An Approach to Solutions of Fractal and Fractional Time Derivative Fokker-Planck Equation

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Abstract

An approach to finding the exact solution of ordinary, fractal and fractional Fokker-Planck equation FPE, based on transforming it to a system of first- order PDEs, together with using the extended unified method, is presented. Reduction of the fractal and fractional derivatives to the classical ones with time dependent coefficient is performed via similarity transformations. Some explicit solutions of the classical, fractal and fractional time derivative FPE, are obtained . It is shown that the solution of the FPE is mixed- Gaussian's. It is worthy to mention that the mixture of Gaussians is a powerful tool in machine learning. Further,it is found that the friction coefficient plays a significant role in lowering the magnitude of the distribution function. While changing the order of the fractal and fraction time derivative has a slight effects and the mean and mean square of the velocity vary slowly.

Key words. reduction of fractal and fractional derivatives, non autonomous FPE, exact solutions. extended unified method, mixed -Gaussians

1 Introduction

The FPE deals with fluctuations of systems which stem from disturbances, each of which changes the variables of the system in an unpredictable way. When macroscopic particles are immersed in fluid, they are pulled by the fluid and the position of particles is an unpredictable. So that they fluctuate about an expected position with a certain probability to find the particle in a given region. The probability density function can be determined via the FPE.This equation is used in many different fields in natural science, in solid-state physics, quantum optics, chemical physics, and mathematical biology. Anomalous diffusion is an eminent behavior and characterizes the transport processes of matter in different systems [1–4].The features of diffusion can be classified to sub- and super- diffusion. They may be measured by evaluating the variance, relevant to the displacement of particles in the medium, which behaves as $x^{\beta}$where x is the space variable. When $0 < \beta < 1$, this stands to sub-advection (or in transition state) and when

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$1 < \beta < 2$, this case is sub-diffusion (or the transition from advection to diffusion). While in the case when $2 < \beta < 3$ describes super-diffusion (or the transition from diffusion to dispersion). The first case assigns to spatially disordered or fractal media and also in fractional Brownian motion and random processes [5,6]. To this end the fractal and fractional time-space derivatives, of orders α, $0 < \alpha < 1$ and $\beta, 1 < \beta < 2$, respectively, do emerge in the formulation of diffusion equations relevant to the study of the transport process. By using the time time fractional derivative, the effects of time distributed delay in the transport process are then taken into consideration. While fractal derivative reflect instantaneous frangibility effect.

Fractional derivatives which was proposed in the literature assume numerous definitions; Caputo, Caputo-Fabrizio {7,8], and a very recently Atangana-Baleneau derivatives [9] and slso Riemann-Liouville, Riesz (see [21]),In the last three, only the Kernel of the integral is different depending to when it is singular or not. Analytical investigation of the fractional KPE equation that describes anomalous diffusion of energetic particles [10] have been accomplished. An approach was proposed in [11–13] to solve analytically the fractional force-less one-dimensional FPE. The Klein–Kramers equation [14,15] that describes the transport of energetic particles in turbulent magnetic fields can be reduced to the force-less homogeneous one-dimensional FPE.

The Klein–Kramers equation [36,37], which describes the Brownian motion of particles in the presence of an external force $F(z)$ is

$$\frac{\partial}{\partial t}W(z,v,t) = (-v\frac{\partial}{\partial z} + \frac{\partial}{\partial v}(\eta v - \frac{F(z)}{m}) + B\frac{\partial^2}{\partial v^2})W(z,v,t), \tag{1}$$

where $W(z,v,t)$ is the probability density function of particles, $B = \frac{\eta K_B T}{m}$,K_B is the Boltzmann constant, T is the absolute temperature and m is the mass of a diffusing particle, v its velocity, η denotesthe friction coefficient, $F(z)$ is an external force field

To describe the distribution in the velocity space one can evauate the mean and mean square of the particle position in the absence of the external force. In this case (1) reduces to the FPE that inspect the diffusion of a test particle in the phase space,

$$\frac{\partial}{\partial t}f(v,t) = \eta\frac{\partial}{\partial v}(vf(v,t)) + B\frac{\partial^2}{\partial v^2}f(v,t), \tag{2}$$

where $f(v,t) = \int_{-\infty}^{\infty} W(z,v,t)dz$ designates the distribution function.

Here, we consider the fractal and fractional time KPE of (2) that generalizes the energetic particles transport equations, that is by replacing the ordinary derivatives using the fractal , Caputo and Caputo-Fabrizio fractional time derivatives [7,8] to make a prediction of the evolution of the particle distribution function in phase space. We mention that the results obtained incorporate the effects of the distributed time delay.

2 Fractal derivative

The fractional derivative was introduced in [16-18]

$$\frac{d}{dt^{\alpha}}f(t) = Limit_{t_1\to t}\frac{f(t_1)-f(t)}{t_1^{\alpha}-t^{\alpha}},\ t>0. \tag{3}$$

When $f\epsilon C^1(\mathbb{R}^+)$, the RHS of (1) Reduces to

$$\frac{d}{dt^{\alpha}}f(t) = \alpha^{-1}t^{1-\alpha}f'(t). \tag{4}$$

From (2) we find that the fractal derive is nothing else but the conformable fractional derivative up to multiplication by α^{-1}. On the other hand is is identical to the LHS of (2) by writing $dt^{\alpha} = \alpha t^{\alpha-1}dt$. That is dt^{α} is reducible. This fact suggests to define the fractal derive by

$$\frac{d}{dt^{\alpha}}f(t) = Limit_{\varepsilon\to 0}\frac{f(t+\varepsilon\alpha^{-1}t^{1-\alpha})-f(t)}{\varepsilon},\ t>0. \tag{5}$$

3 Fractional derivative

The Caputo fractional derivative is

$$D_t^{\alpha C}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-t_1)^{-\alpha+(m-1)}f^{(m)}(t_1)dt_1, m-1<\alpha<mt>0, \tag{6}$$

provided that f is Hoder continuous , $f\epsilon H^{m,\alpha-(m-1)}(\mathbb{R}^+)$ and the integral exists. The Caputo-Fabrizio fractional derivative CFFD is

$$D_t^{\alpha CF}f(t) = \frac{2\alpha}{(1-\alpha)(2-\alpha)}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-t_1)}f'(t_1)dt_1,\ 0<\alpha<1\, t>0, \tag{7}$$

provided that $f\epsilon H^{1,\alpha}(\mathbb{R}^+)$.

