ${}^2\mathcal{L}_{j_1}(X) = \{b,d\}$ and ${}^2\mathcal{U}_{j_1}(X) = U$.

**Definition 29** Let $(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively, and $X \subseteq \Omega$. Then the $m$-boundary, $m$-positive and $m$-negative regions of $H$ using $j$-neighborhoods are denoted by ${}^m\mathcal{B}_j$, ${}^m\mathcal{P}_j$ and ${}^m\mathcal{N}_j$, respectively, $\forall j \in \{j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$, and defined as with respect to $\Gamma$, where $\Gamma$=$\{\mathscr{T}_1, \mathscr{T}_2, ..., \mathscr{T}_n\}$, respectively, as follows:
(1) ${}^m\mathcal{B}_j(X) ={}^m \mathcal{U}_j(X) -{}^m \mathcal{L}_j(X)$,
(2) ${}^m\mathcal{P}_j(X) ={}^m \mathcal{L}_j(X)$,
(3) ${}^m\mathcal{N}_j(X) = \Omega -{}^m \mathcal{U}_j(X)$.

*Example 21* From Examples 18 and 20. Take $j = j_1$ (and also $j \in \{j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). So, we have
(1) ${}^2\mathcal{B}_{j_1}(X) = \{a,c\}$.
(2) ${}^2\mathcal{P}_{j_1}(X) = \{b,d\}$.
(3) ${}^2\mathcal{N}_{j_1}(X) = \phi$.

**Theorem 18** *Let $(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively, and $X \subseteq \Omega$. We have ${}^m\mathcal{L}_j(X) = \underline{\sum_{i=1}^{n} {}_{\mathscr{P}} R_i(X)}$, ${}^m\mathcal{U}_j(X) = \overline{\sum_{i=1}^{n} {}^{\mathscr{P}} R_i(X)}$.*

*Proof* Follows from Definition 11 and 28.

According to the above propositions of multigranulation rough sets using neighborhood systems, we easily obtain the following results.

**Theorem 19** *$(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively, and $X \subseteq \Omega$. Then, with respect to the operators ${}^m\mathcal{L}_j$, we have*
*(i)* ${}^m\mathcal{L}_j(\Omega) = \Omega$,
*(ii)* ${}^m\mathcal{L}_j(\phi) = \phi$,
*(iii)* ${}^m\mathcal{L}_j(X) \subseteq X$,
*(iv)* if $X \subseteq Y \implies {}^m\mathcal{L}_j(X) \subseteq{}^m \mathcal{L}_j(Y)$,
*(v)* ${}^m\mathcal{L}_j({}^m\mathcal{L}_j(X)) ={}^m \mathcal{L}_j(X)$.

*Proof* (1) Since $\phi \in \mathscr{T}_i$ and by Definition 28,

$${}^m\mathcal{L}_j(\phi) = \{A \in \mathscr{T}_i : \bigvee(\mathscr{P}_j(A) \subseteq \phi), i \in n\} = \phi.$$

Thus, ${}^m\mathcal{L}_j(\phi) = \phi$.

(2) Follows from Definition 28 and from (1).

(3) Follows from Definition 28.

(4) By Definition 28 and if $X \subseteq Y$, then

$$\begin{aligned} {}^m\mathcal{L}_j(X) &= \{A \in \mathscr{T}_i : \bigvee(\mathscr{P}_j(A) \subseteq X), i \in n\} \\ &\subseteq \{A \in \mathscr{T}_i : \bigvee(\mathscr{P}_j(A) \subseteq Y), i \in n\} \\ &={}^m \mathcal{L}_j(Y). \end{aligned}$$

(5) Follows from (3) and Definition 28.

**Theorem 20** *$(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively, and $X \subseteq \Omega$. Then, with respect to the operators ${}^m\mathcal{U}_j$, we have*
*(i)* ${}^m\mathcal{U}_j(\Omega) = \Omega$,
*(ii)* ${}^m\mathcal{U}_j(\phi) = \phi$,
*(iii)* $X \subseteq{}^m \mathcal{U}_j(X)$,
*(iv)* if $X \subseteq Y \implies {}^m\mathcal{U}_j(X) \subseteq{}^m \mathcal{U}_j(Y)$,
*(v)* ${}^m\mathcal{U}_j({}^m\mathcal{U}_j(X)) ={}^m \mathcal{U}_j(X)$.

*Proof* Analogue to Theorem 19.

**Theorem 21** *Let $(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively, and $X, W \subseteq \Omega$. Then*
*(1)* ${}^m\mathcal{L}_j(X \cap Y) ={}^m \mathcal{L}_j(X) \cap{}^m \mathcal{L}_j(Y)$,

*(2)* ${}^m\mathcal{L}_j(X \cup Y) \supseteq^m \mathcal{L}_j(X) \cup^m \mathcal{L}_j(Y)$.

