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INTERNAL BERNSTEIN FUNCTIONS AND LÉVY-LAPLACE EXPONENTS

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Summary Bertoin, Roynette et Yor [1] described new connections between the class $\mathcal{LE}$ of Lévy-Laplace exponents Ψ (also called the class (sub)critical branching mechanism) and the class of Bernstein functions ($\mathcal{BF}$) which are internal, i.e. those Bernstein functions ϕ s.t. $\Psi\circ\phi$ remains a Bernstein function for every Ψ. We complete their work and illustrate how the class of internal function is rich from the stochastic point of view. It is well known that every $\phi\in\mathcal{BF}$ corresponds univocally to: (i) a subordinator $(X_t)_{t\geq 0}$ (or equivalently to transition semigroups $\big(\mathbb{P}(X_t\in dx)\big)_{t\geq 0}$; (ii) a Lévy measure μ (which controls the jumps of the subordinator). It is also known that, on $(0,\infty)$, the measure $\mathbb{P}(X_t\in dx)/t$ converges vaguely to $\mathrm{d}\delta_0(dx)+\mu(dx)$ as $t\to 0$, where d is the drift term, but rare are the situations where we can compare the transition semigroups with the Lévy measure. Our extensive investigations on the composition of Lévy-Laplace exponents Ψ with Bernstein functions show, for instance, this remarkable facts: ϕ is internal is equivalent to: (a) $\phi^2\in\mathcal{BF}$ or to (b) $t\mu(dx)-\mathbb{P}(X_t\in dx)$ is a positive measure on $(0,\infty)$. We also provide conditions on μ insuring that ϕ is internal. We also show Lévy-Laplace exponents are closely connected to the class of Thorin Bernstein function and provide conditions on μ insuring that ϕ is internal.

KEYWORDS:

Complete monotonicity, Laplace exponent, Bernstein functions, spectrally negative Lévy process, Generalized Gamma convolution, Positive stable density.

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Dedicated to the memory of Marc Yor (24 July 1949 – 9 January 2014)

1. The class $\mathcal{BF}$ of Bernstein functions and some subclasses

We first recall the class of completely monotone function denoted $\mathcal{CM}$. A function $f:(0,\infty)\to(0,\infty)$ is completely if it is infinitely differentiable there satisfies

$$(-1)^n f^{(n)}(x)\geq 0,\quad \text{for all}\, n\in\mathbb{N},\, x>0. \tag{1.1}$$

By Bernstein characterisation, $f\in\mathcal{CM}$ if, and only if, it is the Laplace transform $\mathcal{L}_\tau$ of some measure $\tau(dx)$ on $[0,\infty)$:

$$f(\lambda):=\mathcal{L}_\tau(\lambda)=\int_{[0,\infty)} e^{-\lambda x}\,\tau(dx),\quad \lambda>0. \tag{1.2}$$

The class of Stieltjes functions is defined by $\mathcal{S}:=\mathcal{L}(\mathcal{CM})$ the class of double Laplace transforms. That means that $f\in\mathcal{S}$ if it is represented by

$$f(\lambda):=a+\frac{b}{\lambda}+\int_{(0\,\infty)}\frac{\mathbb{1}}{\lambda+x}\,\tau(dx),\quad \lambda>0, \tag{1.3}$$

where a, $b \geq 0$ and τ is some measure that integrates $1/x$ at infinity. The class of Bernstein function $\mathcal{BF}$, is the class of functions $\phi : [0,\infty) \to [0,\infty)$ such that $\phi' \in \mathcal{CM}$, and then, it is characterized by

$$\mathcal{BF} = \left\{ \phi(\lambda) = \mathrm{q} + \mathrm{d}\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\, \mu(dx) \right\}, \tag{1.4}$$

where in last definitions, $\mathrm{q}, \mathrm{d} \geq 0$, $(\mathrm{q}, \mathrm{d}, \mu)$ is the so-called triplet of characteristics and μ is the Lévy measure of ϕ, i.e. a positive measure on $(0,\infty)$ necessarily satisfying the integrability condition

$$\int_{(0,\infty)} (x \wedge 1)\, \mu(dx) < \infty. \tag{1.5}$$

The following observation constitutes another link between $\mathcal{BF}$ and $\mathcal{CM}$ and will be useful in the sequel: with the notations

$$\overline{\mu}(t) = \mu(t,\infty),\ t > 0 \quad k(x) = \int_0^x \overline{\mu}(t)dt,\ x \geq 0, \tag{1.6}$$

we have the following representations, valid for $\lambda > 0$:

$$\frac{\phi(\lambda)}{\lambda} = \int_0^\infty e^{-\lambda x} \left(\mathrm{d}\delta_0(dx) + (\overline{\mu}(x) + \mathrm{q})\, dx \right), \qquad \frac{\phi(\lambda)}{\lambda^2} = \int_0^\infty e^{-\lambda x} \left(\mathrm{d} + \mathrm{q}x + k(x) \right) dx. \tag{1.7}$$