The Atangana-Baleanu fractational derivative ABFD, in the Caputo sense, is [9,20]

$$D_t^{\alpha AB}f(t) = \frac{B(\alpha)}{1-\alpha}\int_0^t E_{\alpha}(-\frac{\alpha}{1-\alpha}(t-t_1)^{\alpha})f'(t_1)dt_1,\ 0<\alpha<1\, t>0, \tag{8}$$

where $B(\alpha)>0$ is a normalization function satisfying $B(0)=B(1)=1$, and $E_{\alpha}(t)$ is the Mittag–Leffler function and $f\epsilon H^{1,\alpha}(\mathbb{R}^+)$. It is worth noticing that this function is not invariant under the CFD. The function which is invariant is $e_\alpha(t)$ [26] where

$$E_\alpha(t)=\sum_{n=0}^{\infty}\frac{t^n}{\Gamma(\alpha n+1)}, e_\alpha(t):=e_{\alpha,1}(t)=\sum_{n=0}^{\infty}\frac{t^{\alpha n}}{\Gamma(\alpha n+1)}, \tag{9}$$

and the last function generalizes to

$$e_{\alpha,1}(\lambda,t)=\sum_{n=0}^{\infty}\frac{\lambda^n t^{\alpha n}}{\Gamma(\alpha n+\beta)} \tag{10}$$

We define a new fractional derivative, Gawad's definition

$$D_t^{\beta G}f(t)=\frac{2\lambda^{\frac{1}{\beta}}}{\lambda+2}\int_0^t e^{-\lambda(t-t_1)^\beta}f'(t_1)dt_1,\ \beta>0\, t>0, \tag{11}$$

provided that $f\epsilon H^{1,\beta}(\mathbb{R}^+)$. We mention that the CFFD is a particular case from (3), when $\beta=1$and $\lambda=\frac{\alpha}{1-\alpha}$.The Kernel in the fraction derivative (3) is of interest) in the theory of distributions. As when $0<\beta\leq 1$, the fractional exponential distribution is $f(t)=\lambda^{1/\beta}e^{-\lambda t^\beta}$ generalizes the classical exponential distribution and when $1<\beta\leq 2$, the fractional Gaussian distribution is $f(t)=\frac{\lambda^{1/\beta}}{\sqrt{\pi}}e^{-\lambda t^\beta}$. This later distribution may be considered as transition between the exponential and the Gaussian.

3.1 Reduction of the fractional derivatives

Here we shall reduce the FD's to ordinary derivatives with time dependent coefficients [22-24].

We consider (1) when $0<\alpha<1$

$$D_{t_1}^{\alpha C}f(t_1)=\frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t_1-t_2)^{-\alpha}f'(t_2)dt_2. \tag{12}$$

Theorem. The Caputo FD is reduced to

$$D_t^{\alpha C}f(t)=\frac{1}{\Gamma(2-\alpha)}(T-t)^{1-\alpha}f'(t),\ 0\leq t\leq T_0. \tag{13}$$

Proof.. In (4), operating by the integral on t_1 on $[0,t]$, it holds

$$\int_0^t D_{t_1}^{\alpha C}f(t_1)dt_1=\int_0^t(\frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t_1-t_2)^{-\alpha}f'(t_2)dt_2)dt_1\ . \tag{14}$$

In the RHS of (5) we permute the outer with the inner integral and get

$$\int_0^t D_{t_1}^{\alpha C}f(t_1)dt_1=\begin{array}{c}\int_0^t(\frac{1}{\Gamma(1-\alpha)}(\int_{t_2}^t(t_1-t_2)^{-\alpha}dt_1)f'(t_2)dt_2\\ =\int_0^t(\frac{1}{\Gamma(1-\alpha)}\frac{(t-t_2)^{1-\alpha}}{1-\alpha})f'(t_2)dt_2\end{array}. \tag{15}$$

From (6), it holds that

$$D_t^{\alpha C} f(t) = \frac{1}{\Gamma(2-\alpha)} (T-t)^{1-\alpha} f'(t) \; 0 \leq t \leq T_0. \tag{16}$$

,

which is identically (5).

By using the transformation $f(t) := \tilde{f}(\tau)$, (8) can be rewritten:

$$D_t^{\alpha C} f(t) := \frac{d}{d\tau} \tilde{f}(\tau), \; \tau = \frac{\Gamma(2-\alpha)}{\alpha} (T_0^\alpha - (T_0 - t)^\alpha) \tag{17}$$

The equations (8) and (9) are some of the main results in this work.

By the same way, we find

$$\begin{array}{c} D_t^{\alpha CF} f(t) = \frac{2}{(2-\alpha)} (1 - e^{-\frac{\alpha}{1-\alpha}(T_0 - t)}) f'(t) := \frac{d}{d\tau} \tilde{f}(\tau), \\ \tau = \frac{(2-\alpha)(1-\alpha)}{2\alpha} Log(\frac{e^{\frac{\alpha}{1-\alpha}(T_0-t)} - 1}{e^{\frac{\alpha}{1-\alpha}T_0} - 1}) \; 0 \leq t \leq T_0, \end{array} , \tag{18}$$

$$D_t^{\alpha AB} f(t) = \frac{B(\alpha)}{(1-\alpha)} (T_0 - t) e_{\alpha,2}(-\frac{\alpha}{1-\alpha}, (T - t_1)) f_0'(t), \tag{19}$$

where (see [25])

$$e_{\alpha,\beta}(\sigma, x) = \sum_{n=0}^{\infty} \frac{\sigma^n x^{\alpha n}}{\Gamma(\alpha n + \beta)}. \tag{20}$$

And

$$\begin{array}{c} D_t^{\beta G} f(t) = \frac{2}{\lambda+2} \gamma(\frac{1}{\beta}, \lambda(T_0 - t)^\beta) f'(t) := \frac{d}{d\tau} \tilde{f}(\tau), \\ \tau = \frac{(\lambda+2)}{2} \int_0^t \frac{1}{\gamma(\frac{1}{\beta}, \lambda(T_0 - t_1)^\beta)} dt_1, \; 0 \leq t \leq T_0, 0 < \beta < 1, \end{array} \tag{21}$$

where $\gamma(m, t)$ is the incomplete lower Gamma function. We mention that

$$\gamma(a, x) + \mathit{\Gamma}(a, x) = \Gamma(a), \; a > 0, \; x > 0, \; \mathit{\Gamma}(a, x) = \int_x^\infty e^{-y} y^{a-1} dy. \tag{22}$$

,By using (11), we can prove the following theorem.

