FREE BOUNDARY PROBLEM FOR ONE-DIMENSIONAL COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH TEMPERATURE DEPENDENT VISCOSITY AND HEAT CONDUCTIVITY

TUOWEI CHEN AND YONGQIAN ZHANG

ABSTRACT. We prove the existence and uniqueness of global strong solution to the free boundary problem in one dimensional compressible Navier-Stokes system for the viscous and heat conducting ideal polytropic gas flow, when the viscosity and heat conductivity depend on temperature in power law of Chapman-Enskog and the data is in the neighborhood of some background solution at initial time. We also study the large time behavior of the solution and obtain its decay property.

1. INTRODUCTION

In this paper, we consider the free boundary value problem of one dimension compressible heat-conducting Navier-Stokes equations as follows:

$$\begin{cases} \rho_\tau + (\rho u)_y = 0, & (1.1)\\ (\rho u)_\tau + (\rho u^2 + p)_y = (\mu u_y)_y, & (1.2)\\ (\rho(e + \frac{1}{2}u^2))_\tau + (\rho u(e + \frac{1}{2}u^2) + pu)_y = (\mu u u_y)_y + (\kappa\theta_y)_y, & (1.3) \end{cases}$$

for $\tau > 0$ and $a_1(\tau) < y < a_2(\tau)$, with the boundary condition

$$e_y(\tau, d) = 0,\ (\mu u_y - p)(\tau, d) = 0,\ \tau \geq 0,\ d = a_1(\tau), a_2(\tau). \tag{1.4}$$

Here the unknown functions ρ, u and θ represent the density, the fluid velocity and the temperature, respectively. e is the internal energy, and p is the pressure. μ and κ are the viscosity coefficient and the heat conductivity coefficient. In this paper, we focus on ideal polytropic gas and the constitution relation reads

$$p(\rho, \theta) = R\rho\theta,\ e = c_v\theta,\ c_v = \frac{R}{\gamma - 1}, \tag{1.5}$$

where the specific gas constant R and the heat capacity c_v are positive constants, and $\gamma > 1$ is the adiabatic constant.

The free boundaries $y = a_1(\tau)$ and $y = a_2(\tau)$ are the interfaces separating the gas from the vacuum, which are described by

$$\begin{cases} \frac{da_i(\tau)}{d\tau} = u(a_i(\tau), \tau) \quad \tau > 0,\\ a_i(0) = a_i, \end{cases} \tag{1.6}$$

where a_i are some constants and $i = 1, 2$.

Date: February 19, 2020.

2010 *Mathematics Subject Classification.* 35Q30; 76N10; 76N15; 35B40; 35D35; 35L65;

Key words and phrases. Compressible Navier-Stokes equations, global strong solution, free boundary, temperature dependent transport coefficients, Chapman-Enskog law, asymptotic solution.

To get the fixed boundary valued problem, we will rewrite the problems (1.1)–(1.4) by using Lagrangian mass coordinates. This means, set

$$x = \int_{a_1(\tau)}^{y} \rho(\tau, z)dz, \quad t = \tau, \tag{1.7}$$

and then the free boundaries $y = a_1(\tau)$ and $y = a_2(\tau)$ become $x = 0$ and $x = \int_{a_1(\tau)}^{a_2(\tau)} \rho(z, \tau)dz = \int_{a_1}^{a_2} \rho_0(y)dy$ by the conservation of mass. Without loss of generality, we assume $\int_{a_1}^{a_2} \rho_0(z)dz = 1$ and $R = 1$. Let $v = \frac{1}{\rho}$ be the specific volume, then $p = \frac{\theta}{v}$. Hence, the problem (1.1)–(1.4) is transformed into the system:

$$\begin{cases} v_t - u_x = 0, & (1.8) \\ u_t + (\dfrac{\theta}{v})_x = (\dfrac{\mu u_x}{v})_x, & (1.9) \\ \dfrac{\theta_t}{\gamma - 1} + \dfrac{\theta}{v}u_x = (\dfrac{\kappa\theta_x}{v})_x + \dfrac{\mu u_x^2}{v}, & (1.10) \end{cases}$$

for $(t, x) \in (0, \infty) \times (0, 1)$, with the boundary condition

$$\theta_x(t, d) = 0, \ (\mu u_x - \theta)(t, d) = 0, \ t \geq 0; \ d = 0, 1. \tag{1.11}$$

And we impose the following initial conditions

$$(v, u, \theta)(0, x) = (v_0, u_0, \theta_0)(x), \ \text{ for } \ x \ \in \ [0, 1]. \tag{1.12}$$

In a view of physics, when one derives the full compressible Navier-Stokes equations from the Boltzmann equation by using the Chapman-Enskog expansion, viscosity μ and heat conductivity κ are functions of temperature. According to [1], the following relations hold:

$$\mu = \bar{\mu}\theta^b, \ \kappa = \bar{\kappa}\theta^b, \qquad b \in (\frac{1}{2}, \infty), \tag{1.13}$$

where $\bar{\mu}$ and $\bar{\kappa}$ are two positive constants, and such relations lead to possible degeneracy and strong nonlinearity in viscosity and heat diffusion.

Many known mathematical results mainly focus on the case when viscosity μ and heat conductivity κ are positive constants. For examples, in the one space dimension case, for large initial data away from vacuum, Kazhikhov et al. [8, 9] obtained global smooth solutions by employing the representation for specific volume v, and recently, Jing Li and Zhilei Liang [10] showed the large-time behavior of solutions with large data. There are also many results for the case when viscosity μ depends on density and heat conductivity κ depends on density and temperature, both of which are non-degenerate, and we refer readers to [5, 17, 20] and the references therein. For isentropic flow, the dependence on the temperature in (1.13) is translated into the dependence of viscosity on the density,

$$\mu(\rho) = A\rho^a, \quad A > 0, \ a > 0. \tag{1.14}$$

It is shown by Liu, Xin and Yang in [13] that, at least locally in time, the Navier-Stokes equations for one-dimensional isentropic viscous gas with a jump to the vacuum initially and with condition (1.14), has a physically relevant solution. We also refer readers to [6, 18, 22, 23] for some global existence results of isentropic case.

However, there is few result for the non-isentropic case with relation (1.13). To our knowledge, for the Cauchy problem of one-dimensional compressible Navier-Stokes equations for the viscous and heat conducting ideal polytropic gas flow with degenerate temperature dependent transport coefficients, Liu and Yang et al. in [14] obtained the global non-vacuum classical solutions with smallness assumption that γ is close to 1, which is the first result with large data for the case with relation (1.13). Later, under assumption

$$\mu = \tilde{\mu}h(v)\theta^\alpha, \ \kappa = \tilde{\kappa}h(v)\theta^\alpha, \tag{1.15}$$

where $\tilde{\mu}$, $\tilde{\kappa}$ are positive constants and $h(v)$ is some non-degenerate smooth function, Wang and Zhao in [21] obtained global non-vacuum classical solutions with smallness assumptions for $\mid \alpha \mid$. For the initial-boundary value problem, Jenssen and Karper in [4] proved the global existence of a weak solution under the assumption

$$\mu = \bar{\mu},\ \kappa = \bar{\kappa}\theta^b, \quad b \in [0, \frac{3}{2}), \tag{1.16}$$

where $\bar{\mu}$ and $\bar{\kappa}$ are positive constant. We also refer readers to [16], in which Pan and Zhang proved the existence of global strong solutions for the initial-boundary value problem under assumptions

$$\mu = \bar{\mu},\ \kappa = \bar{\kappa}\theta^b, \quad b \in [0, \infty), \tag{1.17}$$

where $\bar{\mu}$ and $\bar{\kappa}$ are positive constants. Huang and Shi in [3] improved Pan and Zhang's result [16] on the regularity conditions of initial data. For the case with presence of vacuum, even in one dimension, the global well-posedness theory for the full compressible Navier-Stokes equations with degenerate temperature dependent transport coefficient is still an open problem. As a view of physics, it is reasonable to consider the free boundary problem. Recently, some progress in this direction have been obtained when considering some simplified relations. Li and Guo in [11] established the global existence of strong and classical solutions to free boundary problem under assumptions

$$\mu = \bar{\mu},\ \kappa = \bar{\kappa}\theta^b, \quad b \in (0, \infty), \tag{1.18}$$

where $\bar{\mu}$ and $\bar{\kappa}$ are positive constant. When considering the fluid is in the low Mach number regime, Li, Ma and Qu in [12] established the global existence and uniqueness of strong solutions for the free boundary problem under assumptions

$$\mu = \bar{\mu}(1 + \rho^\beta),\ \kappa = \bar{\kappa}\theta^q, \quad \beta \in [0, \infty),\ q \in (0, \infty), \tag{1.19}$$

where $\bar{\mu}$ and $\bar{\kappa}$ are positive constants. We mention that for the free boundary problem based on physics consideration, although some results (such as [4, 11, 12, 16]) have been obtained under some simplified relations, it is still far away from a theory for (1.1)–(1.4) under condition (1.13).