*Proof*
(1) It is sufficient to show

$$ {}^m\mathcal{L}_j(X \cap Y) =^m \mathcal{L}_j(X) \cap^m \mathcal{L}_j(Y). $$

By Definition 15, we have

$$ {}^m\mathcal{L}_j(X \cap Y) = \{A \in \mathscr{T}_i : \bigvee(\mathscr{P}_j(A) \subseteq X \cap Y), i \in n\}, $$

since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$ implies that $\mathscr{P}_j(A) \subseteq X$ and $\mathscr{P}_j(A) \subseteq Y$. Thus, ${}^m\mathcal{L}_j(X \cap Y) \subseteq^m \mathcal{L}_j(X)$ and ${}^m\mathcal{L}_j(X \cap Y) \subseteq^m \mathcal{L}_j(Y)$ by (3). Therefore,

$$ {}^m\mathcal{L}_j(X) \cap^m \mathcal{L}_j(Y) $$

$$ = \{A \in \mathscr{T}_i : \bigvee(\mathscr{P}_j(A) \subseteq X), i \in n\} $$

and

$$ \{A \in \mathscr{T}_i : \bigvee(\mathscr{P}_j(A) \subseteq Y), i \in n\} $$

$$ = \{A \in \mathscr{T}_i : \bigvee(\mathscr{P}_j(X) \cap \mathscr{P}_j(Y) \subseteq \Omega), i \in n\} $$

$$ =^m \mathcal{L}_j(X \cap Y) =^m \mathcal{L}_j(X) \cap^m \mathcal{L}_j(Y). $$

(2) Analogue to (1).

*Example 22* Let $\Omega = \{a, b, c, d, e\}$ and we have three binary relations $R_1 = \{(a, a), (b, b), (c, b), (c, c), (d, a)\}$ and $R_2 = \{(a, b), (b, a), (a, c), (c, a), (c, d), (d, c), (d, b)$. Then the topology determined by $R_1$ is

$$ \mathscr{T}_1 = \{\Omega, \phi, \{c\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\} $$

and by $R_2$ is

$$ \mathscr{T}_2 = \{\Omega, \phi, \{b\}, \{c\}, \{b, c\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}\}. $$

Therefore if $X_1 = \{a\}$ and $X_2 = \{d\}$, then we have ${}^2\mathcal{L}_{j_1}(X_1 \cup X_2) = \{a, b\}$ and ${}^2\mathcal{L}_{j_1}(X_1) \cup^2 \mathcal{L}_{j_1}(X_2) = \phi$. Thus ${}^m\mathcal{L}_j(X_1 \cup X_2) \neq^m \mathcal{L}_j(X_1) \cup^m \mathcal{L}_j(X_2)$.

**Theorem 22** *Let $(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively, and $X, Y \subseteq \Omega$. Then*
*(1)* ${}^m\mathcal{U}_j(X \cup Y) = m\mathcal{U}_j(X) \cup m\mathcal{U}_j(Y)$,
*(2)* $m\mathcal{U}_j(X \cap Y) \subseteq m\mathcal{U}_j(X) \cap m\mathcal{U}_j(Y)$,

*Proof* Analogue to Theorem 21.

*Example 23* From 22, if we have $X_1 = \{a, b\}$ and $X_2 = \{b, d\}$, then we have ${}^2\mathcal{U}_{j_1}(X_1 \cap X_2) = \{b\}$ and ${}^2\mathcal{U}_{j_1}(X_1) \cap^2 \mathcal{U}_{j_1}(X_2) = \{a, b, d\}$. Thus ${}^m\mathcal{U}_j(X_1 \cap X_2) \subseteq^m \mathcal{U}_j(X_1) \cap^m \mathcal{U}_j(X_2)$.

**Theorem 23** *Let $(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively, and $X, Y \subseteq \Omega$. Then, $m\mathcal{L}_j$ and $m\mathcal{U}_j$ are interior and closure operators, respectively.*

*Proof* Its obvious.

**Theorem 24** *Let $(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively, and $X \subseteq \Omega$. Then, with respect to ${}^m\mathcal{B}_j(X)$, we have*
*(1)* ${}^m\mathcal{B}_j(X) =^m \mathcal{U}_j(X) \cap^m \mathcal{U}_j(X^c)$,
*(2)* ${}^m\mathcal{B}_j(X) =^m \mathcal{B}_j(X^c)$,
*(3)* ${}^m\mathcal{U}_j(X) = X \cup^m \mathcal{B}_j(X)$,
*(4)* ${}^m\mathcal{L}_j(X) = X \setminus^m \mathcal{B}_j(X)$,
*(5)* ${}^m\mathcal{B}_j(X) \cap^m \mathcal{L}_j(X) = \phi$,
*(6)* ${}^m\mathcal{B}_j(X_1 \cup X_2) \subseteq^m \mathcal{B}_j(X_1) \cup^m \mathcal{B}_j(X_2)$,
*(7)* ${}^m\mathcal{B}_j(X_1 \cap X_2) \subseteq^m \mathcal{B}_j(X_1) \cup^m \mathcal{B}_j(X_2)$,
*(8)* ${}^m\mathcal{B}_j({}^m\mathcal{U}_j(X)) \subseteq^m \mathcal{B}_j(X)$,
*(9)* ${}^m\mathcal{B}_j({}^m\mathcal{L}_j(X)) \subseteq^m \mathcal{B}_j(X)$.

*Proof* (1) By Theorem 19 and Definition 29, we have

$$^m\mathcal{B}_j(X) =^m \mathcal{U}_j(X) \cap (^m\mathcal{L}_j(X))^c =^m \mathcal{U}_j(X) \cap^m \mathcal{U}_j(X^c).$$

(2) Follows from (1), where

$$\begin{aligned} ^m\mathcal{B}_j(X^c) &=^m \mathcal{U}_j(X^c) \cap (^m\mathcal{U}_j(X^c)^c) \\ &=^m \mathcal{U}_j(X^c) \cap^m \mathcal{U}_j(X) =^m \mathcal{B}_j(X). \end{aligned}$$

