Optimal time decay rates of solutions for the 2D generalized magneto-micropolar equations

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Abstract

This study is concerned with the optimal time rates of weak solutions for the 2D magneto-micropolar equations with only micro-rotational dissipation and magnetic diffusion. Due to some new observations, we obtain the optimal time decay rates of weak solutions $\|\nabla u(t)\|_{L^2} + \|\nabla w(t)\|_{L^2} \leq C(1+t)^{-2}$ and $\|\nabla b(t)\|_{L^p} \leq C(1+t)^{-\frac{1}{2}-(1-\frac{1}{p})}$ with $p \in [2, \infty)$.

Key words: magneto-micropolar equations, partial dissipation, optimal decay rates

1 Introduction and main result

In this study, we consider the 2D magneto-micropolar equations with only micro-rotational dissipation and magnetic diffusion

$$\begin{cases} u_t + u \cdot \nabla u + (\mu + \chi)u + \nabla p = b \cdot \nabla b + 2\chi \nabla \times w, \\ w_t + u \cdot \nabla w + 4\chi w = \kappa \Delta w + 2\chi \nabla \times u, \\ b_t + u \cdot \nabla b = \nu \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,y,0) = u_0,\, w(x,y,0) = w_0,\, b(x,y,0) = b_0, \end{cases} \tag{1.1}$$

where $u = (u_1(x,y,t), u_2(x,y,t))$, $w = w(x,y,t)$, $b = (b_1(x,y,t), b_2(x,y,t))$ and $p = p(x,y,t)$ with $(x,y) \in \mathbb{R}^2$ and $t \geq 0$ denote the velocity of the fluid, micro-rotational velocity, the magnetic field and the hydrostatic pressure, respectively. μ, χ and ν are separately, kinematic viscosity, vortex viscosity and magnetic diffusivity constants. It's worth noting that $\nabla \times u = \partial_x u_2 - \partial_y u_1$ is a scalar function representing the vorticity, and $\nabla \times w = (\partial_y w, -\partial_x w)$.

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The magneto-micropolar equations describe the motion of an incompressible, electrically conducting micropolar fluids in the presence of an arbitrary magnetic field [1,3]. It has attracted considerable attention from the community of mathematical fluids. We refer to [2, 8, 11, 15] for global regularity for 2D problems, [9, 13, 14] for the existence and time decay rates of 3D global small solution. Recently, Shang and Gu [11,12] investigated the time decay rates of weak solutions for the 2D magneto-micropolar equations (1.1), and obtained the following important results.

Lemma 1.1 *([11, 12]) Let* $\kappa > \frac{4\chi^2}{\mu+\chi}$ *and suppose* (u, w, b) *is a global weak solution of the system* (1.1) *with* $(u_0, w_0) \in H^1(\mathbb{R}^2)$ *and* $b_0 \in L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$, *then the solution have the following time decay rates*

$$\begin{aligned} &\|u(t)\|_{L^2} + \|w(t)\|_{L^2} \le C(1+t)^{-2}, \quad &&\|\nabla u(t)\|_{L^2} + \|\nabla w(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}}, \\ &\|b(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}}, &&\|b(t)\|_{L^\infty} + \|\nabla b(t)\|_{L^2} \le C(1+t)^{-1}. \end{aligned}$$

The decay rates of $\|u\|_{L^2}$ and $\|w\|_{L^2}$ in Lemma 1.1 are optimal since they are derived from the sharp estimates of b and ∇b (see [12] for details). However, there exist two terms $(\mu+\chi)u$ and $4\chi w$ in (1.1), the faster decay rates of $\|\nabla u\|_{L^2}$ and $\|\nabla w\|_{L^2}$ may also be expected.