Theorem 1.The GFD satisfies the following:

(i) $D_t^{\beta G}(f(t) + g(t)) = D_t^{\beta G} f(t) + D_t^{\beta G} g(t)$.

(ii) $D_t^{\beta G}(f(t)g(t)) = f(t) D_t^{\beta G} g(t) + g(t) D_t^{\beta G} f(t)$.

(iii)$D_t^{\beta G}(\frac{f(t)}{g(t)})) = \frac{g(t) D_t^{\beta G} f(t) - f(t) D_t^{\beta G} g(t)}{g(t)^2}$.

Further by using (11) the function $f(t)$ which is invariant under the FD $D_t^{\beta G}$,that is $D_t^{\beta G} e_{\beta,G}(t) = e_{\beta,G}(t)$, is found directly:

$$e_{\beta,G}(t) = e^{\frac{(\lambda+2)}{2} \int_0^t \frac{1}{\gamma(\frac{1}{\beta}, \lambda(T_0 - t_1)^\beta)} dt_1,}. \tag{23}$$

4 Solutions of the FPE.

We present the approach to solve of (2) as follows. Here we use the transformations $f_v(v,t) = F(v,t)\,f(v,t)$ and $f_t(v,t) = G(v,t)\,f(v,t)$.

Thus (2) is written

$$\begin{array}{c} f_v(v,t) - F(v,t)\,f(v,t) = 0,\ f_t(v,t) - G(v,t)\,f(v,t) = 0, \\ G(v,t) - \eta(1 + vF(v,t)) - B(F_v(v,t) + F(v,t)^2) = 0. \end{array} \tag{24}$$

On the other hand , the unified method, which asserts that the solutions of the nonlinear PDEs can be expressed by rational forms in an auxiliary function that satisfies an auxiliary equation.

We find a class of solutions of (2).

4.1 Case of Linear auxiliary equations

(I) We assume that the auxiliary equations are linear. In (28) and take the solutions in the form.

$$\begin{array}{c} f(v,t) = \frac{s_1(v)g(v,t)+s_0(v)}{a_1(v)g(v,t)+a_0(v)},\ F(v,t) = \frac{b_1(v)g(v,t)+b_0(v)}{s_1(v)g(v,z,t)+s_0(v)}, \\ G(v,t) = \frac{d_1(v)g(v,t)+d_0(v)}{s_1(v)g(v,t)+s_0(v)}, \end{array} \tag{25}$$

together with linear auxiliary equation.

$$g_t(v,t) = \mu\,(c_1 g(v,t) + c_0),\ g_v(v,t) = h(v)\,(c_1 g(v,t) + c_0), \tag{26}$$

It is worth noticing that in (26), the compatibly equation $g_{tv}(v,t) = g_{vt}(v,t)$ holds.

By inserting (25) into (24), and by using (30), we get a system of coupled ODEs of first order in $a_i, b, d_i, i = 1, 2$..We found that the calculations are not straight forward due to two facts. (i)The equations obtained are nonlinear (ii) It arises that we can have two equations, for example,$a_j(v)$ and $a'_j(v)$, $j = 0, 1$, so we have to use the compatibly equation, $b'_j(v) - (b_j(v))' = 0$.

We have the following equations

$$\begin{array}{c} a'_0(v) = \frac{1}{s_0(v)}(-c_0 a_1(v)h(v)s(v) + a_0(v)(-b_0(v) + c_0 h(v)s_1(v) + s'_0(v))), \\[2ex] s'_1(v) = b_1(v),\ \ s'_0(v) = b_0(v) + h(v)(-c_0 s_1(v) + c_1 s_0(v)), \\[2ex] b'_0(v) = \frac{1}{m\,B}(b_0(v)(-m\,v\,\eta + F_0 + Bc_1 mh(v)) - m(Bc_0 b_1(v)h(v) - c_0(k_0 v + \mu)s_1(v) \\ +(c_1 k_0 v + \eta + c_1\mu)s_0(v))), \\ b'_1(v) = \frac{1}{m\,B}(b_1(v)(-m\,v\,\eta + F_0) - m\,\eta s_1(v)) \end{array} \tag{27}$$

and

$$d_1(v) = 0,\ d_0(v) = \frac{c_1\mu}{a_1(v)}(s_1(v) - a_1(v)s_0(v)),\ a_1(v) = c_1/c_0,\ a_0(v) = 1. \tag{28}$$

By rewriting the second equation in (27) as $b_1(v) = s_1'(v)$,the compatibly equation$b_1'(v) - (b_1(v))' = 0$, gives rise to

$$m\,\eta s_1(v) + mv\eta s_1'(v) + B\,ms_1''(v) = 0, \tag{29}$$

where (29) solves to

$$s_1(v) = A_2 e^{-\frac{v^2\eta}{2\,B}} + A_1\sqrt{\frac{\pi\,B}{2\eta}}e^{\frac{v^2\eta}{2\,B}-\frac{F_0^2}{(2Bm^2\eta}}erfi(\frac{m\,v\,\eta]}{\sqrt{2B\eta}m}),\ erfi(x) = e^{-x^2}\int_0^x e^{y^2}dy. \tag{30}$$