(3) By (1) and Theorem 19, we have

$$\begin{aligned} X \cup^m \mathcal{B}_j(X) &= X \cup (^m\mathcal{U}_j(X) \cap^m \mathcal{U}_j(X^c)) \\ &= (X \cup^m \mathcal{U}_j(X)) \cap (X \cup^m \mathcal{U}_j(X^c)) \\ &=^m \mathcal{U}_j(X) \cap [X \cup (^m\mathcal{L}_j(X))^c] \\ &=^m \mathcal{U}_j(X) \cap \Omega \\ &=^m \mathcal{U}_j(X). \end{aligned}$$

(4)
$$\begin{aligned} X \setminus^m \mathcal{B}_j(X) &= X \setminus (^m\mathcal{U}_j(X) \cap^m \mathcal{U}_j(X^c)) \\ &= X \wedge ([^m\mathcal{U}_j(X) \cap^m \mathcal{U}_j(X^c)]^c) \\ &= X \wedge ([^m\mathcal{U}_j(X)]^c \cup [^m\mathcal{U}_j(X^c)]^c) \\ &= [X \cap^m \mathcal{L}_j(X^c)] \cup [X \cap^m \mathcal{L}_j(X)] \\ &= \phi \cap^m \mathcal{L}_j(X) \\ &=^m \mathcal{L}_j(X). \end{aligned}$$

(5) It is clear.

(6) By Theorem 20 and Definition 29, we have

$$\begin{aligned} ^m\mathcal{B}_j(X_1 \cup X_2) &=^m \mathcal{U}_j(X_1 \cup X_2) \cap (^m\mathcal{U}_j(X_1 \cup X_2)^c) \\ &\subseteq [^m\mathcal{U}_j(X_1) \cup^m \mathcal{U}_j(X_2)] \cap [^m\mathcal{U}_j(X_1^c) \cap^m \mathcal{U}_j(X_2^c)] \\ &= [^m\mathcal{U}_j(X_1) \cup^m \mathcal{U}_j(X_2) \cap^m \mathcal{U}_j(X_1^c) \cap^m \mathcal{U}_j(X_2^c)] \\ &= [^m\mathcal{U}_j(X_1) \cap^m \mathcal{U}_j(X_1^c)] \cup [^m\mathcal{U}_j(X_2) \cap^m \mathcal{U}_j(X_2^c)] \\ &=^m \mathcal{B}_j(X_1) \cup^m \mathcal{B}_j(X_2). \end{aligned}$$

(7) Analogue to (6).

(8)
$$\begin{aligned} ^m\mathcal{B}_j(^m\mathcal{U}_j(X)) &=^m \mathcal{U}_j(^m\mathcal{U}_j(X) \cap^m \mathcal{U}_j^c(X) \\ &\subseteq^m \mathcal{U}_j(X) \cap^m \mathcal{U}_j(^m\mathcal{U}_j^c(X)) \\ &\subseteq^m \mathcal{U}_j(X) \cap^m \mathcal{U}_j(X^c) \\ &=^m \mathcal{B}_j(X). \end{aligned}$$

(9) Analogue to (8).

**Definition 30** Let $R_1$ and $R_2$ be any binary relations on a finite universe $\Omega$, $f_1$ and $f_2$ are the subbase of $R_1$ and $R_2$ respectively. Then we define a intersection mapping $\mathcal{F}_\cap : \Omega \to 2^\Omega$ satisfies

$$\mathcal{F}_{\mathscr{P}\cap}(x) = f_1(x) \cap f_2(x).$$

Where $f_1(x) = \bigcup\{(\mathscr{P}_j(x))_{R_1} : x \in \Omega\}$ and $f_2(x) = \bigcup\{(\mathscr{P}_j(x))_{R_2} : x \in \Omega\}$.

**Definition 31** Let $\mathscr{T}_1, \mathscr{T}_2$ are two topologies induced by $R_1$ and $R_2$. Then we can define $\sqcap$ between two topologies which is defined as follows

$$\mathscr{T}_1 \sqcap \mathscr{T}_2 = \bigcup\{\bigcap \mathcal{F}_{\mathscr{P}\cap}(x) : x \in \Omega\}.$$

**Theorem 25** *If $\mathscr{T}_1, \mathscr{T}_2$ are two topologies induced by $R_1$ and $R_2$, then $\mathscr{T}_1 \sqcap \mathscr{T}_2$ is a topology.*