A stochastic interpretation of the class $\mathcal{BF}$ is the following: if $\phi \in \mathcal{BF}$ and $\phi(0) = 0$, then ϕ is the Lévy-Khintchine exponent of a subordinator $X = (X_t)_{t\geq 0}$, i.e. a non-decreasing Lévy process starting from 0, s.t.

$$\mathbb{E}[e^{-\lambda X_t}] = \int_{[0,\infty)} e^{-\lambda x}\, \mathbb{P}(X_t \in dx) = e^{-t\,\phi(\lambda)}, \qquad \text{for all } t,\ \lambda \geq 0.$$

A generic example of subordinators is given by the well know α-stable subordinators $(S_t^\alpha)_{t\geq 0}$, $0 < \alpha < 1$: the function $\phi(\lambda) = \lambda^\alpha,\ \lambda \geq 0$, is a Bernstein function (actually it is a Thorin Bernstein function, see (1.13) below) which is associated to the so-called positive stable random variable $\mathbf{S}_\alpha := S_1^\alpha$ through the representations:

$$\lambda^\alpha = \int_0^\infty (1 - e^{-\lambda x}) \frac{c_\alpha}{x^{\alpha+1}}\, dx, \qquad c_\alpha = \frac{\alpha}{\Gamma(1-\alpha)} = \frac{1}{\pi}\Gamma(\alpha+1)\sin(\pi\alpha) \tag{1.8}$$

$$e^{-\lambda^\alpha} = \mathbb{E}[e^{-\lambda \mathbf{S}_\alpha}] = \int_0^\infty e^{-\lambda s} f_\alpha(s)\, ds, \tag{1.9}$$

where f_α is the probability density function of $\mathbf{S}_\alpha$. Notice that f_α is explicit only for $\alpha = 1/2$ where it is given by

$$f_{1/2}(x) = \frac{e^{-\frac{1}{4x}}}{2\sqrt{\pi x^3}}, \qquad x > 0. \tag{1.10}$$

For $\alpha \neq 1/2$, formula [10, (2.4.8, p. 90)] gives an evaluation of f_α through the series expansion:

$$f_\alpha(x) = \frac{1}{\pi} \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{\Gamma(n\alpha+1)}{\Gamma(n+1)} \sin(\pi n\, \alpha)\, x^{-(n\alpha+1)}. \tag{1.11}$$

An important subclass of $\mathcal{BF}$ is the class $\mathcal{CBF}$ of complete Bernstein functions which corresponds of those functions $\phi \in \mathcal{BF}$ s.t. the associated Lévy measure μ has the form $\mu(dx) =$

$l(x)dx$, where $l \in \mathcal{CM}$. The class $\mathcal{CBF}$ has tight connection with the class of Stieltjes functions $\mathcal{S}$ because

$$\phi \in \mathcal{CBF} \Longleftrightarrow \lambda \mapsto \frac{\phi(\lambda)}{\lambda} \in \mathcal{S} \Longleftrightarrow \lambda \mapsto \frac{1}{\phi(\lambda)} \in \mathcal{S} \Longleftrightarrow \lambda \mapsto \phi\left(\frac{1}{\lambda}\right) \in \mathcal{S}. \tag{1.12}$$

The class $\mathcal{TBF}$ of Thorin Bernstein functions corresponds of those functions $\phi \in \mathcal{CBF}$ s.t. their Lévy measure μ has the form $\mu(dx) = l(x)dx$, where $x \mapsto xl(x) \in \mathcal{CM}$. The class $\mathcal{TBF}$ has also a tight connection with the class $\mathcal{S}$, since

$$\phi \in \mathcal{TBF} \Longleftrightarrow \phi \geq 0 \quad \text{and} \quad \phi' \in \mathcal{S}. \tag{1.13}$$

See [7, Chpters 6 and 8] for more account on $\mathcal{CBF}$ and $\mathcal{TBF}$. From Corollary 2.4 [3] (cited as Proposition 3.6 in the book of Schilling & al. [7]), we have that $\lambda \mapsto \sqrt{\lambda}\phi(\sqrt{\lambda})$ is always complete Bernstein function for ever $\phi \in \mathcal{BF}$. We improve the last result , with a method different from the one of [3] and [7] by the following:

Theorem 1.1. *Let ϕ be a Bernstein function with triplet of characteristics $(\mathrm{q}, \mathrm{d}, \mu)$. Then, it holds that*

1) $\lambda \mapsto \sqrt{\lambda}\,\phi(\sqrt{\lambda}) \in \mathcal{CBF}$;
2) If furthermore $\mu(dx) = k(x)dx$ where k is a non-increasing function, then

$$\lambda \mapsto \sqrt{\lambda}\,\phi(\sqrt{\lambda}) \in \mathcal{TBF}.$$

The following lemma is the key for proving Theorem 1.1.