In (27), we can evaluate $b_1(v) = s_1'(v)$..By the same way, we rewrite the third equation in(31) as

$$b_0(v) = -s_0'(v) + h(v)(-c_0 s_1(v) + c_1 s_0(v)), \tag{31}$$

we find that the compatibly equation $b_0'(v) - (b_0(v))' = 0$ gives rise to and

$$(m\,v\,\eta)h(v) + Bc_1 mh(v)^2 + m(+\mu - Bh'(v)) = 0.. \tag{32}$$

The solutions of (32) is given by

$$c_1 = \frac{n\,\eta}{\mu},\ h(v) = \frac{P_1(v)}{Q_1},\ P_1(v) = -\sqrt{\frac{2B}{\eta}}\mu B_0 H_{n-1}(\frac{v\sqrt{\eta}}{\sqrt{2B}}) + v\mu_1 F_1(1-\frac{n}{2},\frac{3}{2},\frac{v^2\eta}{2B})),$$
$$Q_1(v) = BB_0 H_n(\frac{v\sqrt{\eta}}{\sqrt{2B}}) + {}_1F_1(-\frac{n}{2},\frac{1}{2},\frac{\eta v^2}{2B}), \tag{33}$$

where $H_n(v\sqrt{\frac{\eta}{2B}})$ and ${}_1F_1(-\frac{n}{2},\frac{1}{2},\frac{\eta v^2}{2B})$ are the Hermite polynomial and the hyper geometric functions respectively.

Further the solution of the auxiliary equations give rise to

$$g(v,t) = -\frac{c_0}{c_1} + B_3 e^{c1\int h(v)dv+\mu t}. \tag{34}$$

We mention that$\int h(v)dv$ can not be directly evaluated. To this end we assume that (see (33))

$$(Q_1 + s(v))' = P_1, \tag{35}$$

and the calculations give

$$s(v) = \frac{(n\eta+\mu)}{n\eta}(-BB_0 H_n(\frac{v\sqrt{\eta}}{\sqrt{2B}}) + B(1 -_P F_Q(\{-\frac{n}{2}\},\{\frac{1}{2}\},\frac{\eta v^2}{2B}))\ , \tag{36}$$

where ${}_PF_Q$ is the generalized hyper geometric function. Finally, we get

$$\int h(v)dv = Log(\mid P_1(v) + s(v) \mid). \tag{37}$$

By substituting for $s_i(v), a_i(v)$ and by using (32)-(41) in the first equation in (29) we get the required solution. It is too lengthy to be produced here.

4.2 case of quadratic auxiliary equations

We assume that the auxiliary equations are quadratic. By using (24), the solutions have the form

$$\begin{gathered} f(v,t) = \frac{s_1 g(v,t)+s_0}{a_1(v)g(v,t)+a_0(v)},\ F(v,t) = \frac{b_1(v)g(v,t)+b_0(v)}{s_1 g(v,t)+s_0}, \\ G(v,t) = \frac{d_1(v)g(v,t)+d_0(v)}{s_1 g(v,t)+s_0}, \end{gathered} \tag{38}$$

together withe the auxiliary equation

$$g_t(v,t) = \mu\,(c_2 g(v,t)^2 + c_1 g(v,t) + c_0),\ g_v(v,t) = h(v)\,(c_2 g(v,t)^2 + c_1 g(v,t) + c_0). \tag{39}$$

It is worth noticing that in (39), the equation $g_{tv}(v,t) = g_{vt}(v,t)$ holds.

By inserting (38) into (25) and by using (39), we find that the calculations are not straight forward due to two facts. (i) The equations obtained are nonlinear (ii) It arises that we can have two equations for $a_j(v)$ and $a_j'(v)$, $j = 0, 1$, so we have to use the compatibly equation, $a_j'(v) - (a_j(v))' = 0$.

We have the following

$$\begin{gathered} a_0'(v) = \frac{1}{s_0}(-ao(v)bo(v) - c_0 s_0 a_1(v)h(v) + c_0 s_1 a_0(v)h(v)),\ c_0 = \frac{c_1^2-k_0^2}{4c_2}, \\ a_1'(v) = \frac{1}{s_1}(-a_1(v)b_1(v) - c_2 s_0 a_1(v)h(v) + c_2 s_1 a_0(v)h(v)), \\ a_1(v) = \frac{a_0(v)((c_1^2-k_0^2)s_1\mu - 4c_2 d_0(v))}{(c_1^2-k_0^2)s_0\mu},\ d_1(v) = -\frac{(d_0(v)((c_1^2-k_0^2)s_1\mu - 4c_1c_2s_0\mu - 4c_2 d_0[v]))}{(c_1^2-k_0^2)s_0\mu}, \\ \\ b_1(v) = \frac{1}{4c_2 s_0}(4c_2 s_1 b_0(v) - (c_1^2 - k_0^2)s_1^2 h(v) + 4c_1c_2s_1s_0h(v) - 4c_2{}_2^2 so^2 h(v), \\ d_0(v) = -\frac{1}{4c_2}((c_1+k_0)(-c_1s_1 + k_0s_1 + 2c_2s_0)\mu), \\ b_0'(v) = -\frac{1}{4B^2c_2h[v)^2}(B^2c_2k_0^2s_0h(v)\hat{}4 + c2s_0(\mu - Bh'^2(v) + h(v)^2(c_2s_0(4B\eta \\ -v\hat{}2\eta^2 + 2Bk_0\mu) + B^2(-c_1^2s_1 + k_0^2s_1 + 2c_1c_2s_0)h'(v)) \\ b_0(v) = \frac{1}{4Bc_2h(v)}(2c_2s_0\mu - 2c_2s_0v\eta h(v) + Bc1\hat{}2s_1h^2(v) - Bko\hat{}2s1h^2(v) \\ -2Bc_1c_2s_0h^2(v) - 2Bc_2s_0h'(v)). \end{gathered} \tag{40}$$

The compatibly equation between $b_0'(v)$ and $b_0(v)$ gives rise to

$$\begin{gathered} s_0(\mu^2 + (-v^2\eta^2 + 2B(\eta + k_0\mu))h^2(v) + B^2k_0^2h^4(v) - 4B\mu h'(v) = 0 \\ +3B^2h'^2(v) - 2B^2h(v)h''(v)) \end{gathered}\ . \tag{41}$$

The equation has the first integral

$$h'(v) = \frac{1}{B}(\eta + v\eta h(v) + Bk_0 h^2(v)), \tag{42}$$

that integrates to

$$h(v) = -\tfrac{P(v)}{Q(v)},\ P(v) = \mu(\sqrt{B}\sqrt{2}n\sqrt{\eta}B_0 H_{n-1}(\tfrac{v\sqrt{\eta}]}{\sqrt{2B}}) - nv\eta\, {}_1F_1(1-\tfrac{n}{2},\tfrac{3}{2},\tfrac{v^2\eta}{2B})),$$
$$Q(v) = n\,\eta B(B_0 H_n(\tfrac{v\sqrt{\eta}]}{\sqrt{2B}}) - nv\eta\, {}_1F_1(-\tfrac{n}{2},\tfrac{1}{2},\tfrac{v^2\eta}{2B})),\ k_0 = \tfrac{n\,\eta}{\mu}, \tag{43}$$

where $H_n(x)$ and ${}_1F_1(a, b, x)$ are the Hermite polynomial and hyper-geometric function respectively.