*Proof* Assume that $\Omega$ be a finite universe, $R_1$ and $R_2$ two binary relations, and
$\mathscr{T}_1$= $\{\Omega, \phi, (\mathscr{P}_j(x_{i1}))_{R_1}, (\mathscr{P}_j(x_{i2}))_{R_1}, \cdots, (\mathscr{P}_j(x_{ik}))_{R_1},$
$\bigcup_{i=1}^{k}(\mathscr{P}_j(x_{ik}))_{R_1}\}, \bigcap_{i=1}^{k}(\bigcup_{i=1}^{k} \mathscr{P}_j(x_{ik}))_{R_1}$, $\mathscr{T}_1$=$\{\Omega, \phi, (\mathscr{P}_j(x_{j1}))_{R_2}, (\mathscr{P}_j(x_{j2}))_{R_2}, \cdots, (\mathscr{P}_j(x_{jl}))_{R_2},$
$\bigcup_{i=1}^{k}(\mathscr{P}_j(x_{il}))_{R_2} \bigcap_{i=1}^{k}(\bigcup_{i=1}^{k} \mathscr{P}_j(x_{il}))_{R_2}\}$,
induced by $R_1$ and $R_2$, $k, l \leq |Y|$, where $|.|$ is cardinality of $\Omega$.
(i) Based on the definition of $\mathscr{T}_1 \sqcap \mathscr{T}_2$, obviously, $\phi \in \mathscr{T}_1 \sqcap \mathscr{T}_2$, $\Omega \in \mathscr{T}_1 \sqcap \mathscr{T}_2$.
(ii) Assume that $X, Y \in \mathscr{T}_1 \sqcap \mathscr{T}_2$, then there exists two classes $(\mathscr{P}_j(x_1))_{R_1} \in \tau_1$, $(\mathscr{P}_j(x_2))_{R_2} \in \tau_2$ such that $X \subseteq (\mathscr{P}_j(x_1))_{R_1}$, $Y \subseteq (\mathscr{P}_j(x_2))_{R_2}$. Hence $X \cap Y \in (\mathscr{P}_j(x_1))_{R_1} \cap (\mathscr{P}_j(x_2))_{R_2} \in \mathscr{T}_1 \sqcap \mathscr{T}_2$.
(iii) Let $\mathscr{T} \in \mathscr{T}_1 \sqcap \mathscr{T}_2$, suppose that $\bigcup_{X \in \mathscr{T}} X \notin \mathscr{T}_1 \sqcap \tau_2$. Then there at least exists an element $x \in X \in \mathscr{T}$, we have an class $\mathscr{P}_j(x)$ consisting $x$ in $\mathscr{T}_1 \sqcap \mathscr{T}_2$ such that $\mathscr{P}_j(x) \notin \mathscr{T}_1 \sqcap \mathscr{T}_2$.
Thus $\mathscr{P}_j(x) = ((\mathscr{P}_j(x_1))_{R_1} \cap (\mathscr{P}_j(x_2))_{R_2})$ holds, a contraction. Therefore, $\mathscr{T}_1 \sqcap \mathscr{T}_2$ is still a topology.

*Example 24* Consider $R_1$ is defined on Example 1 and $R_2 = \{(a,b),(b,a),(a,c),(c,a),(c,d),(d,c),(d,b)\}$. Then we obtain by a relation $R_1$
$f_1(a) = \{a,d\}, f_1(b) = \{b\}, f_1(c) = \{c\}, f_1(d) = \{a,d\}$.
And the relation $R_2$
$f_2(a) = \{a\}, f_2(b) = \{b,c\}, f_2(c) = \{b,c\}$ and $f_2(d) = \{d\}$.
Thus $\mathcal{F}_{\mathscr{P}\cap}(a) = \{a,d\}, \mathcal{F}_{\mathscr{P}\cap}(b) = \phi, \mathcal{F}_{\mathscr{P}\cap}(c) = \{c\}$ and $\mathcal{F}_{\mathscr{P}\cap}(d) = \{a,d\}$.
Therefore

$$\mathscr{T}_1 \sqcap \mathscr{T}_2 = \{\Omega, \phi, \{c\}, \{a,d\}, \{a,d,c\}\}.$$

Similarly, we can prove that the intersection of the finite topologies is topology, i.e., $\sqcap_{i=1}^{m}\mathscr{T}_i$ is a topology with respect to $\mathscr{T}_1, \mathscr{T}_2, \cdots, \mathscr{T}_n$, denoted by $\sqcap_{i=1}^{m}\mathscr{T}_i = \Gamma$.

**Definition 32** Let $(\Omega, \mathscr{T}_1), (\Omega, \mathscr{T}_2), \cdots, (\Omega, \mathscr{T}_n)$ be $n$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively. Then $(Y, \sqcap_{i=1}^{n}\mathscr{T}_i)$ is called a MGTRS, denoted as $(\Omega, \sqcap_{i=1}^{n}\mathscr{T}_i) = (\Omega, \Gamma)$.

**Corollary 4** *Let $\mathscr{T}_1, \mathscr{T}_2$ be two topologies on $\Omega$, if for any $X \in \mathscr{T}_1$, there exists $Y \in \mathscr{T}_2$ such that $X \subseteq Y$. Then we call $\mathscr{T}_1$ finer than $\mathscr{T}_2$, denoted by $\mathscr{T}_1 \leq^{\mathscr{T}} \mathscr{T}_2$. If $\mathscr{T}_1$ is strictly finer than $\mathscr{T}_2$, denoted by $\mathscr{T}_1 <^{\mathscr{T}} \mathscr{T}_2$. If and only if $X = Y$, then $\mathscr{T}_1 = \mathscr{T}_2$. Similarly, let $\Gamma_1, \Gamma_2$ be two multi-granulation topological rough spaces on $X$, if for any $\mathscr{T}_1 \in \Gamma_1$, there exists $\mathscr{T}_2 \in \Gamma_2$ such that $\mathscr{T}_1 \leq^{\mathscr{T}} \mathscr{T}_2$, then we call $\Gamma_1$ than $\Gamma_2$, denoted by $\Gamma_1 \leq^{\Gamma} \Gamma_2$. If $\Gamma_1$ is strictly finer than $\Gamma_2$, denoted by $\Gamma_1 <^{\Gamma} \Gamma_2$. If and only if $\mathscr{T}_1 = \mathscr{T}_2$, then $\Gamma_1 = \Gamma_2$.*

**Theorem 26** *Let $\mathscr{T}_1, \mathscr{T}_2, \cdots, \mathscr{T}_n$ be $n$ topologies on $\Omega$ induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively. If $\mathscr{T}_1 <^{\mathscr{T}} \mathscr{T}_2 <^{\mathscr{T}} \cdots <^{\mathscr{T}} \mathscr{T}_n$, then $\Gamma = \mathscr{T}_1$.*

Therefore, from the above definition, we know $(\Omega, \Gamma)$ is finer than each topology on $\Omega$.