In this paper, we consider the optimal decay rates of $\|\nabla u\|_{L^2}$ and $\|\nabla w\|_{L^2}$. Denote $\Omega \triangleq \nabla \times u$ and $j \triangleq \nabla \times b$. Then from (1.1), we have

$$\begin{cases} \Omega_t + u \cdot \nabla \Omega + (\mu+\chi)\Omega = b \cdot \nabla j - 2\chi \Delta w, \\ \nabla w_t + \nabla(u \cdot \nabla w) + 4\chi \nabla w = \kappa \Delta \nabla w + 2\chi \nabla \Omega, \\ j_t + u \cdot \nabla j = \nu \Delta j + b \cdot \nabla \Omega + T(\nabla u, \nabla b), \end{cases} \tag{1.2}$$

where $T(\nabla u, \nabla b) = 2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1)$. It is worth mentioning that since the first equation of (1.2) has no dissipation on Ω, the Fourier splitting method which relies on the dissipation in order to decompose the whole space into two time-dependent sub-domains does not apply. Furthermore, without smallness assumption, Kato's method [7] does not work due to the difficulty of constructing an iterative procedure. Some more recent new time decay methods such as the one by Guo and Wang [4] involving Sobolev space of negative indices and the one by [6] for dual equation technique do not apply to our circumstance.

In order to overcome these difficulties, we take a new method due to some observations to the structure of the problem (1.2). Specifically, we firstly see that

$$(1+t)^n \|\Omega(t)\|_{L^2}^2 \quad \text{and} \quad (1+t)^n (\|\nabla w(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2)$$

for large positive integer n satisfy some Gronwall type inequalities. Then we can obtain an interesting implicit decay estimate (see Lemma 3.1 below),

$$\int_0^t (1+s)^n (\|\Omega\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla j\|_{L^2}^2)\, \mathrm{d}s \le C(1+t)^{n-2}, \quad t > 0, \tag{1.3}$$

which essentially implies that $\|\nabla u\|_{L^2}$ and $\|\nabla w\|_{L^2}$ admit the faster upper bounds of decay rates. This key estimate (1.3) allows us to derive the desired optimal decay rates. In addition, this practice may be useful for more related decay problems.

Our results read as follows.

Theorem 1.1 *Let $\kappa > \frac{4\chi^2}{\mu+\chi}$ and suppose (u, w, b) is a global weak solution of the system* (1.1) *with $(u_0, w_0) \in H^1(\mathbb{R}^2)$ and $b_0 \in L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$. Then we have the following optimal upper bounds of decay rates*

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2} + \|\nabla w(t)\|_{L^2} \leq C(1+t)^{-2},\\
&\|\nabla b(t)\|_{L^p} \leq C(1+t)^{-\frac{1}{2}-(1-\frac{1}{p})}, \quad p \in [2, \infty).
\end{aligned}$$

Remark 1.1 *Obviously, the decay rates of ∇b in L^p are optimal in the sense it coincides with those of linear equations. In addition, the decay rates of $\|\nabla u\|_{L^2}$ and $\|\nabla w\|_{L^2}$ in Theorem 1.1 are also optimal. We take the decay rate of $\|\nabla u\|_{L^2}$ for example to illustrate this point. Taking the L^2 inner product of the first equation of* (1.2) *with Ω and using Hölder inequality, it follows that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Omega(t)\|_{L^2}^2 + \|\Omega(t)\|_{L^2}^2 \leq C\|b \cdot \nabla j\|_{L^2}^2 + C\|\Delta w\|_{L^2}^2.$$

For large positive integer n, there exists $t_0 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^t(1+t)^n\|\Omega(t)\|_{L^2}^2\Big) \leq Ce^t(1+t)^n(\|b\|_{L^\infty}^2\|\nabla j\|_{L^2}^2 + \|\Delta w\|_{L^2}^2), \quad t > t_0. \tag{1.4}$$

Integrating (1.4) *in time from t_0 to t and applying* (1.3)*, we formally obtain*

$$\begin{aligned}
\|\Omega(t)\|_{L^2}^2 &\leq C(1+t)^{-n}\int_{t_0}^t e^{s-t}(1+s)^n\|b\|_{L^\infty}^2\|\nabla j\|_{L^2}^2\,\mathrm{d}s + f(t)\\
&\leq C(1+t)^{-n}\left(\int_0^t (1+s)^{n-2}\|\nabla j\|_{L^2}^2\,\mathrm{d}s\right) + f(t)\\
&\leq C(1+t)^{-4} + f(t),
\end{aligned} \tag{1.5}$$

where $f(t) = Ce^{-t} + C(1+t)^{-n}\int_0^t e^{s-t}(1+s)^n\|\Delta w\|_{L^2}^2\,\mathrm{d}s$. It seems that the estimate of ∇j in (1.3) *is optimal in the sense that it coincides with the one of linear equation. So comparing* (1.5) *with the one in Theorem 1.1, it's easy to see that the decay rate of $\|\nabla u\|_{L^2}$ we obtained is optimal to some degree.*

This paper is organized as follows. In the next section, we introduce some auxiliary estimates. The proof of Theorem 1.1 is given in Section 3.