Lemma 1.2. *If $f \in \mathcal{CM}$ and its representative measure given by (1.2) has the form $\tau(dx) = a\delta_0(dx) + L(x)dx$, $x > 0$, where $a \geq 0$ and L is a non-increasing function, then*

(1) $\lambda \mapsto f(\sqrt{\lambda}) \in \mathcal{S}$;
(2) If further $L(x) = \int_x^\infty l(t)dt$, and l is also a non-increasing function, then

$$\lambda \mapsto \lambda f(\sqrt{\lambda}) \in \mathcal{TBF}.$$

Proof. 1) We start by observing that

$$x \quad \mapsto \quad g(x) = \frac{1 - e^{-1/x}}{\sqrt{x}} \in \mathcal{CM} \tag{1.14}$$

$$\lambda \quad \mapsto \quad h(\lambda) := \frac{1 - e^{-\sqrt{\lambda}}}{\sqrt{\lambda}} = \frac{1}{2\sqrt{\pi}}\mathcal{L}_g\left(\frac{\lambda}{4}\right) \in \mathcal{S} \tag{1.15}$$

Assertion (1.14) and (1.15) could be extracted from identity 3.723 (2) p. 424 [**?**]: we have

$$1 - e^{-\lambda} = \frac{2\lambda}{\pi}\int_0^\infty \frac{1 - \cos u}{\lambda^2 + u^2}du, \quad \lambda \geq 0 \tag{1.16}$$

which yields

$$g(\lambda) := \frac{1 - e^{-1/\lambda}}{\sqrt{\lambda}} = \frac{2}{\pi}\int_0^\infty \frac{\sqrt{\lambda}}{(\lambda^2 u^2 + 1)}(1 - \cos u)du, \quad \lambda > 0.$$

Notice that we can not conclude for g at this stage because the function $\lambda \mapsto \sqrt{\lambda}/(\lambda^2+1)$ is not in $\mathcal{CM}$ because it is null at 0 and at infinity. We will avoid this problem using again representation (1.16) and notice that

$$h(\lambda) := \frac{1-e^{-\sqrt{\lambda}}}{\sqrt{\lambda}} = \frac{2}{\pi}\int_0^\infty \frac{1-\cos u}{\lambda+u^2}du \in \mathcal{S}.$$

Now, taking the Laplace transform of g and performing the change of variable $u = 1/v$, we get

$$\begin{aligned}
\mathcal{L}(g)(\lambda) &= \int_0^\infty e^{-\lambda u}\frac{1-e^{-1/u}}{\sqrt{u}}du = \frac{\sqrt{\pi}}{\sqrt{\lambda}} - J(\lambda) \\
J(\lambda) &:= \int_0^\infty e^{-\lambda u}\frac{e^{-1/u}}{\sqrt{u}}du = \frac{1}{\sqrt{\lambda}}\int_0^\infty e^{-1/v}\frac{e^{-\lambda v}}{\sqrt{v^{3/2}}}du =: \frac{K(\lambda)}{\sqrt{\lambda}}.
\end{aligned}$$

Notice that $K'(\lambda) = -J(\lambda)$, $\lambda > 0$, and that $K(0) = \sqrt{\pi}$ which gives $K(\lambda) = \sqrt{\pi}e^{-2\sqrt{\lambda}}$. Finally deduce (1.15) through

$$\lambda \mapsto \mathcal{L}_g(\lambda) = \sqrt{\pi}\frac{1-e^{-2\sqrt{\lambda}}}{\sqrt{\lambda}} = 2\sqrt{\pi}\,h(4\lambda) \in \mathcal{S},$$

which is equivalent, by definition of the class $\mathcal{S}$, to $g \in \mathcal{CM}$. Now, we can tackle the proof of the first assertion in the Lemma: defining the measure ρ by $\rho(x,\infty) = L(x) - L(\infty)$, $x > 0$ and using Fubini's theorem, we obtain that f is represented by

$$f(\lambda) = a + \frac{L(\infty)}{\lambda} + \int_{(0,\infty)} \frac{1-e^{-\lambda t}}{\lambda}\rho(dt) \tag{1.17}$$

so that, with h given in (1.15), we have

$$\lambda \mapsto f(\sqrt{\lambda}) = a + \frac{k(\infty)}{\sqrt{\lambda}} + \int_{(0,\infty)} h(\lambda t^2)\,t\,\rho(dt) \in \mathcal{S}$$

because the functions $\lambda \mapsto 1/\sqrt{\lambda}$, $h(\lambda t^2)$, $t > 0$, belong to the convex cone $\mathcal{S}$.