As for the free boundary problem with constant viscosity and heat conductivity, Okada in [15] studied the asymptotic behavior of the solutions in the neighborhood of a given background solution of the free boundary value problem of one dimensional model system associated with compressible viscous and heat-conducting fluid, and he obtained the decay property of those solutions under smallness conditions of the initial data and $\gamma - 1$.

In this paper, inspired by [15], we will define some special solution to (1.8)–(1.11) as a background solution and show the global existence and uniqueness of strong solution in the neighborhood of such background solution under the following assumption

$$\mu = \bar{\mu}\theta^b,\ \kappa = \bar{\kappa}\theta^a, \quad a, b \in (\frac{1}{2}, \infty), \tag{1.20}$$

where $\bar{\mu}$ and $\bar{\kappa}$ are positive constants, and the case with physical relation (1.13) is included.

Before we state the main theorem, we introduce some simplified notations as follows. Let $I \triangleq [0, 1]$ be the domain of space. For $p \geq 1$, $L^p = L^p(I)$ denotes the L^p space with norm $| \cdot |_{L^p}$. For $p \geq 1$ and $p \geq 1$, $W^{k,p} = W^{k,p}(I)$ denotes the Sobolev space with norm $| \cdot |_{W^{k,p}}$, and particularly, $H^k = W^{k,2}(I)$. Moreover, for $f \in L^\infty(0, t; H^1)$, $\mid f \mid_{0,T} \triangleq \sup_{t \in [0,T]} \mid f(t) \mid_{L^\infty}$. And without loss of generality, we assume that $\bar{\mu} = \bar{\kappa} = 1$ throughout this paper.

Set

$$(V(t), U(x), \bar{\theta}) = (\frac{\bar{\theta}}{\bar{\theta}^b} t + v_1, \frac{\bar{\theta}}{\bar{\theta}^b} x + u_1, \bar{\theta}), \tag{1.21}$$

for some constants $\bar\theta > 0$, $v_1 > 0$ and u_1. Then it is easy to verify that $(V(t), U(x), \bar\theta)$ is a solution to (1.8)–(1.11). We consider the solution in the neighborhood of $(V(t), U(x), \bar\theta)$, and we are also concerned with its decay property. Now, we shall state the main result as follows.

Theorem 1.1. *Let $(V(t), U(x), \bar\theta)$ be given as in (1.21) and assume that the condition (1.20) holds and that the initial data $(v_0, u_0, \theta_0)(x)$ satisfies*

$$(v_0, u_0, \theta_0)(x) \in H^1 \times H^2 \times H^2, \tag{1.22}$$

$$(\theta_0^b u_{0,x} - \theta_0)(d) = 0,\ \theta_{0,x}(d) = 0, \ \text{ for } d = 0, 1, \tag{1.23}$$

$$\int_I u_0(x)dx = \int_I U(x)dx, \tag{1.24}$$

$$\text{and} \quad \int_I (\theta_0(x) + \frac{1}{2}u_0(x)^2)dx = \int_I (\bar\theta + \frac{1}{2}U(x)^2)dx. \tag{1.25}$$

There exists a positive constant θ_* only depending on a, b, γ and u_1, such that if $\bar\theta > \theta_*$, then the following statement holds. There exist positive constants ε_1 and α_1, both of which are only depending on a, b, γ, u_1 and $\bar\theta$, such that if $I_0 \triangleq \mid v_0/v_1 - 1,\ u_0 - U,\ \theta_0/\bar\theta - 1 \mid_{H^1} \leq \varepsilon$ for $\varepsilon \in (0, \varepsilon_1)$, then there exists a global unique strong solution (v, u, θ) to (1.8)–(1.12), satisfying

$$v, u, \theta \in C([0, T]; H^1 \times H^2 \times H^2),\ \ \theta^b u_x - \theta, \theta_x \in L^2(0, T; H_0^1 \cap H^2),\ \ \forall T > 0. \tag{1.26}$$

Moreover, for $\alpha \in [0, \alpha_1]$, we have

$$V(t)^\alpha \mid \frac{v(t)}{V(t)} - 1, u(t) - U, \frac{\theta(t)}{\bar\theta} - 1 \mid_{H^1}^2 \leq v_1^\alpha I_0^2 \quad \text{for } \ t \in [0, \infty), \tag{1.27}$$

$$\textit{and} \quad \mid \frac{v(t)}{V(t)} - 1, u(t) - U, \frac{\theta(t)}{\bar\theta} - 1 \mid_{L^\infty} = O(t^{-\frac{\alpha}{2}}) \quad \text{as } \ t \to \infty. \tag{1.28}$$

Remark 1.1. *In the frame of Lagrangian mass coordinates, the condition (1.24) and (1.25) imply that the initial momentum and energy are equal to the momentum and energy of the background solution $(V(t), U(x), \bar\theta)$, respectively. And by classical embedding theorem, the condition (1.22) and (1.23) imply that the initial data is compatible with the boundary condition. More precisely, if (v_0, u_0, θ_0) are smooth functions on I, they satisfy the boundary condition (1.11). Similarly, (1.26) implies that the solution (v, u, θ) is compatible with the boundary condition (1.11).*

Remark 1.2. *In fact, we will choose ε_1 suitably small to ensure that the $\theta_0(x)$ has positive lower bound and there is no vacuum at initial time, and then (1.27) implies that vacuum will never be developed in our case.*

Remark 1.3. *We remark that our method in this paper also can be applied to the cases that equations (1.8)–(1.10) with the following two different free boundary conditions:*

$$u(t, 0) = 0,\ (\mu u_x - \theta)(t, 1) = 0,\ \theta_x(t, d) = 0,\ t \geq 0,\ d = 0, 1, \tag{1.29}$$

or

$$u(t, 0) = 0,\ (\mu u_x - \theta)(t, 1) = 0,\ \theta(t, 0) = \bar\theta,\ \theta_x(t, 1) = 0,\ t \geq 0, \tag{1.30}$$

where $\bar\theta$ is some positive constant. For the free boundary problem with small data in both above cases, in a similar way, we can obtain global existence and uniqueness results under some analogous assumptions on the background solution as in Theorem 1.1.

The remaining of this paper is organized as follows. In Section 2, we present some useful lemmas and show the local existence of solutions to (1.8)–(1.12). In Section 3, we will use energy methods to derive a priori H^1-estimate and then prove the main theorem.

2. Preliminary

In this section, we shall state the following lemmas, which are useful to get the necessary energy estimates.

Lemma 2.1. (Poincaré's inequality). *If $f \in H^1(I)$, the following inequality is satisfied.*

$$| f |_{L^2}^2 \leq 2 | f_x |_{L^2}^2 + 2 | \int_I f dx |^2 . \tag{2.1}$$

Lemma 2.2. (Sobolev's inequality). *Let $f \in H^1(I)$. Then $f \in C^0(I)$ and we can obtain the following estimate*

$$| f |_{L^\infty}^2 \leq 2 | f |_{L^2} | f_x |_{L^2} + | f |_{L^2}^2 . \tag{2.2}$$

Moreover, when there is some $x_0 \in I$ such that $f(x_0) = 0$, we can establish

$$| f |_{L^\infty}^2 \leq 2 | f |_{L^2} | f_x |_{L^2} . \tag{2.3}$$

As for the details of the above two lemmas, we refer readers to [2].