The compatibly equation between $a_1'(v)$ and $a_1(v)$ holds identically. It remains to find $a_0(v)$ by using (40), we get

$$a_{:0}(v) = e^{\frac{v^2\eta}{2B}} B_1,\ a_1(v) = \frac{2c_2 a_0(v)}{c_1 - k_0},. \tag{44}$$

The solution of the auxiliary equation (3) gives rise to

$$g(v,t) = \frac{(n\,\eta - e^{n\,\eta(t+\int h(v)dv)} n\,\eta - c_1\mu - c_1 e^{n\,\eta(t+\int h(v)dv)})\mu)}{2c_2\mu(1 + e^{(n\,\eta(t+\int h(v)dv)})}. \tag{45}$$

By substituting from (44) and (45) into the first equation in (38) we have

$$f(v,t) = \tfrac{P_1}{Q_1},\ P_1 = e^{\frac{v^2\eta}{2B}}(c_1 - k_0)((-1 + e^{n\,\eta(t+\int h(v)dv)})ns_1\eta +$$
$$(1 + e^{n\,\eta(t+\int h(v)dv)})(c_1 s_1 - 2c_2 s_0)\mu),\ Q_1 = 2B_1 c_2 n((-1 + e^{n\,\eta(t+\int h(v)dv)})n\eta$$
$$+(1 + e^{n\,\eta(t+\int h(v)dv)})k_0\mu). \tag{46}$$

Now we evaluate $\int h(v)dv$, to this end we use (43) and assume that

$$P(v) = (Q(v) + S(v))', \tag{47}$$

where $S(v)$ is to be determined. Calculations show that

$$S(v) \quad = \quad \frac{n\eta + \mu)B}{(n\sqrt{\eta}}(n\sqrt{\eta})(-B_0 H_n(\frac{v\sqrt{\eta}]}{\sqrt{2B}}) + (1 -_P F_Q(\{-\frac{n}{2}\},\{\frac{1}{2}\},\frac{v^2\eta}{2B}))), \tag{48}$$

where ${}_P F_Q\ (a, b, x)$ is the generalized hyper geometric function [19]. Finally we get

$$\int h(v)dv = -Log(\mid Q(v) + S(v) \mid), \tag{49}$$

where $Q(v)$ and $S(v)$ are give by (43) and (48) respectively.

4.3 Self Similar solution

now we consider the similarity transformations $z = v\,\omega(t)$, $t := t$;and $f(v,t) = \tilde{f}(z,t)$ so the equation (2) is

$$\frac{\partial}{\partial t}\tilde{f}(z,t) = \eta\frac{\partial}{\partial z}(z\tilde{f}(z,t)) + B\omega(t)^2\frac{\partial^2}{\partial z^2}\tilde{f}(z,t), \tag{50}$$

and as in section 4.1, (50) is transformed to

$$\begin{array}{c} \tilde{f}_z(z,t) - F(z,t)\,\tilde{f}(z,t) = 0,\ \tilde{f}_t(z,t) - G(z,t)\,\tilde{f}(z,t) = 0, \\ G(z,t) - \eta(1 + zF(z,t)) - B\omega(t)^2(F_z(z,t) + F(z,t)^2) = 0. \end{array}. \tag{51}$$

We assume that the solutions, by using (51), have the form

$$\begin{array}{c} \tilde{f}(z,t) = \frac{s_1 g(z,t)+s_0}{a_1(z,t)g(z,t)+a_0(z,t)},\ F(z,t) = \frac{b_1(z,t)g(z,t)+b_0(z,t)}{s_1 g(z,,t)+s_0}, \\ G(z,t) = \frac{d_1(z,t)g(z,t)+d_0(z,t)}{s_1 g(z,,t)+s_0}, \end{array} \tag{52}$$

and

$$\begin{array}{c} g_t(z,t) = \mu(t)\,(c_2 g(z,t)^2 + c_1 g(z,t) + c_0), \\ g_z(z,t) = h(z)\,(c_2 g(z,t)^2 + c_1 g(z,t) + c_0) \end{array}. \tag{53}$$

By inserting (52) and (53) into (51), we have following equations .

$$\begin{array}{c} a_{0z}(z,t) = \frac{1}{s_0\omega(t)}(-a_0(z,t)b(z,t) - c_0 s_0 a_1(z,t]h(z)\omega(t) \\ +c_0 s_1 a_0(z,t)\,h(z)\omega(t), \\ a_{1z}(z,t) = \frac{1}{s_1\omega(t)}(-a_1(z,t])b_1(z,t) - c_2 s_0 a_1(z,t0\,h(z)\omega(t) \\ +c_2 s_1 a_0(z,t)\,h(z)\omega(t)), \\ b_1(z,t) = \frac{1}{s_0}(s_1 b_0(z,t) - c_0 s_1^2 h(z)\omega(t) + c_1 s_1 s_0 h(z)\omega(t) \\ -c_2 s_0^2 h(z)\omega(t)), \\ a_{0t}(z,t) = \frac{1}{s_0}(-a_0(z,t)d_0(z,t) - c_0 s_0 a_1(z,t)\mu(t) \\ +c_0 s_1 a_0(z,t)\mu(t)), \\ a_{1t}(z,t) = \frac{1}{s_1}(-a_1(z,t)d_1(z,t) - c_2 s_0 a_1(z,t)\mu(t) + c_2 s_1 a_0(z,t)\mu(t)) \\ b_{0z}(z,t) = \frac{1}{4B^2 h[z)^2\omega(t)^3}(-s_0\mu(t)^2 + \\ 2B\mu(t)\omega(t)^2((-2c_0 s_1 + c_1 s_0)h(z)^2 + s_0 h'(z)) \\ +\omega(t)^2(-B^2(c_1^2 - 4c2co)s_0 h(z)^4\omega(t)^2 - B^2 s_0\omega(t)^2 h'^2(z) \\ +h(z)^2(s_0\eta(-4B + z^2\eta) + 4Bd_0(z,t) \\ +2B^2(2c_0 s_1 - c_1 s_0)\omega(t)^2 h'(z))), \\ b_0(z,t) = \frac{1}{2Bh(z))\omega(t)}(s_0\mu(t) - s_0 z\eta h(z)\omega(t) + 2Bc_0 s_1 h(z)^2\omega(t)^2 \\ -Bc_1 s_0 h(z)^2\hat{}2\omega(t)^2 - Bs_0\omega(t)^2 h'(z)). \end{array} \tag{54}$$