2 Some auxiliary estimates

We first in this section recall the classic $L^p - L^q$ decay estimates for the heat semigroup $e^{t\Delta}$.

Lemma 2.1 *Let $1 \le p \le q \le \infty$ and $s \ge 0$, the following estimates are valid for $f \in L^p(\mathbb{R}^2)$,*

$$\|\Lambda^s e^{t\Delta} f\|_{L^q(\mathbb{R}^2)} \le C t^{-(\frac{1}{p}-\frac{1}{q})-\frac{s}{2}} \|f\|_{L^p(\mathbb{R}^2)}, \quad t > 0,$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$ denotes the Zygmund operator.

The following fractional Gagliardo-Nirenberg interpolation inequality is necessary to prove the estimate of $\|\nabla b(t)\|_{L^p}$ with $p \in [2, \infty)$.

Lemma 2.2 *([5]) Let $1 < p, p_0, p_1 < \infty$, $\alpha, \beta \in \mathbb{R}$, $0 \le \theta \le 1$. Then the fractional Gagliardo-Nirenberg interpolation inequality*

$$\|\Lambda^\alpha u\|_{L^p} \le C \|u\|_{L^{p_0}}^{1-\theta} \|\Lambda^\beta u\|_{L^{p_1}}^{\theta} \tag{2.1}$$

is true in $\mathbb{R}^2$ if and only if

$$\frac{1}{p} - \frac{\alpha}{2} = \left(\frac{1}{p_1} - \frac{\beta}{2}\right)\theta + \frac{1-\theta}{p_0} \text{ and } \alpha \le \theta\beta.$$

Let's recall the following estimates which have been proved in [11].

Lemma 2.3 *([11]) Under the same conditions of Theorem 1.1, the weak solutions (u, w, b) obey the following global H^1-bound,*

$$\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \int_0^t \left(\|\nabla w(\tau)\|_{L^2}^2 + \|\nabla b(\tau)\|_{L^2}^2\right) \mathrm{d}\tau \le C,$$

$$\|\Omega(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \int_0^t \left(\|\Omega(\tau)\|_{L^2}^2 + \|\Delta w(\tau)\|_{L^2}^2 + \|\nabla j(\tau)\|_{L^2}^2\right) \mathrm{d}\tau \le C,$$

where the constant C depends on μ, χ, κ, ν and the initial data.

3 Proof of Theorem 1.1

As usual, we assume that the solutions are regular enough. The process of obtaining the decay for weak solutions from regular approximations is standard, for details see [11].

Proof of Theorem 1.1. The proof of Theorem 1.1 is divided into three steps.

Step 1. In this step, we will give some auxiliary decay rates for ∇u, ∇w and ∇b.

Taking the L^2 inner product of equations (1.2) with $(\Omega, \nabla w, j)$, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Omega(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2\right) + (\mu+\chi)\|\Omega(t)\|_{L^2}^2 \\
&+ 4\chi\|\nabla w(t)\|_{L^2}^2 + \kappa\|\nabla w(t)\|_{L^2}^2 + \kappa\|\Delta w\|_{L^2}^2 + \nu\|\nabla j\|_{L^2}^2 \\
&= -4\chi \int_{\mathbb{R}^2} \Delta w \, \Omega \, \mathrm{d}x - \int_{\mathbb{R}^2} \nabla(u \cdot \nabla w) \cdot \nabla w \, \mathrm{d}x + \int_{\mathbb{R}^2} T(\nabla u, \nabla b) \cdot j \, \mathrm{d}x,
\end{aligned} \tag{3.1}$$

where we use the equalities

$$\int_{\mathbb{R}^2} (b \cdot \nabla j) \cdot \Omega \, \mathrm{d}x + \int_{\mathbb{R}^2} (b \cdot \nabla \Omega) \cdot j \, \mathrm{d}x = 0,$$

and

$$-2\chi \int_{\mathbb{R}^2} \Delta w \, \Omega \, \mathrm{d}x = 2\chi \int_{\mathbb{R}^2} \nabla \Omega \cdot \nabla w \, \mathrm{d}x.$$