The existence and uniqueness of local in time solution to (1.8)–(1.11) can be obtained by a standard Banach fixed point argument, c.f. [7, 19]. Thus, we state the following lemma for local solution and omit the detailed proof for it.

Lemma 2.3. *Suppose that* (1.20) *holds, and if the initial data (v_0, u_0, θ_0) satisfies* (1.22),(1.23) *and*

$$\inf_{x\in I} v_0(x) > 0, \ \inf_{x\in I} \theta_0(x) > 0, \tag{2.4}$$

then there exists a positive constant T_1 only depending on $| (v_0, u_0, \theta_0) |_{H^1}$, $\inf_{x\in I} v_0(x)$ and $\inf_{x\in I} \theta_0(x)$, such that there exists a unique local strong solution (v, u, θ) to (1.8)–(1.12) *on $[0, T_1] \times I$, satisfying*

$$\begin{aligned}
&(v, u, \theta) \in C([0, T_1]; H^1 \times H^2 \times H^2),\\
&\theta u_x - \theta^b \in L^2(0, T_1; H_0^1), \ \theta_x \in L^2(0, T_1; H_0^1),\\
&C^{-1} \leq v(t, x) \leq C, \ C^{-1} \leq \theta(t, x) \leq C,\\
&| (v, u, \theta)(t, \cdot) |_{H^1}^2 + \int_0^t | (v, u, \theta)(s, \cdot) |_{H^1}^2 + | (u_{xx}, \theta_{xx})(\cdot, s) |_{L^2}^2 \, ds \leq C,
\end{aligned} \tag{2.5}$$

for any $(t, x) \in [0, T_1] \times I$, where $C > 0$ is some constant only depending on $| (v_0, u_0, \theta_0) |_{H^1}$, $\inf_{x\in I} v_0(x)$, $\inf_{x\in I} \theta_0(x)$ and T_1.

3. A Priori Estimate and Global Existence

In order to prove our main theorem, we will rewrite our problem as follows. Set

$$q(t, x) = \frac{v(t, x)}{V(t)} - 1, \ r(t, x) = u(t, x) - U(x), \ h(t, x) = \frac{\theta(t, x)}{\bar{\theta}} - 1. \tag{3.1}$$

By using the changes of variables in (3.1), the problem (1.8)–(1.12) is reduced to the following system:

$$
\begin{cases}
q_t + \dfrac{\bar\theta}{\bar\theta^b}\dfrac{q}{V} - \dfrac{r_x}{V} = 0, & (3.2)\\
r_t - \dfrac{1}{V}[\dfrac{\bar\theta^b(h+1)^b r_x}{1+q}]_x + \dfrac{1}{V}[\dfrac{\bar\theta h}{1+q}]_x - \dfrac{\bar\theta}{V}[\dfrac{(h+1)^b-1}{1+q}]_x = 0, & (3.3)\\
\dfrac{\bar\theta h_t}{\gamma-1} - \dfrac{1}{V}[\dfrac{\bar\theta^{a+1}(h+1)^a h_x}{1+q}]_x + \dfrac{\bar\theta^2 h}{\bar\theta^b V(1+q)} & \\
= \dfrac{\bar\theta(h+1)^b r_x^2}{V(1+q)} + \dfrac{\bar\theta[2(h+1)^b-1]r_x}{V(1+q)} - \dfrac{\bar\theta h r_x}{V(1+q)} + \dfrac{\bar\theta^2[(h+1)^b-1]}{\bar\theta^b V(1+q)}, & (3.4)
\end{cases}
$$

for $(t,x)\in(0,\infty)\times(0,1)$, with boundary condition

$$
\begin{cases}
\big(\bar\theta^b(h+1)^b r_x - \bar\theta h + \bar\theta[(h+1)^b-1]\big)(t,d) = 0,\\
h_x(t,d) = 0, \quad \text{for } \ t\geq 0, \ d=0,1,
\end{cases} \tag{3.5}
$$

and initial condition

$$
(q,r,h)(0,x) = (q_0,r_0,h_0)(x), \ \text{ for } \ x\in I, \tag{3.6}
$$

where $q_0 = v_0/v_1 - 1$, $r_0 = u_0 - U$, $h_0 = \theta_0/\bar\theta - 1$.

3.1. **A Priori Estimate.**

To prove theorem 1.1, it suffices to derive the following priori estimate.

Proposition 3.1. *Let any $T>0$ be fixed. Assume that the same conditions as in Theorem 1.1 hold. Let $(q,r,h)\in C([0,T];H^1)$ with $(r_{xx},h_{xx})\in L^2(0,T;L^2)$ be the unique strong solution to (3.2)–(3.6) on $[0,T]\times I$. There exists a positive constant θ_* only depending on a, b, γ and u_1, such that if $\bar\theta\geq\theta_*$, then there exist positive constants $M\leq\frac{1}{2}$ and α_1, both of which are only depending on a, b, γ, u_1 and $\bar\theta$, but independent of T, such that if $I_0\leq M\leq\frac{1}{2}$ and if*

$$
\mid q,r,h\mid_{0,T}\leq M\leq\frac{1}{2}, \tag{3.7}
$$

then

$$
V^\alpha(t)\mid (q,r,h)(t)\mid_{H^1}^2\leq v_1^\alpha I_0^2, \tag{3.8}
$$

$$
\int_0^t \mid (r_{xx},h_{xx})(s)\mid_{L^2}^2\,ds\leq I_0^2, \tag{3.9}
$$

for any $\alpha\in[0,\alpha_1]$ and $t\in[0,T]$.

To prove the above proposition, We begin with some elementary observation. From the hypothesis (3.7), we could easily obtain that

$$
\begin{aligned}
&\mid q(t,x)\mid\leq\frac{1}{2}, \ \mid r(t,x)\mid\leq\frac{1}{2}, \ \mid h(t,x)\mid\leq\frac{1}{2},\\
&\frac{1}{2}\leq h(t,x)+1\leq 2, \ \frac{1}{2}\leq q(t,x)+1\leq 2 \ \text{ for } \ \forall \ (t,x)\in[0,T]\times I.
\end{aligned} \tag{3.10}
$$

Then, as in [15], we will deduce two important inequalities based on the conservation of momentum and energy. Integrating the momentum equation (1.9) over $[0,t]\times I$, and using the condition (1.24), we have

$$
\int_I u(t,x)dx = \int_I u_0(x)dx = \int_I U(x)dx. \tag{3.11}
$$

And (3.11) yields that

$$\int_I r(t,x)dx = \int_I r_0(x)dx = 0. \tag{3.12}$$

Thus, by using Poincaré's inequality (2.1), we deduce from (3.12) that

$$| r(t) |_{L^2}^2 \leq | r_x(t) |_{L^2}^2 . \tag{3.13}$$

Integrating the energy equation (1.10) over $[0,t]\times I$, and using the condition (1.25), we have

$$\int_I \frac{\bar\theta h(t,x)}{\gamma-1} + \frac{1}{2}r(t,x)^2 + (\frac{\bar\theta}{\bar\theta^b}x + u_1)r(t,x)dx = 0. \tag{3.14}$$

Thus, by using Hölder inequality and (3.10), it follows from (3.14) that

$$\begin{aligned} | \int_I h(t,x)dx |^2 &= | \frac{\gamma-1}{\bar\theta}\int_I \frac{1}{2}r(t,x)^2 + (\frac{\bar\theta}{\bar\theta^b}x + u_1)r(t,x)dx |^2 \\ &\leq \frac{(\gamma-1)^2}{2\bar\theta^2} | r(t) |_{L^2}^2 + 2\int_I | \frac{\bar\theta}{\bar\theta^b}x + u_1 |^2 dx \, | r(t) |_{L^2}^2 \\ &\leq (\frac{4\bar\theta^2}{4\bar\theta^{2b}} + 4u_1^2 + \frac{1}{2}) | r(t) |_{L^2}^2 . \end{aligned} \tag{3.15}$$

Using Poincaré's inequality (2.1) again, we deduce from (3.15) and (3.13) that

$$\begin{aligned} | h(t) |_{L^2}^2 &\leq 2 | \int_I h(t,x)dx |^2 + 2 | h_x(t) |_{L^2}^2 \\ &\leq 2 | h_x(t) |_{L^2}^2 + (\gamma-1)^2[\frac{(8u_1^2+1)}{\bar\theta^2} + \frac{4}{3\bar\theta^{2b}}] | r_x(t) |_{L^2}^2 . \end{aligned} \tag{3.16}$$

Now, we are going to employ energy method to prove Proposition 3.1 with the help of (3.13), (3.16). And in what follows, we denote by C a general constant only depending on a, b, γ, u_1 and $\bar\theta$, which may be different between line to line.