Now the compatibly equation $b_{0z}(z,t) - (b_0(z,t))_z = 0$, solves to

$$\begin{array}{c} d_0(z,t) = \frac{1}{4Bh(z)^2\omega(t)^2}(s_0\omega(t)^2 + 2B\mu(t)\omega(t)^2((2c_0 s_1 - c_1 s_0)h(z)^2 \\ -2s_0 h'(z)) + s_0\omega(t)^2(B^2(c_1^2 - 4c_2 c_0)h(z)^4\omega(t)^2 \\ -\eta h(z)^2(-4A + z\hat{}2\eta + 2B\omega(t)) \\ +3B^2\omega(t)^2 h'^2(z) - 2B^2 h(z)\omega(t)^2 h''(z))). \end{array} \tag{55}$$

The compatibly equations $(a_{iz}(z,t))_t - (a_{it}(z,t))_z = 0$, give rise to

$$\begin{gathered}(-B^2(c_1^2-4c_2c_0)h(z)^4\omega(t)^5h'(z)+\omega(t)h(z)(\mu(t)^{2\mu(t)} \\ -4B\mu(t)\omega(t)^2h'(z)+3B^2\omega(t)^4h'^2(z)+z\eta h(z)^3\omega(t)(\eta\omega(t)^2+\omega'(t)) \\ +2Bh(z)\omega(t)^3(\mu(t)-2B\omega(t)^2h'(z))h''(z) \\ +h[z]\hat{}2(\omega(t)\mu'(t)-2\mu(t)\omega(t))+B^2\omega(t)^5h^{(3)}(z)))=0.\end{gathered} \tag{56}$$

We find that (60) holds when

$$\begin{gathered}\mu(t)=A_0\omega^2(t),\ \omega'(t)=-\eta\omega(t)^2,\ c_0=(c_1^2-p_3^2)/(4c2),\ . \\ h'(z)=\tfrac{A_0}{A}+p_1h(z)+p_2h(z)^2\end{gathered} \tag{57}$$

It remains to evaluate $a_j(z,t),\ j=0,1$, where detailed calculations yield

$$a_1(z,t)=\tfrac{B_3\sqrt{B_0+t\eta}}{\sqrt{B_0}}e^{(\frac{1}{4}(2p_1z-4t\eta+(z^2\eta B_0+t\eta))/B-(Bp_1^2t)/(B_0^2+B_0t\eta)+(4A_0p_2t)/(B_0^2+Bot\eta)-4c_1\int h(z)dz))},$$

$$a_0(z,t)=e^{\int_0^t\frac{1}{4}((\eta(-4+\frac{z^2\eta}{B})+2\eta\omega(t_1)-Bp_1^2\omega(t_1)^2)dt_1}e^{(\frac{p_1z}{2}+\frac{B_0z^2\eta}{4B}}B_1,\ c_0=0,\ c_2=0. \tag{58}$$

The solution of the auxiliary equation is

$$g(z,t)=B_2e^{c_1(\int_0^t\mu(t_1)dt_1+\int h(z)dz)}, \tag{59}$$

together with

$$h(z)=-\frac{-p1\sqrt{B}-rtanh(\frac{r(z+A_1)}{2\sqrt{B}})}{2c_1\sqrt{B}},r=\sqrt{Bp_1^2+4A_0c_1}. \tag{60}$$

Finally we get the solution of (51) which is

$$\begin{gathered}f(z,t)=\tfrac{P_2}{Q_2},\ P_1=\sqrt{B_0}e^{-\frac{p_1z}{2}-\frac{z^2\eta(Bo+t\eta)}{4B}+t(\eta+\frac{Bp_1^2}{4B_0^2+4B_0t\eta})}, \\ (B_2e^{\frac{1}{2}(\frac{(2A_0(-1+t)c_1)}{(B_0+\eta)(B_0+t\eta)}-p_1z-2Log(cosh((\frac{\sqrt{Bp_1^2+4A_0c_1}(z+A_1)}{2\sqrt{B}}))}s_1+s_0)\qquad, \\ Q_1=(B_1+B_2B_3e^{\frac{A_0(B_0(c_1(-1+t)-c_1t)-c_1t\eta)}{B_0(B_0+\eta)(Bo+t\eta)}})\sqrt{Bo+t\eta},\ z=v,\ \omega(t)=\tfrac{v}{\sqrt{Bo+t\eta}}.\end{gathered} \tag{61}$$

5 Solutions of fractal and fractional FPE

The study of fractional evolution equations occupies a remarkable area in the literature [29-32]. Here the fractal and fractional time derivative FPE is reduced to the classical one's by using the similarity transformations $f(v,t)=\tilde{f}(v,\tau)$ and τ is given by

(a) In the case of fractal derivative $\tau = t^{\beta}$.

(b) In the case CFD $\tau = \frac{\Gamma(2-\alpha)}{\alpha}(T_0^{\alpha}-(T_0-t)^{\alpha})$.