2) For the second assertion, notice that $L(\infty) = 0$, and necessarily $l(\infty) = 0$, otherwise, L would not be defined. Then, denote by η the measure defined by $\eta(t,\infty) = l(t)$ and insert $\rho(dt) = l(t)dt$ in (1.17) so that

$$\lambda f(\sqrt{\lambda}) = a\sqrt{\lambda} + \sqrt{\lambda}\int_0^\infty (1-e^{-\sqrt{\lambda}t})\int_t^\infty \eta(ds)\,dt,$$

Thus, Fubini's theorem gives

$$\lambda f(\sqrt{\lambda}) = a\sqrt{\lambda} + \int_{(0,\infty)} H(\lambda s)\,\eta(ds), \quad \text{with} \quad H(\lambda) := e^{-\sqrt{\lambda}} - 1 + \sqrt{\lambda}.$$

Now, notice that $H' = h$, where h is the Stieltjes function given by (1.15), use (1.13) to conclude that $H \in \mathcal{TBF}$ and finally use the last representation of $\lambda \mapsto \lambda f(\sqrt{\lambda})$, to conclude that it belongs to the convex cone $\mathcal{TBF}$. $\square$

Proof of Theorem 1.1. Since $\lambda \mapsto \sqrt{\lambda}$ and λ are both Thorin functions and then, complete Bernstein functions, there is no loss of generality to take the characteristics q, d both equal to 0. Using formula (1.6), we have the

$$\sqrt{\lambda}\phi(\sqrt{\lambda}) = \lambda f(\sqrt{\lambda}), \quad f(\lambda) = \int_0^\infty e^{-\lambda x}\overline{\mu}(x)dx.$$

Lemma 1.2 applied on the function f, with $k(x) = \overline{\mu}(x)$, asserts that $\lambda \mapsto f(\sqrt{\lambda}) \in \mathcal{S}$, and we conclude by (1.12). For the second assertion conclude similarly with the second statement of Lemma 1.2. □

2. The class $\mathcal{LE}$ of Lévy-Laplace exponents and some subclasses

The class of Lévy-Laplace exponents has been analytically considered in [1] and is also called the class of branching mechanisms for (sub)critical continuous state branching processes. It is defined by the convex cone

$$\mathcal{LE} := \{\Psi(\lambda) = \mathrm{a} + \mathrm{q}\lambda + \mathrm{d}\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x)\nu(dx), \quad \lambda \geq 0\} \tag{2.1}$$

where $\mathrm{a}, \mathrm{q}, \mathrm{d} \geq 0$ and ν is a positive measure on $(0,\infty)$ fulfilling

$$\int_{(0,\infty)} \left(x \wedge x^2\right)\nu(dx) < \infty. \tag{2.2}$$

and we shall call $(\mathrm{a}, \mathrm{q}, \mathrm{d}, \mu)$ is the quadruple of characteristics of Ψ. Notice that $\Psi' \in \mathcal{BF}$ and an integration by parts gives that $\lambda \mapsto \Psi(\lambda)/\lambda \in \mathcal{LE}$: represented by

$$\Psi(\lambda) = \lambda\phi(\lambda), \quad \text{where} \quad \phi(\lambda) = \mathrm{q} + \mathrm{d}\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\overline{\nu}(x)dx, \tag{2.3}$$

where, as in (1.6), $\overline{\nu}(x) := \nu(x,\infty)$. A stochastic interpretation of the class $\mathcal{LE}$ is the following: if $\Psi \in \mathcal{LE}$ and $\Psi(0) = 0$, then it is the Laplace exponent of a spectrally negative Lévy process $Z = (Z_t)_{t\geq 0}$ i.e. a Lévy process with no positive jumps that do not drift to $-\infty$. This terminology stems from the fact the Lévy measure of Z, $\Pi(dx)$ in [4, Theorem 1.6] gives no mass to $(0,\infty)$. The measure $\nu(dx)$ in the representation (2.1) is simply obtained as the image of $\Pi(dx)$ by $x \mapsto -x$. So, we always have the Laplace representation

$$e^{t\Psi(\lambda)} = \mathbb{E}[e^{\lambda Z_t}], \quad \text{for } t,\ \lambda \geq 0. \tag{2.4}$$

A first tight link of $\mathcal{LE}$ with $\mathcal{BF}$ is the following: Since any function Ψ is a bijection, then its inverse $\varphi = \Psi^{-1}$ is in $\mathcal{BF}$, i.e is the Lévy-Khintchine exponent of a subordinator, namely the *first passage time* process $X = (X_t)_{t\geq 0}$ defined by $X_t := \inf\{s > 0,\ \text{s.t. } Z_s > t\}$, see [4, Corollary 3.14]. Another known fact is that this function φ is also a special Bernstein function, i.e.

$$\varphi \in \mathcal{BF} \quad \text{and} \quad \lambda \mapsto \frac{\lambda}{\varphi(\lambda)} \in \mathcal{BF}.$$

An analytical interpretation of $\mathcal{LE}$ is that any of its member Ψ are non-negative anti-derivative of functions in $\mathcal{BF}$. This is easily seen by representations (1.4) and (2.1).