Proof of Proposition 3.1. First, for $t \in [0,T]$, multiplying (3.3) and (3.4) by $\bar\theta^b V^\alpha r$ and $\bar\theta^b V^\alpha h$, respectively, summing them up, integrating over I, and using the boundary condition (3.5), we have

$$\begin{aligned} &\{\int_I \frac{1}{2}\bar\theta^b V^\alpha r^2 + \frac{1}{2(\gamma-1)}V^\alpha h^2 dx\}_t + V^{\alpha-1}\int_I \frac{\bar\theta^2 h^2}{1+q}dx \\ &+ V^{\alpha-1}\int_I \frac{\bar\theta^{2b}(h+1)^b r_x^2}{1+q}dx + V^{\alpha-1}\int_I \frac{\bar\theta^{a+b+1}(h+1)^a h_x^2}{1+q}dx \\ \leq& \frac{\alpha\bar\theta V^{\alpha-1}}{2}\int_I r^2 dx + \frac{\alpha\bar\theta^2 V^{\alpha-1}}{2(\gamma-1)}\int_I h^2 dx + V^{\alpha-1}\int_I \frac{\bar\theta^{b+1}[(h+1)^b-1]r_x}{1+q}dx \\ &+ V^{\alpha-1}\int_I \frac{\bar\theta^{b+1}[2(h+1)^b]hr_x}{1+q}dx + V^{\alpha-1}\int_I \frac{\bar\theta^{b+1}(h+1)^b h r_x^2}{1+q}dx \\ &- V^{\alpha-1}\int_I \frac{\bar\theta^{b+1}h^2 r_x}{1+q}dx + V^{\alpha-1}\int_I \frac{\bar\theta^2[(h+1)^b-1]h}{1+q}dx \\ \equiv& \sum_{i=1}^{7} I_i. \end{aligned} \tag{3.17}$$

Among the terms on the right hand side of (3.17), due to the hypothesis (3.7), we use (3.13), (3.16) and Schwarz's inequality to get the estimations on I_1, I_2, I_5, I_6 and I_7 as follows,

$$I_1=\frac{\alpha\bar\theta V^{\alpha-1}}{2}\int_I r^2dx\le \alpha CV^{\alpha-1}\int_I r_x^2dx, \tag{3.18}$$
$$I_2=\frac{\alpha\bar\theta^2 V^{\alpha-1}}{2(\gamma-1)}\int_I h^2dx\le \alpha CV^{\alpha-1}(\int_I r_x^2dx+\int_I h_x^2dx), \tag{3.19}$$
$$I_5=V^{\alpha-1}\int_I\frac{\bar\theta^{b+1}(h+1)^bhr_x^2}{1+q}dx\le MCV^{\alpha-1}\int_I r_x^2dx, \tag{3.20}$$
$$I_6=-V^{\alpha-1}\int_I\frac{\bar\theta^{b+1}h^2r_x}{1+q}dx\le MCV^{\alpha-1}(\int_I r_x^2dx+\int_I h_x^2dx), \tag{3.21}$$
$$\begin{aligned}\text{and }\ I_7=V^{\alpha-1}\int_I\frac{\bar\theta^2[(h+1)^b-1]h}{1+q}dx&\le 2^{|b-1|+1}b\bar\theta^2V^{\alpha-1}\int_I h^2dx\\ &\le 2^{|b-1|+2}b\bar\theta^2V^{\alpha-1}\mid h_x\mid_{L^2}^2+2^{|b-1|+1}b(\gamma-1)^2[(8u_1^2+1)+\frac{4\bar\theta^2}{3\bar\theta^{2b}}]V^{\alpha-1}\int_I r_x^2dx,\end{aligned} \tag{3.22}$$

where we we have already used the following inequality

$$\mid(h+1)^b-1\mid\le\mid\int_1^{h+1}bs^{b-1}ds\mid\le 2^{|b-1|}b\mid h\mid. \tag{3.23}$$

In addition, due to (3.7), we use (3.13), (3.16) and Young's inequality to estimate I_3 and I_4 as follows,

$$\begin{aligned}I_3=&V^{\alpha-1}\int_I\frac{\bar\theta^{b+1}[(h+1)^b-1]r_x}{1+q}dx\le(\frac12)^{b+3}V^{\alpha-1}\bar\theta^{2b}\int_I r_x^2dx+2^{2|b-1|+b+3}b^2V^{\alpha-1}\bar\theta^2\int_I h^2dx\\ \le&(\frac12)^{b+3}V^{\alpha-1}\bar\theta^{2b}\int_I r_x^2dx+2^{2|b-1|+b+4}b^2V^{\alpha-1}\bar\theta^2\int_I h_x^2dx\\ &+2^{2|b-1|+b+3}b^2(\gamma-1)^2[(8u_1^2+1)+\frac{4\bar\theta^2}{3\bar\theta^{2b}}]V^{\alpha-1}\int_I r_x^2dx,\end{aligned} \tag{3.24}$$
$$\begin{aligned}\text{and }\ I_4=V^{\alpha-1}\int_I\frac{\bar\theta^{b+1}[2(h+1)^b]hr_x}{1+q}dx&\le(\frac12)^{b+3}V^{\alpha-1}\bar\theta^{2b}\int_I r_x^2dx+2^{3b+3}\bar\theta^2V^{\alpha-1}\int_I h^2dx\\ &\le(\frac12)^{b+3}V^{\alpha-1}\bar\theta^{2b}\int_I r_x^2dx+2^{3b+4}\bar\theta^2V^{\alpha-1}\int_I h_x^2dx\\ &+2^{3b+3}(\gamma-1)^2[(8u_1^2+1)+\frac{4\bar\theta^2}{3\bar\theta^{2b}}]V^{\alpha-1}\int_I r_x^2dx.\end{aligned} \tag{3.25}$$

Substituting (3.18)–(3.22) and (3.24)–(3.25) into (3.17) and using (3.10), we obtain

$$\begin{aligned}&\{\int_I\frac12\bar\theta^bV^\alpha r^2+\frac{1}{2(\gamma-1)}V^\alpha h^2dx\}_t+\frac12\bar\theta^2V^{\alpha-1}\int_I h^2dx+\eta_1V^{\alpha-1}\int_I r_x^2dx+\eta_2V^{\alpha-1}\int_I h_x^2dx\\ \le&(\alpha+M)CV^{\alpha-1}(\int_I r_x^2dx+\int_I h_x^2dx),\end{aligned} \tag{3.26}$$

where $\eta_1=\eta_1(\bar\theta,b,\gamma,u_1)\triangleq(\frac12)^{b+2}\bar\theta^{2b}-(2^{3b+3}+2^{|b-1|+1}b+2^{2|b-1|+b+3}b^2)(\gamma-1)^2[(8u_1^2+1)+\frac{4\bar\theta^2}{3\bar\theta^{2b}}]$, and $\eta_2=\eta_2(\bar\theta,a,b)\triangleq(\frac12)^{a+1}\bar\theta^{a+b+1}-(2^{3b+4}+2^{|b-1|+2}b+2^{2|b-1|+b+4}b^2)\bar\theta^2$.