By Hölder and Young inequalities, we have

$$4\chi \int_{\mathbb{R}^2} \Delta w \, \Omega \, \mathrm{d}x \le 4\chi \|\Delta w\|_{L^2} \|\Omega\|_{L^2} \le (\mu + \chi - \epsilon) \|\Omega\|_{L^2}^2 + \frac{4\chi^2}{\mu + \chi - \epsilon} \|\Delta w\|_{L^2}^2,$$

where $\epsilon > 0$ is sufficiently small such that $\kappa > \frac{4\chi^2}{\mu+\chi-\epsilon} + \epsilon$. Using Hölder inequality, Young inequality and Gagliardo-Nirenberg interpolation inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \nabla(u \cdot \nabla w) \cdot \nabla w \, \mathrm{d}x &= \int_{\mathbb{R}^2} \nabla w \cdot \nabla u \cdot \nabla w \, \mathrm{d}x \\
&\le \|\nabla w\|_{L^4}^2 \|\Omega\|_{L^2} \\
&\le C \|\Delta w\|_{L^2} \|\nabla w\|_{L^2} \|\Omega\|_{L^2} \\
&\le \epsilon \|\Delta w\|_{L^2}^2 + C \|\nabla w\|_{L^2}^2 \|\Omega\|_{L^2}^2.
\end{aligned}$$

Similarly, we also obtain

$$\int_{\mathbb{R}^2} T(\nabla u, \nabla b) \cdot j \, \mathrm{d}x \le \|j\|_{L^4}^2 \|\nabla u\|_{L^2} \le \frac{\nu}{2} \|\nabla j\|_{L^2}^2 + C \|j\|_{L^2}^2 \|\Omega\|_{L^2}^2,$$

Inserting these estimates into (3.1), we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \Big(\|\Omega(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 \Big) + 2\epsilon \|\Omega\|_{L^2}^2 + 8\chi \|\nabla w\|_{L^2}^2 \\
&\qquad + 2\big(\kappa - \frac{4\chi^2}{\mu + \chi - \epsilon} - \epsilon\big) \|\Delta w\|_{L^2}^2 + \nu \|\nabla j\|_{L^2}^2 \le C \Big(\|\nabla w\|_{L^2}^2 + \|j\|_{L^2}^2 \Big) \|\Omega\|_{L^2}^2.
\end{aligned}$$

Let $\sigma = \min\{2\epsilon, 8\chi\}$, then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \Big(\|\Omega(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 \Big) &+ \sigma \Big(\|\Omega\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 \Big) \\
&\le C \Big(\|\nabla w\|_{L^2}^2 + \|j\|_{L^2}^2 \Big) \|\Omega\|_{L^2}^2. \qquad (3.2)
\end{aligned}$$

We now apply the generalized Fourier splitting methods ([10]) to examine the decay estimate of $\|j(t)\|_{L^2}$. Dividing the domain $\mathbb{R}^2$ into $B(t)$ and $B(t)^c$ obeys

$$\begin{aligned}
\|\nabla j(t)\|_{L^2}^2 &= \int_{\mathbb{R}^2} |\xi|^2 |\hat{j}(\xi, t)|^2 \, \mathrm{d}\xi \ge \frac{n}{1+t} \int_{B(t)^c} |\hat{j}(\xi, t)|^2 \, \mathrm{d}\xi \\
&= \frac{n}{1+t} \left(\int_{\mathbb{R}^2} |\hat{j}(\xi, t)|^2 \, \mathrm{d}\xi - \int_{B(t)} |\hat{j}(\xi, t)|^2 \, \mathrm{d}\xi \right).
\end{aligned}$$