As a second consequence of Theorem 1.1, we deduce a second tight link between the classes $\mathcal{LE}$ and $\mathcal{BF}$:

Corollary 2.1. *Every $\Psi \in \mathcal{LE}$ satisfies $\lambda \mapsto \Psi(\sqrt{\lambda}) \in \mathcal{TBF}$.*

Proof. Representation (2.3) gives $\Psi(\sqrt{\lambda}) = \sqrt{\lambda}\phi(\sqrt{\lambda})$ such that the Lévy measure of ϕ has a non-increasing density and the assertion is an immediate consequence of Theorem 1.1. □

In the same direction, we will exhibit a third tight link between the classes $\mathcal{LE}$ and $\mathcal{BF}$ as done by Bertoin, Roynette and Yor [1]:

Definition 2.2. [1] *A Bernstein function ϕ is said to be internal, if*

$$\Psi \circ \phi \in \mathcal{BF}, \quad \textit{for every } \Psi \in \mathcal{LE}.$$

Theorem 2.3. *Let ϕ be a Bernstein function. The following assertions are equivalent:*

(1) ϕ is internal;
(2) $\phi^2 \in \mathcal{BF}$;
(3) $\phi_1(\phi)\phi_2(\phi) \in \mathcal{BF}$ for all $\phi_1, \phi_2 \in \mathcal{BF}$;
(4) For every $\Psi \in \mathcal{LE}$, $\Psi(\phi)$ is the composition of some Thorin Bernstein ψ with ϕ^2.

Proof. The equivalence between (4) $\Longleftrightarrow$ (1) is a consequence of Corollary 2.1 when taking $\psi(\lambda) = \Psi(\sqrt{\Psi})$. For (3) $\Longrightarrow$ (1) just use representation (2.3) with $\phi_1(\lambda) = \lambda$ and $\phi_2 \in \mathcal{JBF}$. So that $\Psi = \phi_1\phi_2 \in \mathcal{LE}$. (1) $\Longrightarrow$ (2) is proved by taking $\Psi(\lambda) = \lambda^2$. (2) $\Longrightarrow$ (3) is treated as follows: using Corollary 3.8 (vi) [7], we know that

$$\lambda \mapsto \phi_1(\sqrt{\lambda})\phi_2(\sqrt{\lambda}) \in \mathcal{BF}$$

and we compose with ϕ^2. □

Corollary 3.9 [7] on closure of the class $\mathcal{BF}$ under pointwise limits, also insures the following:

Proposition 2.4. *Let $\phi : [0,\infty) \to [0,\infty)$. The following assertions are equivalent:*

(a) ϕ is an internal Bernstein function;
(b) Thee exist a sequence $(t_n)_n$ and two Bernstein functions ϕ_1, ϕ_2 such that

$$\lim_{n\to\infty} t_n = 0, \quad \lim_{\lambda\to 0+} \frac{\phi_i(\lambda)}{\lambda} = c_i > 0 \quad \textit{and} \quad \phi_1(t_n\phi)\,\phi_2(t_n\phi) \in \mathcal{BF}.$$

Example 2.5. *Let $\phi : [0,\infty) \to [0,\infty)$. Then ϕ is an internal Bernstein function is internal if one of the following holds for all $t > 0$,*

(a) $\Psi_0(t\phi) \in \mathcal{BF}$ where $\Psi_0(\lambda) = e^{-\lambda} - 1 + \lambda \in \mathcal{LE}$;
(b) $\Psi_1(t\phi) \in \mathcal{BF}$ where $\Psi_1(\lambda) = \lambda - \log(1+\lambda) \in \mathcal{LE}$;
(c) $\Psi_2(t\phi) \in \mathcal{BF}$ where $\Psi_2(\lambda) = \lambda(1 - e^{-\lambda})$.

3. Stochastic characterization of internality

In what follows, we propose a deeper stochastic point of view of internality:

Theorem 3.1. *Let ϕ be a Bernstein function with characteristics $(0, \mathrm{d}, \mu)$ and associated with the subordinator $(X_t)_{t\geq 0}$. The following assertions are equivalent:*

(1) ϕ *is internal;*
(2) The measure $t\,\mu(dx) - \mathbb{P}(X_t \in dx)$ *is positive;*
(3) The function $x \mapsto t\,\overline{\mu}(x) - \mathbb{P}(X_t > x)$ *is nonnegative and non-increasing;*
(4) There exists a subordinator $(Y_t)_{t\geq 0}$ *and a constant* $c \geq 0$ *such that*

$$\mu(dx) = c\int_0^\infty \mathbb{P}(Y_t \in dx)\,\frac{dt}{t^{\frac{3}{2}}}$$

and $(X_t)_{t\geq 0}$ *has the same distribution as* $(Y_t)_{t\geq 0}$ *subordinated by an independent* $\frac{1}{2}$*-stable process* $(S_t^{\frac{1}{2}})_{t\geq 0}$*, viz.*

$$(X_t)_{t\geq 0} \overset{d}{=} \left((Y \circ \mathbf{S}^{\frac{1}{2}})_t\right)_{t\geq 0}.$$