We observe that the condition (1.20) yields $2b>2-2b$ and $a+b+1>2$. Hence, for fixed u_1, there exists a positive constant θ_* only depending on a, b, γ and u_1, such that if $\bar\theta\ge\theta_*$, then $\eta_1>0$ and $\eta_2>0$. Thus, there exist positive constants $\alpha_1\le\frac12$ and m_1 only depending on a, b, γ, u_1 and $\bar{\theta}$, such that for $\alpha \in [0, \alpha_1]$ and $M \leq m_1$, we have

$$\{\int_I \bar{\theta}^b V^\alpha r^2 + \frac{1}{(\gamma-1)} V^\alpha h^2 dx\}_t + V^{\alpha-1}\int_I h^2 dx + V^{\alpha-1}\int_I r_x^2 dx + V^{\alpha-1}\int_I h_x^2 dx \leq 0. \quad (3.27)$$

Multiplying (3.2) by $V^\alpha q$, integrating over I, applying Schwarz's inequality, we have

$$\{\int_I V^\alpha q^2 dx\}_t + (1-\alpha)\bar{\theta}^{1-b} V^{\alpha-1}\int_I q^2 dx \leq V^{\alpha-1}\bar{\theta}^{b-1}\int_I r_x^2 dx. \quad (3.28)$$

Next, we will estimate the higher-order norm of (q, r, h). Differentiating (3.2) with respect to x, multiplying the resulting equation by $V^\alpha q_x$, integrating over I, and applying Schwarz's inequality, we have for any $t \in [0, T]$,

$$\{\int_I V^\alpha q_x^2 dx\}_t + (1-\alpha)\bar{\theta}^{1-b} V^{\alpha-1}\int_I q_x^2 dx \leq V^{\alpha-1}\bar{\theta}^{b-1} \mid r_{xx} \mid_{L^2}^2 . \quad (3.29)$$

From the equation (3.3), we obtain the following equation,

$$\begin{aligned}\bar{\theta}^b V^{-1}(h+1)^b r_{xx} =& (1+q)r_t - b\bar{\theta}V^{-1}(h+1)^{b-1}h_x r_x + \bar{\theta}V^{-1}h_x - b\bar{\theta}V^{-1}(h+1)^{b-1}h_x \\ &+ V^{-1}\frac{\bar{\theta}^b(h+1)^b r_x}{1+q} q_x - V^{-1}\frac{\bar{\theta}h}{1+q} q_x + V^{-1}\frac{\bar{\theta}[(h+1)^b - 1]}{1+q} q_x. \end{aligned} \quad (3.30)$$

Multiplying the above by $V^{\frac{\alpha+1}{2}}$, squaring both sides, integrating over I, and using Sobolev's inequality and Schwarz's inequality, by virtue of (3.7) and (3.23), we get for $t \in [0, T]$,

$$\begin{aligned}&(\frac{1}{2})^{2b}\bar{\theta}^{2b}V^{\alpha-1}\int_I r_{xx}^2 dx \leq V^{\alpha-1}\int_I \bar{\theta}^{2b}(h+1)^{2b} r_{xx}^2 dx \\ \leq& 28V^{\alpha+1}\int_I r_t^2 dx + CV^{\alpha-1}\int_I h_x^2 r_x^2 dx + 7\bar{\theta}^2 V^{\alpha-1}\int_I h_x^2 dx \\ &+ CV^{\alpha-1}\int_I r_x^2 q_x^2 dx + 28(1+b^2 2^{2|b-1|})\bar{\theta}^2 V^{\alpha-1}\int_I h^2 q_x^2 dx \\ \leq& 28V^{\alpha+1} \mid r_t \mid_{L^2}^2 + 7\bar{\theta}^2 V^{\alpha-1} \mid h_x \mid_{L^2}^2 + 28(1+b^2 2^{2|b-1|})M^2\bar{\theta}^2 V^{\alpha-1} \mid q_x \mid_{L^2}^2 \\ &+ \frac{1}{20}(\frac{1}{2})^{2b}\bar{\theta}^{2b}V^{\alpha-1} \mid r_{xx} \mid_{L^2}^2 + CV^{\alpha-1}\{\mid r_x \mid_{L^2}^2 \mid h_x \mid_{L^2}^2 + \mid r_x \mid_{L^2}^2 \mid q_x \mid_{L^2}^2 \\ &+ \mid r_x \mid_{L^2}^2 \mid h_x \mid_{L^2}^4 + \mid r_x \mid_{L^2}^2 \mid q_x \mid_{L^2}^4\}. \end{aligned} \quad (3.31)$$

Therefore, it follows from (3.29) and (3.31) that for $\alpha \in [0, \frac{1}{2}]$, there exists a positive constant m_2 only depending on b and $\bar{\theta}$, such that for $M \leq m_2$ and $t \in [0, T]$, we have

$$\begin{aligned}&(\frac{1}{2})^{2b}\bar{\theta}^{b+1}\{\int_I V^\alpha q_x^2 dx\}_t + (\frac{1}{2})^{2b+1}\bar{\theta}^2 V^{\alpha-1}\int_I q_x^2 dx + (\frac{1}{2})^{2b}\bar{\theta}^{2b}V^{\alpha-1}\int_I r_{xx}^2 dx \\ \leq& 28V^{\alpha+1} \mid r_t \mid_{L^2}^2 + 7\bar{\theta}^2 V^{\alpha-1} \mid h_x \mid_{L^2}^2 + \frac{1}{10}(\frac{1}{2})^{2b}\bar{\theta}^{2b}V^{\alpha-1} \mid r_{xx} \mid_{L^2}^2 \\ &+ CV^{\alpha-1}\{\mid r_x \mid_{L^2}^2 \mid h_x \mid_{L^2}^2 + \mid r_x \mid_{L^2}^2 \mid q_x \mid_{L^2}^2 + \mid r_x \mid_{L^2}^2 \mid h_x \mid_{L^2}^4 + \mid r_x \mid_{L^2}^2 \mid q_x \mid_{L^2}^4\}. \end{aligned} \quad (3.32)$$

Multiplying (3.4) by $(-V^\alpha\bar\theta^b h_{xx})$, integrating over I and using the boundary condition (3.5), we obtain for $t\in[0,T]$ that

$$
\begin{aligned}
&\{\int_I \frac{\bar\theta^{b+1}}{2(\gamma-1)}h_x^2dx\}_t+V^{\alpha-1}\int_I\frac{\bar\theta^{a+b+1}(h+1)^a h_{xx}^2}{1+q}dx-\frac{\alpha\bar\theta^2}{2(\gamma-1)}V^{\alpha-1}\int_I h_x^2dx\\
=&V^{\alpha-1}(-\int_I\frac{a\bar\theta^{a+b+1}(h+1)^{a-1}h_x^2h_{xx}}{1+q}dx+\int_I\frac{\bar\theta^{a+b+1}(h+1)^a q_xh_xh_{xx}}{(1+q)^2}dx\\
&-\int_I\frac{\bar\theta^{b+1}(h+1)^b r_xh_{xx}}{1+q}dx+\int_I\frac{\bar\theta^2hh_{xx}}{1+q}dx-\int_I\frac{\bar\theta^{2b}(h+1)^b r_x^2h_{xx}}{1+q}dx\\
&+\int_I\frac{\bar\theta^{b+1}r_xhh_{xx}}{1+q}dx-\int_I\frac{\bar\theta^{b+1}[(h+1)^b-1]r_xh_{xx}}{1+q}dx-\int_I\frac{\bar\theta^2[(h+1)^b-1]h_{xx}}{1+q}dx)\\
\equiv&\sum_{i=1}^8 J_i.
\end{aligned}\tag{3.33}
$$