**Corollary 5** *If $\mathscr{T}_1 <^{\mathscr{T}} \mathscr{T}_2$, and $\mathcal{S}_1^{\mathscr{P}}, \mathcal{S}_2^{\mathscr{P}}$ are the topology subbase of $\mathscr{T}_1, \mathscr{T}_2$, respectively. Then $\mathcal{S}_1^{\mathscr{P}} <^{\mathscr{T}} \mathcal{S}_2^{\mathscr{P}}$.*

**Corollary 6** *If $\Gamma_1 <^{\Gamma} \Gamma_2$, and $\mathcal{S}_{1M}^{\mathscr{P}}, \mathcal{S}_{2M}^{\mathscr{P}}$ are their family of the topology subbase of $\Gamma_1, \Gamma_2$, respectively. Then $\mathcal{S}_{1M}^{\mathscr{P}} <^{\Gamma} \mathcal{S}_{2M}^{\mathscr{P}}$.*

**Theorem 27** *Let $\mathcal{S}_i^{\mathscr{P}}$ be the topology subbase of $\mathscr{T}_i$, then $\sqcap_{i=1}^{n}\mathcal{S}_i^{\mathscr{P}}$ is a topology subbase of MGTRS $\Gamma$.*

*Proof*
(i) For any $x \in X$, there exists $(\mathscr{P}_j(x))_{R_i} \in \mathcal{S}_i^{\mathscr{P}}$ such that $x \in \cap(\mathscr{P}_j(x))_{R_i}$. Note that $(\mathscr{P}_j(x))_{R_i} \in \sqcap_{i=1}^{n}\mathcal{S}_i^{\mathscr{P}}$. Hence, let $B = (\mathscr{P}_j(x))_{R_i} \in \sqcap_{i=1}^{n}\mathcal{S}_i^{\mathscr{P}}$, we have $x \in B$.
(ii) For any $B_1, B_2 \in \sqcap_{i=1}^{n}\mathcal{S}_i^{\mathscr{P}}$, suppose $x \in B_1 \cap B_2$, but $B_1$ is a set which is $(\mathscr{P}_j(x))_{R_i}$, $B_2$ is a set which is $(\mathscr{P}_j(y))_{R_i}$, then $x = y$, otherwise $(\mathscr{P}_j(x))_{R_i} = (\mathscr{P}_j(x))_{R_i}$. Hence $B_1 \cap B_2 = \phi$. There exists $B_3 = \phi$ obviously, $\phi \subset \phi$ holds.

Therefore $\sqcap_{i=1}^{n}\mathcal{S}_i^{\mathscr{P}}$ is a topology subbase of MGTRS $\Gamma$.

**Definition 33** Let $(\Omega, \Gamma)$ be a MGTRS, $\mathcal{S}_M^{\mathscr{P}}$ is a topology base of $\Gamma$, and $X \subseteq \Omega$. Then the interior operator is defined as

$$\mathcal{I}_j(X) = \bigcup\{\mathcal{G} \in \Gamma : \mathcal{G} \subseteq X\}.$$

**Definition 34** Let $(\Omega, \Gamma)$ be a MGTRS, $\beta_M$ is a topology base of $\Gamma$, and $X \subseteq \Omega$. Then the closure operator is defined as

$$\mathcal{C}_j(X) = \bigcap\{\mathcal{F} \in \Gamma^c : X \subseteq \mathcal{F}\}.$$

*Example 25* From Example 24. Take $j = j_1$ (and also $j \in \{j_2, j_3, j_4, j_5, j_6, j_7, j_8\}$ are similarly). If $X = \{a\}$, then $\mathcal{I}_{j_1}(X) = \phi$ and $\mathcal{C}_{j_1}(X) = \{a,d\}$.

**Proposition 7** *Let $(\Omega, \Gamma)$ be a MGTRS and $X, Y \subseteq \Omega$. Then, with respect to the operators $\mathcal{I}_j$, we have*
*(i)* $\mathcal{I}_j(\phi) = \phi$,
*(ii)* $\mathcal{I}_j(\Omega) = \Omega$,
*(iii)* $\mathcal{I}_j(X) \subseteq X$,
*(iv)* if $X \subseteq Y \implies \mathcal{I}_j(X) \subseteq \mathcal{I}_j(Y)$,
*(v)* $\mathcal{I}_j(\mathcal{I}_j(X)) = \mathcal{I}_j(X)$.
*(vi)* $\mathcal{I}_j(X \cap Y) = \mathcal{I}_j(X) \cap \mathcal{I}_j(Y)$,
*(vii)* $\mathcal{I}_j(X \cup Y) \supseteq \mathcal{I}_j(X) \cup \mathcal{I}_j(Y)$.

*Proof* By Definitions 33 and Theorems 19 and 21.

**Proposition 8** *Let $(\Omega, \Gamma)$ be MGTRS and $X, Y \subseteq \Omega$. Then, with respect to the operators $\mathcal{I}_j$, we have*
*(i)* $\mathcal{C}_j(\phi) = \phi$,

*(ii)* $\mathcal{C}_j(\Omega) = \Omega$,
*(iii)* $X \subseteq \mathcal{C}_j(X)$,
*(iv)* *if* $X \subseteq Y \implies \mathcal{C}_j(X) \subseteq \mathcal{C}_j(Y)$,
*(v)* $\mathcal{C}_j(\mathcal{C}_j(X)) = \mathcal{C}_j(X)$.
*(vi)* $\mathcal{C}_j(X \cup Y) = \mathcal{C}_j(X) \cup \mathcal{C}_j(Y)$,
*(vii)* $\mathcal{C}_j(X \cap Y) \subseteq \mathcal{C}_j(X) \cap \mathcal{C}_j(Y)$.