Remark 3.2. *The results obtained in Theorems and 3.1 merit the following comments:*

(i) Equivalence $(1) \Longleftrightarrow (4)$ *of Theorem 3.1 and the implication (1) $\Longrightarrow$ (2) of Theorem 3.2 have been given in* [1, Theorem 4] *with a different proof based on the transition of the stable distributions.*
(ii) Controls of the type (2) or (3) in Theorem 3.2 are of big importance in the theory of Lévy process. Indeed, rare are the cases where controls of the finite-dimensional distributions of a subordinator (see [2] for instance) or even conditions insuring absolute continuity (see [9] for instance) are available. It is known that, on $(0,\infty)$*, the measure* $\mathbb{P}(X_t \in dx)/t$ *converges vaguely to* $\mathrm{d}\delta_0(dx) + \mu(dx)$ *as* $t \to 0$*, where* d *is the drift term, but rare are the situations where we can compare the transition semigroup* $\mathbb{P}(X_t \in dx)$ *with the Lévy measure. Theorem 3.1 actually shows that the internality condition, the measure* $t\mu(dx)$ *dominates* $\mathbb{P}(X_t \in dx)$*.*

It is clear that every stable Bernstein function $\phi(\lambda) = \lambda^\alpha$, $\alpha \in (0, \frac{1}{2}]$, is internal. So, using representations (1.8) and (1.11) we immediately get this additional information from Theorem 3.1:

Corollary 3.3. *Let f_α the positive stable density given by (1.9). The following conditions are equivalent:*

(1) $0 < \alpha \leq 1/2$;
(2) $x^{\alpha+1}\, f_\alpha(x) \leq \dfrac{\alpha}{\Gamma(1-\alpha)}$, *for all* $x > 0$;
(3) $\displaystyle\sum_{n=2}^{+\infty} (-z)^n\, \frac{\Gamma(n\alpha+1)}{\Gamma(n+1)}\, \sin(\pi n\,\alpha) \geq 0$, *for all* $z > 0$.

Proof of Theorem 3.1. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3): Observe that ϕ is a Bernstein function, with characteristics $(0, \mathrm{d}, \mu)$,associated to some subordinator $(X_t)_{t\geq 0}$ viz.

$$\mathbb{E}\big[e^{-\lambda X_t}\big] = e^{-t\phi(\lambda)}, \quad \text{for all } \lambda, t \geq 0. \tag{3.1}$$

By Example 2.5, we know that internality of ϕ is equivalent to $\Psi_0(t\phi) \coloneqq t\phi - (1 - e^{-t\phi}) \in \mathcal{BF}$ for every $t > 0$. Then,

$$\begin{aligned}
\Psi_0(t\phi)(\lambda) &= t\phi(\lambda) - (1 - \mathbb{E}[e^{-\lambda X_t}]) \\
&= t\mathrm{d}\lambda + t\int_{(0,\infty)} (1 - e^{-\lambda x})\mu(dx) - \int_{(0,\infty)} (1 - e^{-\lambda x})\mathbb{P}(X_t \in dx) \\
&= t\mathrm{d}\lambda + \int_{[0,\infty)} (1 - e^{-\lambda x})\big(t\mu(dx) - \mathbb{P}(X_t \in dx)\big) \\
&= t\mathrm{d}\lambda + \lambda\int_{(0,\infty)} e^{-\lambda x}\big(t\overline{\mu}(x)\big) - \mathbb{P}(X_t > x)\big)dx,
\end{aligned}$$

where the latter is obtained by the integration by parts. Since $\Psi_0(t\phi) \in \mathcal{BF}$, then, $t\mu(dx) - \mathbb{P}(X_t \in dx)$ is a Lévy measure, by or by [7, Remark 3.3 ii)] this implies that $t\overline{\mu}(x) - \mathbb{P}(X_t > x)$ is the right quue of a Lévy measure, that is a positive and non-increasing function. This latter is equivalent to 2) or 3).

(1) $\Leftrightarrow$ (4): By Theorem 3.2, Internality of ϕ is equivalent to $\phi^2 \in \mathcal{BF}$. Let the subordinator $(Y_t)_{t\geq 0}$ associated to the Bernstein function ϕ^2. Let $\mathrm{q} = 0$, there is no loss of generality in assuming that $\mathrm{d} = 0$. We have

$$\mathbb{E}\big[e^{-\lambda X_t}\big] = e^{-t\phi(\lambda)} = e^{-t\sqrt{\phi^2(\lambda)}} = \mathbb{E}\left[e^{-\lambda (Y\circ \mathbf{S}^{\frac{1}{2}})_t}\right]$$