Due to (3.7), using Young's inequality, Hölder's inequality and Sobolev's inequality, by virtue of (3.13), (3.16) and (3.23), we obtain the estimations on each term on the right hand side of (3.33) as follows. For $t\in[0,T]$,

$$
\begin{aligned}
J_1=&-V^{\alpha-1}\int_I\frac{a\bar\theta^{a+b+1}(h+1)^{a-1}h_x^2h_{xx}}{1+q}dx\le a\bar\theta^{a+b+1}2^{|a-1|+1}V^{\alpha-1}\mid h_x\mid_{L^\infty}\mid h_x\mid_{L^2}\mid h_{xx}\mid_{L^2}\\
\le&\frac{1}{16}\frac{\bar\theta^{a+b+1}}{2^{a+1}}V^{\alpha-1}\mid h_{xx}\mid_{L^2}^2+CV^{\alpha-1}\mid h_x\mid_{L^2}^6,
\end{aligned}\tag{3.34}
$$

$$
\begin{aligned}
J_2=&V^{\alpha-1}\int_I\frac{\bar\theta^{a+b+1}(h+1)^a q_xh_xh_{xx}}{(1+q)^2}dx\le 2^{a+2}\bar\theta^{a+b+1}V^{\alpha-1}\mid h_x\mid_{L^\infty}\mid q_x\mid_{L^2}\mid h_{xx}\mid_{L^2}\\
\le&\frac{1}{16}\frac{\bar\theta^{a+b+1}}{2^{a+1}}V^{\alpha-1}\mid h_{xx}\mid_{L^2}^2+CV^{\alpha-1}\mid h_x\mid_{L^2}^2\mid q_x\mid_{L^2}^4,
\end{aligned}\tag{3.35}
$$

$$
\begin{aligned}
J_3=&-V^{\alpha-1}\int_I\frac{\bar\theta^{b+1}(h+1)^b r_xh_{xx}}{1+q}dx\le 2^{b+1}\bar\theta^{b+1}V^{\alpha-1}\mid h_{xx}\mid_{L^2}\mid r_x\mid_{L^2}\\
\le&\frac{1}{16}\frac{\bar\theta^{a+b+1}}{2^{a+1}}V^{\alpha-1}\mid h_{xx}\mid_{L^2}^2+CV^{\alpha-1}\mid r_x\mid_{L^2}^2,
\end{aligned}\tag{3.36}
$$

$$
J_4=V^{\alpha-1}\int_I\frac{\bar\theta^2hh_{xx}}{1+q}dx\le\frac{1}{16}\frac{\bar\theta^{a+b+1}}{2^{a+1}}V^{\alpha-1}\mid h_{xx}\mid_{L^2}^2+CV^{\alpha-1}\mid h\mid_{L^2}^2,\tag{3.37}
$$

$$
\begin{aligned}
J_5=&-V^{\alpha-1}\int_I\frac{\bar\theta^{2b}(h+1)^b r_x^2h_{xx}}{1+q}dx\le 2^{b+1}\bar\theta^{2b}V^{\alpha-1}\mid r_x\mid_{L^\infty}\mid r_x\mid_{L^2}\mid h_{xx}\mid_{L^2}\\
\le&\frac{1}{16}\frac{\bar\theta^{a+b+1}}{2^{a+1}}V^{\alpha-1}\mid h_{xx}\mid_{L^2}^2+\frac{1}{10}(\frac{1}{2})^{2b}\bar\theta^{2b}V^{\alpha-1}\mid r_{xx}\mid_{L^2}^2+CV^{\alpha-1}(\mid r_x\mid_{L^2}^4+\mid r_x\mid_{L^2}^6),
\end{aligned}\tag{3.38}
$$

$$
\begin{aligned}
J_6=&V^{\alpha-1}\int_I\frac{\bar\theta^{b+1}r_xhh_{xx}}{1+q}dx\le 2\bar\theta^{b+1}V^{\alpha-1}\mid h\mid_{L^\infty}\mid r_x\mid_{L^2}\mid h_{xx}\mid_{L^2}\\
\le&\frac{1}{16}\frac{\bar\theta^{a+b+1}}{2^{a+1}}V^{\alpha-1}\mid h_{xx}\mid_{L^2}^2+CM^2V^{\alpha-1}\mid r_x\mid_{L^2}^2,
\end{aligned}\tag{3.39}
$$

$$
\begin{aligned}
J_7=&V^{\alpha-1}\int_I\frac{\bar\theta^{b+1}[(h+1)^b-1]r_xh_{xx}}{1+q}dx\le 2^{|b-1|+1}b\bar\theta^{b+1}V^{\alpha-1}\mid h\mid_{L^\infty}\mid r_x\mid_{L^2}\mid h_{xx}\mid_{L^2}\\
\le&\frac{1}{16}\frac{\bar\theta^{a+b+1}}{2^{a+1}}V^{\alpha-1}\mid h_{xx}\mid_{L^2}^2+CM^2V^{\alpha-1}\mid r_x\mid_{L^2}^2,
\end{aligned}\tag{3.40}
$$

and

$$J_8 = -V^{\alpha-1}\int_I \frac{\bar\theta^2[(h+1)^b-1]h_{xx}}{1+q}dx \le \frac{1}{16}\frac{\bar\theta^{a+b+1}}{2^{a+1}}V^{\alpha-1} \mid h_{xx}\mid_{L^2}^2 + CV^{\alpha-1}\mid h\mid_{L^2}^2 . \tag{3.41}$$

Substituting (3.34)–(3.41) into (3.33), and using (3.10), we deduce that for $\alpha \in [0,\frac{1}{2}]$, $M \le 1$, there exists a positive constant C_2 only depending on a, b, γ, and $\bar\theta$, such that for any $t\in[0,T]$,

$$\begin{aligned}
&\{\int_I \frac{\bar\theta^{b+1}}{2(\gamma-1)}h_x^2dx\}_t + \frac{\bar\theta^{a+b+1}}{2^{a+2}}V^{\alpha-1}\int_I h_{xx}^2dx\\
\le&\frac{1}{10}(\frac{1}{2})^{2b}\bar\theta^{2b}V^{\alpha-1}\mid r_{xx}\mid_{L^2}^2 + C_2V^{\alpha-1}(\mid r_x\mid_{L^2}^2 + \mid h_x\mid_{L^2}^2 + \mid h\mid_{L^2}^2)\\
&+ CV^{\alpha-1}(\mid r_x\mid_{L^2}^4 + \mid h_x\mid_{L^2}^6 + \mid h_x\mid_{L^2}^2\mid q_x\mid_{L^2}^4 + \mid r_x\mid_{L^2}^6).
\end{aligned}\tag{3.42}$$

Multiplying (3.3) by $V^{\alpha+1}r_t$, integrating over I and using the boundary condition (3.5), we have for $t\in[0,T]$,

$$\begin{aligned}
&\{\int_I V^\alpha\frac{\bar\theta^b(h+1)^br_x^2}{2(1+q)}dx\}_t + \{\int_I V^\alpha\frac{\bar\theta[(h+1)^b-1-h]r_x}{1+q}dx\}_t + \int_I V^{\alpha+1}r_t^2dx\\
=&\alpha V^{\alpha-1}\int_I\frac{\bar\theta^{b+1}(h+1)^br_x^2 - 2\bar\theta^2hr_x + 2\bar\theta^2[(h+1)^b-1]r_x}{2\bar\theta^b(1+q)}dx + V^\alpha\int_I\frac{b\bar\theta^b(h+1)^{b-1}h_tr_x^2}{2(1+q)}dx\\
&+ V^\alpha\int_I\frac{\bar\theta[b(h+1)^{b-1}-1]h_tr_x}{1+q}dx + V^\alpha\int_I dx - V^\alpha\int_I\frac{\bar\theta^b(h+1)^br_x^2q_t}{2(1+q)^2}dx\\
&- V^\alpha\int_I\frac{\bar\theta[(h+1)^b-1-h]r_xq_t}{(1+q)^2}dx\\
\equiv&\sum_{i=1}^5 L_i.
\end{aligned}\tag{3.43}$$