*Proof* By Definitions 34 and Theorems 20 and 22.

**Theorem 28** *Let* $\Omega_R = \{\mathscr{P}_j(A_1), \mathscr{P}_j(A_2), \cdots, \mathscr{P}_j(A_k)\}$. *Then we call*

$$m_{\mathscr{P}}(\mathscr{P}_j(A)) = \frac{|\Omega|}{k}(1 - \frac{1}{k.|\mathscr{P}_j(A)|}).$$

*A measure of granularity of a set* $\mathscr{P}_j(A)$, $k$ *is the number of blocks in* $\Omega_R$, *denoted by* $|\Omega_R| = k$.

*Proof* It is sufficient to show that $m$ meets all the conditions in Definition 5.
(i) Obviously, $m_{\mathscr{P}}(\mathscr{P}_j(A)) = \frac{|\Omega|}{k}(1 - \frac{1}{k.|\mathscr{P}_j(A)|}) \geq 0$.
(ii) If $\mathscr{P}_j(A) \subset \mathscr{P}_j(B)$, then $|P_j(A)| < |P_j(B)|$, then

$$m_{\mathscr{P}}(\mathscr{P}_j(A)) - m(\mathscr{P}_j(B))$$

$= \frac{|\Omega|}{k}(1 - \frac{1}{k.|\mathscr{P}_j(A)|}) - \frac{|\Omega|}{k}(1 - \frac{1}{k.|\mathscr{P}_j(B)|})$
$= \frac{|\Omega|}{k}(1 - \frac{1}{k.|\mathscr{P}_j(A)|} - 1 + \frac{1}{k.|\mathscr{P}_j(B)|})$
$= \frac{|\Omega|}{k}(\frac{1}{k.|\mathscr{P}_j(B)|} - \frac{1}{k.|\mathscr{P}_j(A)|}) < 0$,
i.e., $m(\mathscr{P}_j(A)) < m(\mathscr{P}_j(B))$.
(iii) If $A \sim_s B$, then $|\mathscr{P}_j(A)| \leq |\mathscr{P}_j(B)|$ and $|\mathscr{P}_j(A)| \geq |\mathscr{P}_j(B)|$. Hence $m_{\mathscr{P}}(\mathscr{P}_j(A)) = m_{\mathscr{P}}(\mathscr{P}_j(B))$.

**Proposition 9** *Let* $\Omega_R$ *be a classes of* $\Omega$ *induced by any binary relation on* $\Omega$ *and* $X \in \Omega_R$. *The maximum granularity measure of* $X$ *with respect to* $R$ *is one. This value is achieved if and only if* $k = 1$, $max(m_{\mathscr{P}}(\mathscr{P}_j(A))) = |\Omega|(1 - \frac{1}{|\Omega|})$.

**Proposition 10** *Let* $\Omega_R$ *be a classes of* $\Omega$ *induced by any binary relation on* $\Omega$ *and* $X \in \Omega_R$. *The minimum granularity measure of* $X$ *with respect to* $R$ *is one. This value is achieved if and only if* $k = |\Omega|$, $min(m_{\mathscr{P}}(\mathscr{P}_j(A))) = (1 - \frac{1}{|\Omega|})$.

*Example 26* From Example 24, the first topology have the following subbase $\mathcal{S}^{\mathscr{P}}_{j1} = \{\{a,d\}, \{c\}, \phi\}$. Then $m^{\mathscr{P}}(\{a,d\}) = \frac{4}{3}(1 - \frac{1}{3*2}) = \frac{10}{9}$ and $m^{\mathscr{P}}(\{c\}) = \frac{4}{3}(1 - \frac{1}{3*1}) = \frac{8}{9}$.

**Theorem 29** *Let* $(\Omega, \Gamma)$ *be a MGTRS,* $m_{\mathscr{P}} : 2^{\Omega} \to \mathcal{R}$ *a measure of the granularity of subsets of* $\Omega$, *and* $\mathcal{S}^{\mathscr{P}}_M = \{A_1, A_2, ..., A_n\}$ *a topology subbase of* $\Gamma^{\mathscr{P}}_M$. *Then a measure* $G^{\mathscr{P}}_M(\Gamma) = \sum_{i=1}^{n} m^{\mathscr{P}}(A_i).p^{\mathscr{P}}(A_i)$
*is a topological granularity of* $\Gamma$, *where* $p^{\mathscr{P}}(A_i) = \frac{|A_i|}{|\Omega|}$.