The subordination relation $(X_t)_{t\geq 0} \overset{d}{=} \left((Y \circ \mathbf{S}^{\frac{1}{2}})_t\right)_{t\geq 0}$ together with representation (1.8) and Sato's subordination theorem (Theorem 30.1 [6]), give

$$\begin{aligned}
\phi(\lambda) &= \int_{(0,\infty)} \left(1 - e^{-t\phi^2(\lambda)}\right) \frac{c_{\frac{1}{2}}}{t^{\frac{3}{2}}}\, dt \\
&= \int_{(0,\infty)} \left(1 - \mathbb{E}[e^{-\lambda Y_t}]\right) \frac{c_{\frac{1}{2}}}{t^{\frac{3}{2}}}\, dt \\
&= \int_{(0,\infty)} \left(1 - e^{-\lambda x}\right) \int_0^\infty \mathbb{P}(Y_t \in dx)\, \frac{c_{\frac{1}{2}}}{t^{\frac{3}{2}}}\, dt.
\end{aligned}$$

With $c = c_{\frac{1}{2}} = \frac{1}{2\sqrt{\pi}}$, we deduce that the Lévy measure $\mu(dx)$ is represented by

$$\mu(dx) = c\int_0^\infty \mathbb{P}(Y_t \in dx)\, \frac{dt}{t^{\frac{3}{2}}}.$$

□

4. Conditions on the Lévy measure insuring internality

The following remarkable fact will be used several times in the sequel:

Proposition 4.1. *Let $f:(0,\infty)\to(0,\infty)$. Then*

(1) If f is concave then it is non-decreasing.
(2) f is concave if, and only if, $x\mapsto xf(1/x)$ is concave.

The second statement is Lemma 2.2 in [5]. We were not able to find a reference for the first statement, which is probably known in the literature, and we propose this simple proof.

Proof. Since f has necessarily non-increasing slopes:

$$\frac{f(y)-f(x)}{y-x}\geq\frac{f(z)-f(x)}{z-x},\quad 0\leq x<y<z,$$

then, because $f(z)\geq 0$, we get

$$f(y)\geq f(x)+\frac{y-x}{z-x}(f(z)-f(x))\geq\frac{z-y}{z-x}f(x),$$

for fixed x, y and arbitrary big values of z. It sufficies to let $z\to\infty$. $\square$

We go back to internality and give sufficient conditions for it:

Proposition 4.2. *Let $\mu(dx)$ be a Lévy measure and denote*

$$g(x)=\sqrt{x}\,\overline{\mu}(x),\quad x>0. \tag{4.1}$$

(1) The function $x\mapsto g(x^2)$ is concave if, and only if, $\mu(dx)$ has a density function of the form

$$\frac{\mu(dx)}{dx}=\frac{r(x)}{x^{\frac{3}{2}}},\quad \textit{where } r \textit{ is a non-decreasing function.}$$

(2) If $x\mapsto g(x^2)$ is concave, then $x\mapsto g(ax)\,g(bx)$ is concave for every $a,b>0$.

Proof. Observe that concavity of $x\mapsto g(x^2)$ implies the one of g and, by Proposition 4.1, b is non-decreasing. We will see that actually, g is differentiable on $(0,\infty)$ and

$$\textit{concavity of } x\mapsto g(x^2) \textit{ is equivalent to nonincreaseness of } x\mapsto\sqrt{x}g'(x). \tag{4.2}$$

Proof of (1): For the necessity part, observe that

$$dg(x)=\frac{1}{2\sqrt{x}}\overline{\mu}(x)dx-\sqrt{x}\mu(dx)$$

is a positive measure, then μ admits a density function that we write on the form $x^{-3/2}r(x)$ where r is a nonnegative measurable function and g is represented by

$$g(x)=\sqrt{x}\int_x^\infty\frac{r(t)}{t^{\frac{3}{2}}}dt,\quad x>0. \tag{4.3}$$

Because g is differentiable $\frac{d}{dx}g(x^2)=2xg'(x^2)$ is non-increasing, we find that

$$x\mapsto -2\sqrt{x}g'(x)=2\frac{r(x)}{\sqrt{x}}-\int_x^\infty\frac{r(t)}{t^{\frac{3}{2}}}dt,\quad \text{is non-increasing.}$$

Thereafter, trivial differentiation gives that

$$d(-\sqrt{x}g'(x)) = \frac{dr(x)}{\sqrt{x}}, \quad x > 0, \quad \text{is a positive measure,} \tag{4.4}$$

thus r is non-decreasing. For the sufficiency part, observe that if μ admits a density function that we write on the form $x^{-3/2}r(x)$, then the function g is necessarily represented by (4.3) and reading the arguments from (4.4) backward allows to conclude.
Proof of (2): Differentiating the function $x \mapsto h(x) := g(ax)g(bx) = \sqrt{ab}\,x\,\overline{\mu}(ax)\,\overline{\mu}(bx)$, we obtain, by the discussion in point 1), that

$$x \mapsto h'(x) = \sqrt{ab}\left(\sqrt{ax}g'(ax)\,\overline{\mu}(bx) + \sqrt{bx}g'(bx)\,\overline{\mu}(ax)\right)$$

is a non-increasing function. □

Theorem 4.3. *Let ϕ be a Bernstein function with characteristics $(\mathrm{q},\mathrm{d},\mu)$ and the function g given by (4.1).*

(1) The function ϕ is internal if and only if $\mathrm{d} = 0$ and the convoluted function $\overline{\mu} \star \overline{\mu}$ is concave.
(2) Assume the g-function given in (4.1) is such that $x \mapsto g(x^2)$ is concave, or less restrictively assume that $x \mapsto g(ax)g(bx)$ is concave for every $a, b > 0$. Then ϕ is internal.