Due to (3.7), applying (3.23), Sobolev's inequality, Hölder's inequality and Schwarz's inequality, and using the equations (3.2) and (3.4), we get the following estimations on each term on the right hand side of (3.43). For $t\in[0,T]$,

$$L_1 \le \alpha C(\mid r_x\mid_{L^2}^2 + \mid h\mid_{L^2}^2), \tag{3.44}$$

$$\begin{aligned}
L_2 \le&\frac{1}{290}(\frac{1}{2})^{2b}\bar\theta^{2b}V^{\alpha-1}\mid r_{xx}\mid_{L^2}^2 + \frac{1}{29}\frac{\bar\theta^{a+b+1}}{2^{a+4}}V^{\alpha-1}\mid h_{xx}\mid_{L^2}^2 + C\mid r_x\mid_{L^2}^2 + C\mid r_x\mid_{L^2}^4 + C\mid h_x\mid_{L^2}^4\\
&+ C\mid r_x\mid_{L^2}^6 + C\mid h_x\mid_{L^2}^2\mid r_x\mid_{L^2}^4 + C\mid r_x\mid_{L^2}^2\mid h_x\mid_{L^2}^2\mid q_x\mid_{L^2}^2,
\end{aligned}\tag{3.45}$$

$$\begin{aligned}
L_3 \le&\frac{1}{290}(\frac{1}{2})^{2b}\bar\theta^{2b}V^{\alpha-1}\mid r_{xx}\mid_{L^2}^2 + \frac{1}{29}\frac{\bar\theta^{a+b+1}}{2^{a+4}}V^{\alpha-1}\mid h_{xx}\mid_{L^2}^2 + C\mid r_x\mid_{L^2}^2 + \mid h_x\mid_{L^2}^2 + C\mid r_x\mid_{L^2}^3\\
&+ C\mid r_x\mid_{L^2}^4 + C\mid r_x\mid_{L^2}^2\mid h_x\mid_{L^2}^2 + C\mid r_x\mid_{L^2}^2\mid q_x\mid_{L^2}^2 + C\mid r_x\mid_{L^2}^2\mid q_x\mid_{L^2}^4,
\end{aligned}\tag{3.46}$$

$$L_4 \le\frac{1}{290}(\frac{1}{2})^{2b}\bar\theta^{2b}V^{\alpha-1}\mid r_{xx}\mid_{L^2}^2 + C\mid r_x\mid_{L^2}^2 + C\mid r_x\mid_{L^2}^3 + C\mid r_x\mid_{L^2}^6, \tag{3.47}$$

and $L_5 \le C(\mid r_x\mid_{L^2}^2 + \mid h\mid_{L^2}^2)$. (3.48)

Substituting (3.44)–(3.48) into (3.43) and considering $\alpha \in [0, \frac{1}{2}]$, we have

$$\begin{aligned}
&\{\int_I V^\alpha \frac{\bar\theta^b (h+1)^b r_x^2}{2(1+q)} dx\}_t + \{\int_I V^\alpha \frac{\bar\theta[(h+1)^b - 1 - h] r_x}{1+q} dx\}_t + \int_I V^{\alpha+1} r_t^2 dx \\
\le& \frac{3}{290}(\frac{1}{2})^{2b} \bar\theta^{2b} V^{\alpha-1} \mid r_{xx} \mid_{L^2}^2 + \frac{1}{29} \frac{\bar\theta^{a+b+1}}{2^{a+3}} V^{\alpha-1} \mid h_{xx} \mid_{L^2}^2 + \frac{1}{29} C_3 V^{\alpha-1}(\mid r_x \mid_{L^2}^2 + \mid h \mid_{L^2}^2) \\
&+ C V^{\alpha-1}\{\mid r_x \mid_{L^2}^3 + C \mid r_x \mid_{L^2}^4 + \mid h_x \mid_{L^2}^4 + \mid r_x \mid_{L^2}^2 \mid h_x \mid_{L^2}^2 + \mid r_x \mid_{L^2}^2 \mid q_x \mid_{L^2}^2 + \mid r_x \mid_{L^2}^6 \\
&+ \mid h_x \mid_{L^2}^2 \mid r_x \mid_{L^2}^4 + \mid r_x \mid_{L^2}^2 \mid q_x \mid_{L^2}^4 + \mid r_x \mid_{L^2}^2 \mid h_x \mid_{L^2}^2 \mid q_x \mid_{L^2}^2\},
\end{aligned} \tag{3.49}$$

where C_3 is a positive constant only depending on a, b, γ, u_1 and $\bar\theta$.

Furthermore, we define

$$A(t) \triangleq \int_I V^\alpha \frac{\bar\theta^b (h+1)^b r_x^2}{2(1+q)} dx, \quad B(t) \triangleq \int_I V^\alpha \frac{\bar\theta[(h+1)^b - 1 - h] r_x}{1+q} dx, \tag{3.50}$$

and then by using Young's inequality and Schwarz's inequality, we have

$$\begin{aligned}
&(\frac{1}{2})^{b+3} \bar\theta^b \int_I V(t)^\alpha r_x^2(t) dx - (2^{2|b-1|} b^2 + 1) 2^{b+4} \bar\theta^{2-b} \int V(t)^\alpha h^2 dx \le A(t) + B(t) \\
\le& (2^b \bar\theta^b + \bar\theta) \int_I V(t)^\alpha r_x^2(t) dx + (2^{2|b-1|+1} b^2 + 2) \bar\theta \int V(t)^\alpha h^2 dx.
\end{aligned} \tag{3.51}$$

Last, we do the computation as

$$N \times (3.27) + (3.28) + (3.32) + 29 \times (3.42) + (3.49),$$

where

$$N = \bar\theta^{b-1} + 7\bar\theta^2 + C_1 + C_2 + C_3 + (2^{2|b-1|} b^2 + 1) 2^{b+4} \bar\theta^{2-b},$$

and then we have the following estimate

$$\begin{aligned}
&\{V^\alpha(t) \mid (q, q_x, r, h_x)(t) \mid_{L^2}^2 + A(t) + B(t) + N V^\alpha(t) \mid h(t) \mid_{L^2}^2\}_t \\
&+ V^{\alpha-1}(t)\big(\mid (q, q_x, r_x, h_x, h)(t) \mid_{L^2}^2 + \mid (r_{xx}, h_{xx})(t) \mid_{L^2}^2 \big) + V^{\alpha+1}(t) \mid r_t(t) \mid_{L^2}^2 \\
\le& C V^{\alpha-1}(t)\big(\mid r_x \mid_{L^2}^3 + \mid r_x \mid_{L^2}^4 + \mid h_x \mid_{L^2}^4 + \mid r_x \mid_{L^2}^2 \mid h_x \mid_{L^2}^2 + \mid r_x \mid_{L^2}^2 \mid q_x \mid_{L^2}^2 \\
&+ \mid r_x \mid_{L^2}^6 + \mid h_x \mid_{L^2}^6 + \mid r_x \mid_{L^2}^2 \mid h_x \mid_{L^2}^4 + \mid r_x \mid_{L^2}^2 \mid q_x \mid_{L^2}^4 + \mid h_x \mid_{L^2}^2 \mid r_x \mid_{L^2}^4 \\
&+ \mid h_x \mid_{L^2}^2 \mid q_x \mid_{L^2}^4 + \mid r_x \mid_{L^2}^2 \mid h_x \mid_{L^2}^2 \mid q_x \mid_{L^2}^2 \big),
\end{aligned} \tag{3.52}$$

for $t \in [0, T]$ and $\alpha \in [0, \alpha_1]$, provided that $\bar\theta \ge \theta_*$ and $M \le \min\{\frac{1}{2}, m_1, m_2\}$. Here we put

$$Z(t) \triangleq \mid (q, r, h)(t) \mid_{H^1}^2 . \tag{3.53}$$

Then by (3.51) and (3.13), it follows from (3.52) that

$$\begin{aligned}
&\{V(t)^\alpha Z(t)\}_t + V(t)^{\alpha-1} Z(t)\{1 - C(Z(t)^{\frac{1}{2}} + Z(t) + Z(t)^2)\} \\
&+ V(t)^{\alpha-1} \mid (r_{xx}, h_{xx})(t) \mid_{L^2}^2 + V^{\alpha+1}(t) \mid r_t(t) \mid_{L^2}^2 \le 0,
\end{aligned} \tag{3.54}$$

for any $t \in [0, T]$, $\alpha \in [0, \alpha_1]$. When I_0 is suitably small, solving the above differential inequality and we obtain

$$V(t)^\alpha Z(t) \le v_1^\alpha Z(0), \qquad \forall\, t \in [0, T],\ \alpha \in [0, \alpha_1]. \tag{3.55}$$