where $n>10$ is a large positive integer and

$$B(t)=\left\{\xi\in\mathbb{R}^2;|\xi|^2\le\frac{n}{\sigma(1+t)}\right\}.$$

Then for $t>n$, from (3.2) together with the decay results in Lemma 1.1, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\Omega(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2\Big)+\frac{n}{1+t}\Big(\|\Omega(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2\Big)\\
&\le\frac{n}{\sigma(1+t)}\int_{B(t)}|\hat{j}(t)|^2\,\mathrm{d}\xi+C\Big(\|\nabla w\|_{L^2}^2+\|j\|_{L^2}^2\Big)\|\Omega\|_{L^2}^2\\
&\le\frac{n}{\sigma(1+t)}\int_{B(t)}|\xi|^2|\hat{b}(t)|^2\,\mathrm{d}\xi+C\Big(\|\nabla w\|_{L^2}^2+\|j\|_{L^2}^2\Big)\|\Omega\|_{L^2}^2\\
&\le C(1+t)^{-2}\|b(t)\|_{L^2}^2+C\Big(\|\nabla w\|_{L^2}^2+\|j\|_{L^2}^2\Big)\|\Omega\|_{L^2}^2\\
&\le C(1+t)^{-3}+C\Big(\|\nabla w\|_{L^2}^2+\|j\|_{L^2}^2\Big)\|\Omega\|_{L^2}^2.
\end{aligned}$$

Multiplying the above inequality by $(1+t)^n$ and integrating in time from n to t give

$$\begin{aligned}
&(1+t)^n\Big(\|\Omega(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2\Big)\\
&\le C(1+t)^{n-2}+C\int_0^t(1+s)^n\Big(\|\nabla w\|_{L^2}^2+\|j\|_{L^2}^2\Big)\|\Omega\|_{L^2}^2\,\mathrm{d}s.
\end{aligned}$$

Applying Lemma 2.3 and Gronwall inequality to the above inequality about

$$(1+t)^n\|\Omega(t)\|_{L^2}^2\quad\text{and}\quad(1+t)^n(\|\nabla w(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2),$$

respectively, we obtain that

$$(1+t)^n\Big(\|\Omega(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2+\|j(t)\|_{L^2}^2\Big)\le C(1+t)^{n-2}.$$

That is,

$$\|\nabla u(t)\|_{L^2}+\|\nabla w(t)\|_{L^2}+\|\nabla b(t)\|_{L^2}\le C(1+t)^{-1}. \tag{3.3}$$

Step 2. In this step, we will show the faster decay rates of ∇u and ∇w than these in (3.3). To do this, we first need to establish the following estimate.

Lemma 3.1 *Under the conditions of Theorem 1.1, we have*

$$\int_0^t(1+s)^n(\|\Omega\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla j\|_{L^2}^2)\,\mathrm{d}s\le C(1+t)^{n-2},\quad t>0. \tag{3.4}$$

Proof. Multiplying both sides of (3.2) by $(1+t)^n$ and recalling Lemma 1.1 yield

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big((1+t)^n\big(\|\Omega\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|j\|_{L^2}^2\big)\Big)+\sigma(1+t)^n(\|\Omega\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla j\|_{L^2}^2)\\
&\le n(1+t)^{n-1}\big(\|\Omega\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|j\|_{L^2}^2\big)+C(1+t)^n\Big(\|\nabla w\|_{L^2}^2+\|j\|_{L^2}^2\Big)\|\Omega\|_{L^2}^2\\
&\le C(1+t)^{n-3}+C(1+t)^{n-2}\|\Omega\|_{L^2}^2.
\end{aligned}$$

Since $\|\Omega\|_{L^2}^2$ is integrable in $(0,\infty)$, integrating in time from 0 to t, one has

$$
\begin{aligned}
&(1+t)^n\big(\|\Omega\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|j\|_{L^2}^2\big)+\sigma\int_0^t(1+s)^n(\|\Omega\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla j\|_{L^2}^2)\,\mathrm{d}s\\
&\le C(1+t)^{n-2}+C\int_0^t(1+s)^{n-2}\|\Omega\|_{L^2}^2\,\mathrm{d}s\\
&\le C(1+t)^{n-2},
\end{aligned}
$$

which implies that

$$
\int_0^t(1+s)^n(\|\Omega\|_{L^2}^2+\|\nabla w\|_{L^2}^2+\|\nabla j\|_{L^2}^2)\,\mathrm{d}s\le C(1+t)^{n-2}.
$$