*Proof* It is sufficient to show that $G^{\mathscr{P}}_M$ satisfies all the conditions in Definition 6.
(i) Obviously, $G^{\mathscr{P}}_M(\Gamma) \geq 0$ holds.
(ii) Suppose $\Gamma_1 <^{\Gamma} \Gamma_2$, $\mathcal{S}^{\mathscr{P}}_{1M} \subset \mathcal{S}^{\mathscr{P}}_{2M}$ holds. This means that every class of $\mathcal{S}^{\mathscr{P}}_{2M}$ is a union of one or more blocks of $\mathcal{S}^{\mathscr{P}}_{1M}$ and at least one class of $\mathcal{S}^{'\mathscr{P}}_{M}$ is the union of at least two blocks from $\mathcal{S}^{\mathscr{P}}_{1M}$. By the fact $\Omega$ is a finite universe, there exists a finite sequence of partitions $\mathcal{S}^{\mathscr{P}}_{1M} = \mathcal{S}^{\mathscr{P}}_{M1} \subset \mathcal{S}^{\mathscr{P}}_{M2} \subset ... \subset \mathcal{S}^{\mathscr{P}}_{Mi} = \mathcal{S}^{\mathscr{P}}_{M2}$ such that exactly one block of $\mathcal{S}^{\mathscr{P}}_{j+1}$ is the union of two classes from $\mathcal{S}^{\mathscr{P}}_{j}$ for $j = 1, 2, \cdots, n-1$ and $n \geq 2$. We want to show that $G(\mathcal{S}^{\mathscr{P}}_{j}) < G(\mathcal{S}^{\mathscr{P}}_{j+1})$. Without loss of generality, suppose a class of $\mathcal{S}^{\mathscr{P}}_{j+1}$ is obtained by the union of two classes $A_{j1}$ and $A_{j2}$ of $\mathcal{S}^{\mathscr{P}}_{j}$, that is,$\mathcal{S}^{\mathscr{P}}_{j} = \{A_{j1}, A_{j2}, \cdots, A_{jk}\}$, $k \geq 2$ and $\mathcal{S}^{\mathscr{P}}_{j+1} = \{A_{j1} \cup A_{j2}, \cdots, \cup A_{jk}\}$. According to the definition of $G^{\mathscr{P}}_M(\Omega)$ and monotonicity of $m^{\mathscr{P}}$, we have:
$G^{\mathscr{P}}_M(\mathcal{S}^{\mathscr{P}}_{j}) = \sum_{i=1}^{k} m^{\mathscr{P}}(A_{ji}.p^{\mathscr{P}}(A_{ji}))$
$= m^{\mathscr{P}}(A_{j1}).p^{\mathscr{P}}(A_{j1}) + m^{\mathscr{P}}(A_{j2}).p^{\mathscr{P}}(A_{j2}) + \sum_{i=3}^{k} m^{\mathscr{P}}(A_{ji}.p^{\mathscr{P}}(A_{ji}))$
$< m^{\mathscr{P}}(A_{j1} \cup A_{j2}).p^{\mathscr{P}}(A_{j1}) + m^{\mathscr{P}}(A_{j2} \cup A_{j1}).p^{\mathscr{P}}(A_{j2}) + \sum_{i=3}^{k} m^{\mathscr{P}}(A_{ji}.p^{\mathscr{P}}(A_{ji}))$
$= m^{\mathscr{P}}(A_{j1} \cup A_{j2}).p^{\mathscr{P}}(A_{j1}) + p^{\mathscr{P}}(A_{j1}) + \sum_{i=3}^{k} m^{\mathscr{P}}(A_{ji}.p^{\mathscr{P}}(A_{ji}))$
$= m^{\mathscr{P}}(A_{j1} \cup A_{j2}).p^{\mathscr{P}}(A_{j1} \cup A_{j1}) + \sum_{i=3}^{k} m^{\mathscr{P}}(A_{ji}.p^{\mathscr{P}}(A_{ji}))$
$= G_M(\mathcal{S}^{\mathscr{P}}_{j+1})$.
Then $\Gamma_1 <^{\Gamma} \Gamma_2$ holds.
(iii) Assume that $\Gamma_1 = \Gamma_2$, $\mathcal{S}^{\mathscr{P}}_{1M} \subset \mathcal{S}^{\mathscr{P}}_{2M}$ holds. And from Definition 32, $G(\Gamma_1) = G(\Gamma_2)$ holds.

**Proposition 11** *Let* $(\Omega, \Gamma)$ *be a MGTRS. The maximum topological granularity measure of* $\mathscr{T}$ *with respect to* $\Omega$ *is one. This value is achieved if and only if* $m^{\mathscr{P}} = 1$, $max(G^{\mathscr{P}}_M(\Gamma)) = |\Omega| - 1$.

**Proposition 12** *Let $(\Omega, \Gamma)$ be a MGTRS. The minimum topological granularity of s with respect to $\Omega$ is one. This value is achieved if and only if $m^{\mathscr{P}} = |\Omega|$, $min(G_M^{\mathscr{P}}(\Gamma)) = 1 - \frac{1}{|\Omega|}$.*

Thus, $1 - \frac{1}{|\Omega|} \leq G(\Gamma_M^{\mathscr{P}}) \leq |\Omega| - 1$.

**Theorem 30** *Let $\Gamma_1, \Gamma_2$ be two MGTRS. If $\Gamma_1 <^{\mathscr{T}} \Gamma_2$, then $G_M^{\mathscr{P}}(\Gamma_1) < G_M^{\mathscr{P}}(\Gamma_2)$.*

**Definition 35** Let $(\Omega, \Gamma)$ be a MGTRS and $\mathcal{S}_M^{\mathscr{P}} = \{A_1, A_2, \cdots, A_n\}$ is a topology subbase of $\Gamma$. Then the topological entropy of $\Gamma$ is defined as:

$$E_M^{\mathscr{P}}(\Gamma) = 1 - \frac{1}{|\Omega|} \sum_{i=1}^{q} m^{\mathscr{P}}(A_i).p^{\mathscr{P}}(A_i),$$

where $p^{\mathscr{P}}(A_i) = \frac{|A_i|}{|\Omega|}$.