(c) In the case of CFFD $\tau = \frac{(2-\alpha)(1-\alpha)}{2\alpha} Log(\frac{e^{\frac{\alpha}{1-\alpha}(T_0-t)}-1}{e^{\frac{\alpha}{1-\alpha}T_0}-1})$.

(d) In the case of Gawad's FD $\tau = \frac{(\lambda+2)}{2}\int_0^t \frac{1}{\gamma(\frac{1}{\beta},\lambda(T_0-t_1)^\beta)}dt_1$.

The equation (2) is rewritten

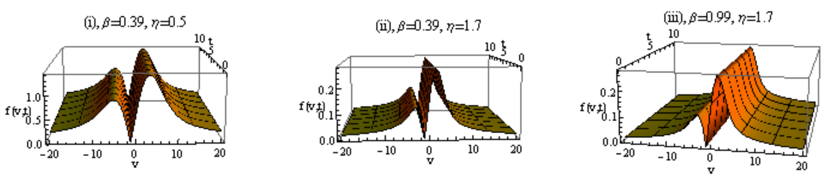
$$\frac{\partial}{\partial \tau}\widetilde{f}(v,\tau) = \eta\frac{\partial}{\partial v}(v\tilde{f}(v,\tau)) + B\frac{\partial^2}{\partial v^2}\tilde{f}(v,\tau), \tag{62}$$

We find two classes of solutions, first by using the similarity transformations and second by considering (54) non autonomous equation in (i) and (ii) respectively.

(I) We use the similarity transformations $f(v,t) = \widetilde{f}(v,\tau)$ and τ is given as follows.

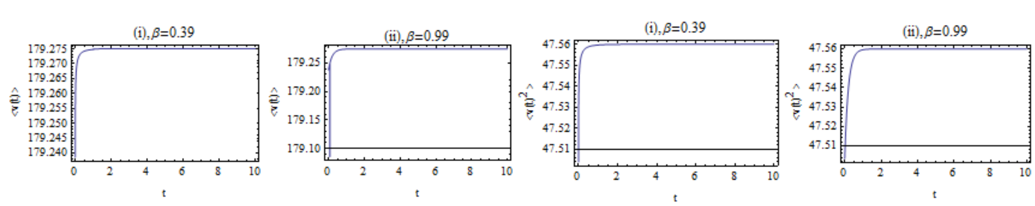
Thus the solutions given in section 5 hold but $t \to \tau$ and τ is replaced by one of the values mentioned in (a)-(d). Now we present some numerical results with relevance to subsection 5.1. We confine ourselves to consider the cases of fractal and CFD.

The results of the solution, given in subsection 5.1 are displayed against v and t, and they are shown in figures 1 (i)-(iii) for different values of β and the friction coefficient η.



Figures 1 (i)-(iii), (i)$B_0 = 1.5, c_{o:0} = 2, n = 10, \eta = 1.7, A_0 = 1.3; A_1 = 2.3, B = 5, m = 2.5, \tau = t^\beta, \beta = 0.39; A_2 = 3, \mu = -0.5, B_1 = 1.9, B_2 = 0.7$. (ii) the same caption as in (i) but $\eta = 1.7$.(iii) the same caption as in (i) but $\beta = 0.99$.

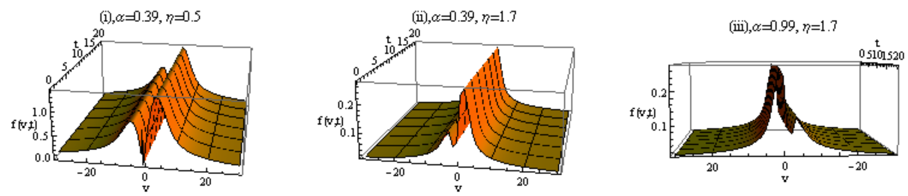
Figures 1 show that the distribution function is mixed-Gaussian's and that the friction coefficient plays a significant role in lowering the magnitude of the distribution density function. While and the effect of vanning the fractal order plays a in lowering the tails



Figures 2 (i) and (ii) show the mean and the mean square of the velocity, when $B_0 = 1.5c_0 = 2, n = 10, \eta = 0.5, A_0 = 1.3, A_1 = 2.3, B = 5, m := 2.5, \tau = t^\beta, \beta = 0.99; A_2 = 3, \mu = -0.5, B_1 = 1.9, B_2 = 0.7$.

The order of the fractal time derivative has no remarkable effect on the mean and the mean square.

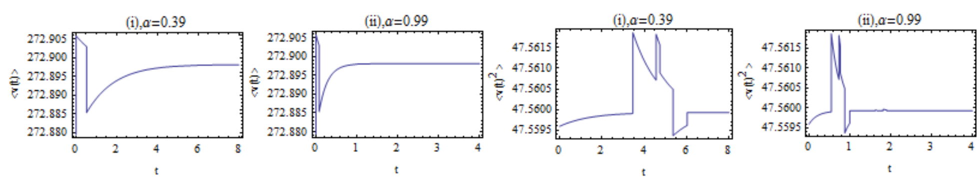
For the solution of fractional FPE in the caputo sense, the results in section 5.1 are displayed against v and t, and they are shown in figures3 (i)-(iii) for different values of α and the friction coefficient η.



Figures 3 (i)-(iii), (i) $B_0 = 1.5, c_0 = 2, n = 10, \eta = 0.5, Ao := 1.3, A_1 = 2.3, B = 5; m = 2.5, \tau = \frac{\Gamma(2-\alpha)}{\alpha}(T_0^{\alpha} - (T_0 - t)^{\alpha}, \alpha = 0.39, T_0 = 20, A_2 := 3, \mu = -0.5, B_1 := 1.9, B_2 := 0.7$. (ii) the same caption as in (i) but $\eta = 1.7$.(iiii) the same caption as in (i) but $\alpha = 0.99$.

Figures 3 show mixed Gaussian's. The friction coefficient plays a dominant role in lowering the magnitude of the distribution function. While and when $\alpha = 0.99$. permutation the Gaussian's occurs.

In figures 4 (i) and (ii0 for the mean and mean square of the velocity are displayed against t, by varying the fractional order.



Figures 4 show the man and mean square for the same caption as in Figs.3.

Figures (i) and (ii), show no remarkable variation in the mean and mean square when varying the fractional order.