So we complete the proof of Lemma 3.1. □

With the help of Lemma 3.1, we now continue to prove Theorem 1.1. Taking the L^2 inner product to the first and second equations of (1.2) with Ω and ∇w and adding the resulting equations together, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\Omega(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2\Big)+(\mu+\chi)\|\Omega(t)\|_{L^2}^2+4\chi\|\nabla w(t)\|_{L^2}^2+\kappa\|\Delta w(t)\|_{L^2}^2\\
&=\int_{\mathbb{R}^2}(b\cdot\nabla j)\cdot\Omega\,\mathrm{d}x-4\chi\int_{\mathbb{R}^2}\Delta w\cdot\Omega\,\mathrm{d}x-\int_{\mathbb{R}^2}\nabla w\cdot\nabla u\cdot\nabla w\,\mathrm{d}x. \qquad (3.5)
\end{aligned}
$$

By Hölder and Young inequalities, we have

$$
\int_{\mathbb{R}^2}(b\cdot\nabla j)\cdot\Omega\,\mathrm{d}x\le\|b\|_{L^\infty}\|\nabla j\|_{L^2}\|\Omega\|_{L^2}\le\frac{\epsilon}{2}\|\Omega\|_{L^2}^2+C\|b\|_{L^\infty}^2\|\nabla j\|_{L^2}^2.
$$

By the similar method with the proof of (3.2), we see that

$$
4\chi\int_{\mathbb{R}^2}\Delta w\cdot\Omega\,\mathrm{d}x\le(\mu+\chi-\epsilon)\|\Omega\|_{L^2}^2+\frac{4\chi^2}{\mu+\chi-\epsilon}\|\Delta w\|_{L^2}^2
$$

and

$$
\int_{\mathbb{R}^2}\nabla w\cdot\nabla u\cdot\nabla w\,\mathrm{d}x\le\epsilon\|\Delta w\|_{L^2}^2+C\|\nabla w\|_{L^2}^2\|\Omega\|_{L^2}^2.
$$

Inserting these estimates into (3.5) and multiplying the resulting inequality by $(1+t)^n$, we have

$$
\begin{aligned}
&(1+t)^n\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\Omega(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2\Big)+\sigma(1+t)^n\Big(\|\Omega(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2\Big)\\
&\le C(1+t)^n\|b\|_{L^\infty}^2\|\nabla j\|_{L^2}^2+C(1+t)^n\|\nabla w\|_{L^2}^2\|\Omega\|_{L^2}^2.
\end{aligned}
$$

Combining the above inequality with (3.3) and Lemma 1.1, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big((1+t)^n(\|\Omega(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2)\Big)\\
&\le C(1+t)^n\|b\|_{L^\infty}^2\|\nabla j\|_{L^2}^2+C(1+t)^n\|\nabla w\|_{L^2}^2\|\Omega\|_{L^2}^2\\
&\le C(1+t)^{n-2}(\|\nabla j\|_{L^2}^2+\|\Omega\|_{L^2}^2). \qquad (3.6)
\end{aligned}
$$

Integrating (3.6) in time from 0 to t and combining with Lemma 3.1, we have

$$\begin{aligned}&(1+t)^n(\|\Omega(t)\|_{L^2}^2+\|\nabla w(t)\|_{L^2}^2)\\&\le C+C\int_0^t(1+s)^{n-2}(\|\nabla j\|_{L^2}^2+\|\Omega\|_{L^2}^2)\,\mathrm{d}s\\&\le C(1+t)^{n-4}.\end{aligned}$$

Then we obtain

$$\|\nabla u(t)\|_{L^2}+\|\nabla w(t)\|_{L^2}\le C(1+t)^{-2}. \tag{3.7}$$

Step 3. In the last step, we prove the estimate of $\|\nabla b(t)\|_{L^p}$ with $p\in[2,\infty)$.