**Definition 36** Let $(\Omega, \Gamma)$ be a MGTRS. Then $\mathscr{T}_k$ is significant in $\Gamma$, if $E_\Gamma^{\mathscr{P}}(\Omega, \sqcap_{i=1}^q \mathscr{T}_i) \neq E_\Gamma^{\mathscr{P}}(\Omega, \sqcap_{i=1, i\neq k}^q \mathscr{T}_i)$. So, $\mathscr{T}_k$ is not significant in $\Gamma$ if $E_\Gamma^{\mathscr{P}}(\Omega, \sqcap_{i=1}^q \mathscr{T}_i) = E_\Gamma^{\mathscr{P}}(\Omega, \sqcap_{i=1, i\neq k}^q \mathscr{T}_i)$.

**Definition 37** Let $(\Omega, \Gamma)$ be a MGTRS. The significance measure of $\mathscr{T}_k$ in $\Gamma$ is defined as

$$S_\Gamma^{\mathscr{P}}(\mathscr{T}_k) = \frac{E_\Gamma^{\mathscr{P}}(\Omega, \sqcap_{i=1}^q \mathscr{T}_i)}{E_\Gamma^{\mathscr{P}}(\Omega, \sqcap_{i=1, i\neq k}^q \mathscr{T}_i)}.$$

**Definition 38** Let $\Gamma = \{\mathscr{T}_1, \mathscr{T}_2, \cdots, \mathscr{T}_q\}$ be $q$ topological spaces induced by any binary relations $R_1, R_2, \cdots, R_n$, respectively on $\Omega$. If there exists a subset $\Gamma_i = \{\mathscr{T}_{i1}, \mathscr{T}_{i2}, \cdots, \mathscr{T}_{iq}\} \subseteq \Gamma$, such that $E_\Gamma^{\mathscr{P}}(\Omega, \sqcap_{i=1}^q \mathscr{T}_i) = E_\Gamma^{\mathscr{P}}(\Omega, \sqcap_{i=k}^q \mathscr{T}_{ik})$, but $E_\Gamma^{\mathscr{P}}(\Omega, \mathscr{T}_{i1} \sqcap \mathscr{T}_{i2} \sqcap \cdots \sqcap \mathscr{T}_{ik} \sqcap \mathscr{T}_{i(k+1)}) \neq E_\Gamma^{\mathscr{P}}(\Omega, \mathscr{T}_1 \sqcap \mathscr{T}_2 \sqcap \cdots \sqcap \mathscr{T}_q)$, then we call $\Gamma_i$ is a granularity reduct of $\Gamma$.

Now, we establish Algorithm 3 to produce the intersection of topologies and Algorithm 4 to make reduction for topologies.

**Algorithm 3.** An algorithm for $\sqcap_{i=1}^n \mathscr{T}_i$

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**Input**: $x \in \Omega$ and Binary Relations $R_i$.
**Output**: $\sqcap_{i=1}^n \mathscr{T}_i$.
**for** $(i = 1; i <= |R_i|; i++)$
  **for** $(k = 1; k <= |R_k|; k++)$
    $f_i(x) = (\mathscr{P}_j(x))_{R_k}$.
    $\mathcal{F}_{\mathscr{P}\cap}^i(x) = \bigcap_{k=1}^n f_i(x)$.
  **endfor**
**endfor**
**for** $(i = 1; i <= |R_i|; i++)$
    $\sqcap_{i=1} \mathscr{T}_i = \bigcup(\bigcap \mathcal{F}_{\mathscr{P}\cap}^i(x))$.
**endfor**
**end**

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**Algorithm 4.** An algorithm for a reduction

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**Input**: $\Omega$, $\Omega_R$ and Multi-source information decision table.
**Output**: make a reduction ( $reduct$ is a set which conserve the selected granularities).
$reduct = \phi$
**for** $(i = 1; i <= n; i++)$
    $m^{\mathscr{P}}(\mathscr{P}_j(A_i)) = \frac{|\Omega|}{k}(1 - \frac{1}{k.|\mathscr{P}_j(A_i)|})$, where $k = |\Omega_R|$
**endfor**
**for** $(i = 1; i <= n; i++)$
    $p^{\mathscr{P}}(A_i) = \frac{|A_i|}{|\Omega|}$
**endfor**
**for** $(i = 1; i <= n; i++)$
    $E_M^{\mathscr{P}}(\Gamma) = 1 - \frac{1}{|\Omega|} \sum_{i=1}^{q} m^{\mathscr{P}}(A_i).p^{\mathscr{P}}(A_i)$
**endfor**
**for** $(i = 1; i <= |R_i|; i++)$

$S_{\Gamma}^{\mathscr{P}}(\mathscr{T}_k) = \frac{E_{\Gamma}^{\mathscr{P}}(\Omega, \sqcap_{i=1}^{q}\mathscr{T}_i)}{E_{\Gamma}^{\mathscr{P}}(\Omega, \sqcap_{i=1, i\neq k}^{q}\mathscr{T}_i)}$, where $k \leq |R_i|$
$reduct = \mathscr{T}_k$
**if**$(E_{\Gamma}^{\mathscr{P}}(\Omega, reduct) = E_{\Gamma}^{\mathscr{P}}(\Omega, \sqcap_{i=1}^{q}\mathscr{T}_i))$
**goto:** end
**endif**
**endfor**
**end**

## Conclusion

To extend the application domain of MGRS, the paper introduced two kinds of Multi-granulation rough set models. These models depend on a special types of a NS and a $j$-adhesion NS. Some basic properties of these models will be studied. We show that new MGRS models are generalized versions of MGRS models from the topological view. The introduced techniques are very useful in application because it opens the way for more topological applications from real life problems.

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