Proof. (1) Recall the form (3) of ϕ and that the function $g(x) = \sqrt{x}\overline{\mu}(x)$, $x > 0$ introduced in Proposition 4.2. Without loss of generality, we may assume that $\mathrm{q} = 0$. The square of ϕ produces a term of the form $d^2\lambda^2$ which will not allow ϕ^2 to be Bernstein function unless $\mathrm{d} = 0$. Now, use representation (3) and write

$$\phi^2(\lambda) = \left(\lambda \int_0^\infty e^{-\lambda x}\,\overline{\mu}(x)\,dx\right)^2 = \lambda^2 \int_0^\infty e^{-\lambda x}\,(\overline{\mu} \star \overline{\mu})(x)\,dx.$$

The function ϕ^2 meets the form (1.7) if and only if $\overline{\mu} \star \overline{\mu}$ is concave.
(2) We have to check that $x \mapsto k(x) = (\overline{\mu} \star \overline{\mu})(x)$, $x > 0$, is concave. For that, simply write, with obvious change of variable

$$\begin{aligned} k(x) &= \int_0^x \overline{\mu}(x-y)\,\overline{\mu}(y)\,dy = \int_0^1 x\overline{\mu}(xz)\,\overline{\mu}(x(1-z))\,dz \\ &= \int_0^1 g(xz)\,g(x(1-z))\,\frac{dz}{\sqrt{z(1-z)}}, \end{aligned}$$

and conclude with point 2) of Proposition 4.2. □

Proposition 4.4. *Let ϕ be a Bernstein function with characteristics $(\mathrm{q},\mathrm{d},\mu)$.*

(1) The function $\lambda \mapsto \lambda\phi'(\lambda)$ is an internal Bernstein function if and only if $\mathrm{d} = 0$ and the convoluted measure $(x\mu) \star (x\mu)$ has a concave density function.
(2) Assume μ that has a density function of the form $\frac{\mu(dx)}{dx} = \frac{r(x)}{x^{\frac{3}{2}}}$. Then $(i) \Rightarrow (ii) \Rightarrow (iii)$, where
  (i) $x \mapsto r(x^2)$ is differentiable, concave and $x \mapsto \frac{r(x)}{\sqrt{x}}$ is non-increasing;
  (ii) $x \mapsto r(ax)r(bx)$ is concave for every $a, b > 0$;

(iii) $\lambda \mapsto \lambda\phi'(\lambda)$ *is an internal Bernstein function.*

(3) If condition (ii) *holds, then* ϕ *is also internal.*

Proof. Reasoning as in the proof of Theorem 4.3, it is immediate that d should be equal to zero. We may assume that q = 0 without loss of generality.

(1) The assertion comes from representation (1.7) and from the following one:

$$(\lambda\phi'(\lambda))^2 = \lambda^2 \int_{(0,\infty)} e^{-\lambda x} \left((x\mu) \star (x\mu)\right)(dx).$$

(2) Making the change of variable $y = xz$ and writing

$$\frac{(x\mu) \star (x\mu)(dx)}{dx} = \int_0^x \frac{r(x-y)}{\sqrt{x-y}} \frac{r(y)}{\sqrt{y}} dy = \int_0^x \frac{r(xz) r\left((1-z)x\right)}{\sqrt{z(1-z)}} dz,$$

we obtain that condition (ii) is sufficient for (iii). Condition (i) yields (ii) due to Proposition 4.1: concavity of $x \mapsto r(x^2)$ implies that $x \mapsto \sqrt{x} r'(x)$ is non-increasing and positive, hence

$$\frac{d}{dx} r(ax)\, r(bx) = \sqrt{ab}\left[\sqrt{ax}\, r'(ax)\, \frac{r(bx)}{\sqrt{bx}} + \sqrt{bx}\, r'(bx)\, \frac{r(ax)}{\sqrt{ax}}\right]$$

is a non-increasing function and finally $x \mapsto r(ax)\, r(bx)$ is concave.

(3) It is enough to check the second assertion of Theorem 4.3: by an elementary change of variables, we obtain the concavity of the function

$$x \mapsto g(ax) g(bx) = \sqrt{ab} x\, \overline{\mu}(ax) \overline{\mu}(bx) = \int_1^\infty \int_1^\infty \frac{r(axs) r(bxt)}{(st)^{\frac{3}{2}}}\, ds\, dt.$$

□

5. The proofs

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