6 Conclusions

An approach for finding solutions of linear PDE's with variable coefficients is presented. It is established by transforming the PDE to a system of first order PDE's and the extended unified method is implemented A class of solutions of fractal and fractional Fokker Planck equations are obtained. The solutions show that the distribution function is mixed-Gaussian's. Further the friction coefficient plays the role of lowering the magnitude of the distribution function. On the other hand, varying the order of the fractional time derivative has the effect of permuting the Gaussian's the distribution function.

References

[11] Kazakeviˇcius, R., Ruseckas, J.: Anomalous diffusion in non homogeneous media: power spectral density of signals generated by time-subordinated nonlinear Langevin equations. Physica A 438, 210–222 (2015).

[2] Compte, A.: Continuous time random walks on moving fluids. Phys. Rev. E 55(6), 6821 (1997).

[3] Uchaikin, V.V.: Anomalous transport equations and their application to fractal walking. Physica A 255, 65–92 (1998).

[4] Klages, R., Radons, G., Sokolov, I.M.: Anomalous transport: foundations and applications.Wiley, New York (2008)

[5] Goychuk, I.: Fractional-time random walk sub diffusion and anomalous transport with finite mean residence times: faster, not slower. Phys. Rev. E 86(2), 021113 (2012).

[6] Anderson, J., Kim, E., Moradi, S.: A fractional Fokker-Planck model for anomalous diffusion. Phys. Plasma 21(12), 122109 (2014).

[7] M. Caputo , M. Fabrizio , A new definition of fractional derivative without singular Kernel, Progr. Fract. Differ. Appl. 1, 73–85 (2015).

[8] J. Losada , J.J. Nieto , Properties of the new fractional derivative without singular Kernel, Progr. Fract. Differ. Appl. 1, 87–92 (2015).

[9] A. Atangana, D. Baleanu., New fractional derivatives with non-local and non-singular kernel: theory and applications to heat transfer model Therm Sci, 20 (2016) 763-769.

[10] Tawfik, A. M., Fichtner H.. A., Elhanbaly A., Schlickeiser R.,An analytical study of fractional Klein–Kramers Approximations for describing anomalous diffusion of energetic particles.,J. of Stat. Phys. 174,830–845 (2019).

[11] Moradi, S., Anderson, J., Weyssow, B.: A theory of non-local linear drift wave transport. Phys. Plasmas 18(6), 062106 (2011).

[12] Barkai, E.: Stable equilibrium based on Lévy statistics: stochastic collision models approach. Phys. Rev. E 68(5), 055104 (2003).

[13] Moradi, S., Anderson, J.: Non-local gyrokinetic model of linear ion-temperature-gradient modes. Phys. Plasmas 19(8), 082307 (2012).

[14] Risken, H.: The Fokker-Planck Equation. Springer Series in Synergetics, vol. 18. Springer, Berlin (1989).

[15] Van Kampen, N.G.: Stochastic processes in chemistry and physics. Amsterdam 1, 120–127 (1981).

[16] Chen W., Timespace fabric underlying anomalous diffusion, Chaos, Solitons and Fractals 28, 923929 (2006).

[17] Chen W., Fractional and fractal derivatives modeling of turbulence [J], Arxiv preprint nlin/0511066, (2005).

[18] R. Kanno, Representation of random walk in fractal space-time, Physica A 248, 165175 (1998) .

[19] Al-Musallam, F. A. and Tuan, V. K. "H-Function with Complex Parameters II: Evaluation." Int. J. Math. Math. Sci. b25, 727-743, (2001).

[20] Syama M.I., Al -Refa M., Fractional differential equations with Atangana–Baleanu fractional derivative: analysis and applications.Chaos, Solitons & Fractals 2, 100013 (2019).

[21] Podlubny, I.: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, toMethods of Their Solution and Some of Their Applications, vol. 198. Academic Press, Cambridge (1998)

[22] H. I. Abdel-Gawad , M. Tantawy, D. Baleanu.,.Fractional KdV and Boussenisq-Burger's equations, reduction to PDE and stability approaches. Math. Meth. Appl. Sci. on line.(2020)

[23] H. I. Abdel-Gawad., Approximate solutions of nonlinear fractional equations., Appl. Math. and Comput. 215 (2010) 4094–4100.

[24] H. I. Abdel-Gawad and Mohamed Osman., Exact solutions of KdV equation with space and time dependent coefficients by the extended unified method., Indian J. Pure Appl. Math., 45(1), 1-11 (2014).

[25] H.. I. Abdel-Gawad. , Towards a unified method for exact Solutions of evolution Equations. An application to reaction diffusion equations with finite memory transport, J. Stat. Phys. 147, 506–521 (2012).

[26] H. I. Abdel-Gawad, , N El-Azab, and M. Osman , Exact solution of the space-dependent KdV equation by the extended unified method, JPSP, 82,044004, (2013).

[27] H. I . Abdel-Gawad, A Biswas, Multi-soliton solutions based on interactions of basic traveling waves with an application to the nonlocal Boussinesq equation, Acta. Phys. Pol .B 47 (4), 1101–1112 (2016).

[28] H. .I. Abdel-Gawad , M. Tantawy and R. E. Abo-Elkhair, On the extension of solutions of the real to complex KdV equation and a mechanism for the construction of rogue waves, Wave Random Complex. 26, 397–406 (2016).

[29] D. Kumar , J. Singh , D. Baleanu , M.A. Qurashi , Analysis of logistic equation pertaining to a new fractional derivative with non- singular kernel, Adv. Mech. Eng. 9 (1), 1–8 (2017).

[30] A. Atangana , B.T. Alkahtani , Analysis of non- homogeneous heat model with new trend of derivative with fractional order, Chaos Soliton Fract. 89, 566–571 (2016).

[31] A. Choudhary , D. Kumar , J. Singh , Analytical Solution of fractional differential equations arising in fluid mechanics by using Sumudu transform method, Nonl. Eng. 3 (3), 133–139 (2014) .

[32] J.-S. Duan, R. Rach, D. Baleanu, A. M. Wazwaz., A review of the Adomian decomposition method and its applications to fractional differential equations., Commun. Frac. Calc. 73 - 99 (2012).