Writing the magnetic field equation of (1.1) into integral form and applying ∇ to the resulting equation, we get that

$$\begin{aligned}\|\nabla b(t)\|_{L^p}&\le\|\nabla e^{\nu t\Delta}b_0\|_{L^p}+\int_0^t\|\nabla e^{\nu(t-s)\Delta}(u\cdot\nabla b-b\cdot\nabla u)(s)\|_{L^p}\,\mathrm{d}s\\&\le Ct^{\frac{1}{p}-\frac{3}{2}}+\int_0^{\frac{t}{2}}\|\nabla\nabla e^{\nu(t-s)\Delta}(u\otimes b-b\otimes u)(s)\|_{L^p}\,\mathrm{d}s\\&\quad+\int_{\frac{t}{2}}^t\|\nabla e^{\nu(t-s)\Delta}(u\cdot\nabla b-b\cdot\nabla u)(s)\|_{L^p}\,\mathrm{d}s.\end{aligned} \tag{3.8}$$

By Lemma 2.1, Hölder inequality and Gagliardo-Nirenberg interpolation inequality, we get

$$\begin{aligned}&\int_0^{\frac{t}{2}}\|\nabla\nabla e^{\nu(t-s)\Delta}(u\otimes b-b\otimes u)(s)\|_{L^p}\,\mathrm{d}s\\&\le C\int_0^{\frac{t}{2}}(t-s)^{\frac{1}{p}-2}\|(u\otimes b-b\otimes u)(s)\|_{L^1}\,\mathrm{d}s\\&\le Ct^{\frac{1}{p}-2}\int_0^{\frac{t}{2}}\|u(s)\|_{L^2}\|b(s)\|_{L^2}\,\mathrm{d}s\\&\le Ct^{\frac{1}{p}-2}\int_0^{\frac{t}{2}}(1+s)^{-4}\,\mathrm{d}s\le Ct^{\frac{1}{p}-2}.\end{aligned} \tag{3.9}$$

Let $\alpha\in(1-\frac{1}{p},1)$. By fractional Gagliardo-Nirenberg interpolation inequality (2.1), Lemma 1.1 and the estimate (3.7), we have

$$\|\Lambda^{1-\alpha}u(s)\|_{L^p}\le\|u(s)\|_{L^2}^{2(\frac{1}{p}-\frac{1-\alpha}{2})}\|\nabla u(s)\|_{L^2}^{1-2(\frac{1}{p}-\frac{1-\alpha}{2})}\le C(1+s)^{-2},$$

$$\|\Lambda^{1-\alpha}b(s)\|_{L^{2p}}\le\|b(s)\|_{L^2}^{2(\frac{1}{2p}-\frac{1-\alpha}{2})}\|\nabla b(s)\|_{L^2}^{1-2(\frac{1}{2p}-\frac{1-\alpha}{2})}\le C(1+s)^{\frac{1}{2p}-\frac{1-\alpha}{2}-1}.$$

Then by Lemma 2.1, we also obtain

$$
\begin{aligned}
&\int_{\frac{t}{2}}^{t} \|\nabla e^{\nu(t-s)\Delta}(u\cdot\nabla b+b\cdot\nabla u)(s)\|_{L^p}\,\mathrm{d}s\\
&\le \int_{\frac{t}{2}}^{t} \|\nabla \Lambda^{\alpha} e^{\nu(t-s)\Delta}\Lambda^{-\alpha}\nabla(u\otimes b-b\otimes u)(s)\|_{L^p}\,\mathrm{d}s\\
&\le C\int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1+\alpha}{2}}\|\Lambda^{1-\alpha}(u\otimes b-b\otimes u)(s)\|_{L^p}\,\mathrm{d}s\\
&\le C\int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1+\alpha}{2}}\Big(\|\Lambda^{1-\alpha}u(s)\|_{L^p}\|b(s)\|_{L^\infty}+\|\Lambda^{1-\alpha}b(s)\|_{L^{2p}}\|u(s)\|_{L^{2p}}\Big)\,\mathrm{d}s\\
&\le C\int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1+\alpha}{2}}\Big((1+s)^{-3}+(1+s)^{\frac{1}{2p}-\frac{1-\alpha}{2}-3}\Big)\,\mathrm{d}s \le C(1+t)^{\frac{1}{2p}-3}.
\end{aligned}
\tag{3.10}
$$

Inserting (3.9) and (3.10) into (3.8), we have

$$
\|\nabla b(t)\|_{L^p} \le C(1+t)^{-\frac{1}{2}-(1-\frac{1}{p})}. \tag{3.11}
$$

Thus we complete the proof of Theorem 1.1. □

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