Thus, from (3.53) and (3.55), we have the following estimate

$$V(t)^\alpha \mid (q, r, h)(t) \mid_{H^1}^2 \le v_1^\alpha I_0^2, \qquad \forall\, t \in [0, T],\ \alpha \in [0, \alpha_1]. \tag{3.56}$$

Furthermore, we could deduce from (3.54) and (3.56) that

$$\int_0^t \mid (r_{xx}, h_{xx})(s) \mid_{L^2}^2 ds \le I_0^2, \qquad \forall\ t \in [0, T],\ \alpha \in [0, \alpha_1]. \tag{3.57}$$

Therefore, (3.8) and (3.9) are satisfied and the proof of Proposition 3.1 is completed. □

3.2. **Global Existence.**

Now, we are ready to prove our main result with the help of Proposition 3.1.

Proof of Theorem 1.1. First, let θ_* and M be the same as in Proposition 3.1, and let $\varepsilon = \frac{1}{4}M$. Next, the local existence result in Lemma 2.3 yields the local existence of unique strong solution (q, r, h) to (3.2)–(3.6) on $[0, T_1] \times I$, satisfying $(q, r, h) \in C([0, T_1]; H^1)$, which together with Sobolev's inequality implies that there exists $T_0 \in (0, T_1)$ such that $\mid q, r, h \mid_{0,T_0} \leq M$. Then by using Proposition 3.1 on $[0, T_0]$, we obtain the estimate (3.8) and (3.9) for any $t \in [0, T_0]$, from which we deduce that $\mid (q, r, h)(T_0) \mid_{L^\infty} \leq M$, $\inf_{x\in I} V(1 + q)(T_0, x) > C$ and $\inf_{x\in I} \bar{\theta}(1 + h)(T_0, x) > C$, where C is some positive constant only depending on a, b, γ, u_1 and $\bar{\theta}$. Therefore, by a standard argument, we can extend the priori estimate (3.8) on $[0, T_0]$ to a global priori estimate, and we refer readers to [15] for the details. Finally, with the help of the global priori estimate (3.8), the existence of global solution (q, r, h) to (3.2)–(3.6) can be established by extending the local strong solution to a global one by a standard continuity argument, for which we refer readers to [11,16]. Let $v = V(1+q)$, $u = U + r$ and $\theta = \bar{\theta}(1+h)$, we could verify that (v, u, θ) is a global strong solution to (1.8)–(1.12), satisfying (1.26) and (1.27). As for the decay property (1.28), it follows from (1.27) by using Sobolev's inequality.

The proof of Theorem 1.1 is completed. □

Acknowledgments: This work is partially supported by NSFC Project 11421061, and by National Science Foundation of Shanghai 15ZR1403900.

References

[1] S. Chapman and T.G. Cowling. *The mathematical theory of non-uniform gases. An account of the kinetic theory of viscosity, thermal conduction and diffusion in gases, Thermal Conduction and Diffusion in Gases.* Cambridge University Press; 1990.

[2] L.C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, second edition; 2010.

[3] B. Huang and X. Shi. Nonlinearly exponential stability of compressible Navier-Stokes system with degenerate heat-conductivity. *J. Differential Equations.* 2020; 268(5):2464-2490.

[4] H. K. Jenssen and T. K. Karper. One-dimensional compressible flow with temperature dependent transport coefficients. *SIAM J. Math. Anal.*. 2010; 42(2):904–930.

[5] S. Jiang. Global Smooth Solutions of the Equations of a Viscous, Heat-Conducting, One-Dimensional Gas with Density-Dependent Viscosity.*Math. Nachr.*. 1998; 190:169–183.

[6] S. Jiang, Z. Xin, P. Zhang, Global weak solutions to 1D compressible isentropic Navier-Stokes equations with density-dependent viscosity. *Methods Appl. Anal.*. 2005; 12(3):239–251.

[7] J. I. Kanel. A model system of equations for the one-dimensional motion of a gas. *Differencial nye Uravnenija.* 1968; 4:721–734.

[8] A.V. Kazhikhov. On the Cauchy problem for the equations of a viscous gas. (Russian) *Sibirsk. Mat. Zh.*. 1982; 23(1):60–64.

[9] A.V. Kazhikhov and V.V. Shelukhin. Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas. *J. Appl. Math. Mech.*. 1977; 41(2):273–282.

[10] Li, Jing and Liang, Zhilei. Some Uniform Estimates and Large-Time Behavior for One-Dimensional Compressible Navier-Stokes System in Unbounded Domains with Large Data. *Arch. Ration. Mech. Anal.*. 2016; 220(3):1195–1208.

[11] Z. Li and Z. Guo. On free boundary problem for compressible Navier-Stokes equations with temperature-dependent heat conductivity. *Discrete Contin. Dyn. Syst. Ser. B.*. 2017; 22(10):3903-3919.

[12] Z. Li, Y. Ma and Y. Ou. Global strong solutions to 1-D vacuum free boundary problem for compressible Navier-Stokes equations with variable viscosity and thermal conductivity. *J. Math. Anal. Appl.*. 2019; 474(2):1153–1177.

[13] T.-P. Liu, Z. Xin and T. Yang. Vacuum states for compressible flow. *Discrete Contin. Dynam. Systems.* 1998; 4(1):1–32.

[14] H. Liu, T. Yang, H. Zhao and Q. Zou, One-dimensional compressible Navier-Stokes equations with temperature dependent transport coefficients and large data. *SIAM J. Math. Anal.*. 2014; 46(3):2185–2228.
[15] M. Okada. Free boundary value problems for equations of one-dimensional motion of compressible viscous fluids. *Japan J. Appl. Math.*. 1987; 4(2):219–235.
[16] R. Pan and W. Zhang. Compressible Navier-Stokes equations with temperature dependent heat conductivity. *Commun. Math. Sci.*. 2015; 13(2):401–425.
[17] X. Qin, and Z.-A. Yao. Global smooth solutions of the compressible Navier-Stokes equations with density-dependent viscosity. *J. Differential Equations*. 2008; 244(8):2041–2061.
[18] X. Qin, Z.-A. Yao and H. Zhao. One dimensional compressible Navier-Stokes equations with density dependent viscosity and free boundaries. *Comm. Pure Appl. Anal.*. 2008; 7(2):373–381.
[19] A. Tani. On the first initial-boundary value problem of compressible viscous fluid motion, *Publ. Res. Inst. Math. Sci. Kyoto Univ.*. 1977; 13:193–253.
[20] D. Wang. On the global solution and interface behaviour of viscous compressible real flow with free boundaries. *Nonlinearity*. 2003; 16(2):719–733.
[21] T. Wang and H. Zhao. Global large solutions to a viscous heat-conducting one-dimensional gas with temperature-dependent viscosity. Preprint at arXiv:1505.05252, 2015.
[22] T. Yang, Z.-A. Yao and C. Zhu. Compressible Navier-Stokes equations with density-dependent viscosity and vacuum. Comm. *Partial Differential Equations*. 2001; 26(5-6):965–981.
[23] C. Zhu. Asymptotic Behavior of compressible Navier-Stokes equations with density-dependent viscosity and vacuum. *Comm. Math. Phys.*. 2010; 293(1):279–299.

Tuowei Chen, School of Mathematical Sciences, Fudan University, Shanghai 200433, P. R. China;

E-mail address: twchen16@fudan.edu.cn

Yongqian Zhang, School of Mathematical Sciences, Fudan University, Shanghai 200433, P. R. China

E-mail address: yongqianz@fudan.edu.cn