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RESEARCH ARTICLE

N-dimensional Heisenberg's uncertainty principle for fractional Fourier transform

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A sharper uncertainty inequality which exhibits a lower bound larger than that in the classical N-dimensional Heisenberg's uncertainty principle is obtained, and extended from N-dimensional Fourier transform domain to two N-dimensional fractional Fourier transform domains. The conditions that reach the equality relation of the uncertainty inequalities are deduced. Example and simulation are performed to illustrate that the newly derived uncertainty principles are truly sharper than the existing ones in the literature. The new proposals' applications in time-frequency analysis and optical system analysis are also given.

KEYWORDS:
Heisenberg's uncertainty principle, fractional Fourier transform, time-frequency analysis, optical system analysis

1 | INTRODUCTION

Uncertainty principle plays an important role in harmonic analysis, quantum mechanics, and time-frequency analysis.[1,2,3] The classical N-dimensional Heisenberg's uncertainty principle is given by the inequality[4,5]

$$\int_{\mathbb{R}^N} \|\mathbf{x}-\mathbf{a}\|^2 |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \int_{\mathbb{R}^N} \|\mathbf{w}-\mathbf{b}\|^2 \left|\hat{f}(\mathbf{w})\right|^2 \mathrm{d}\mathbf{w} \geq \frac{N^2}{16\pi^2}\|f\|_2^4 \tag{1}$$

for any $f(\mathbf{x}) \in L^2(\mathbb{R}^N)$ equipped with a natural norm $\|\cdot\|_2 = \left(\int_{\mathbb{R}^N} |\cdot(\mathbf{x})|^2 \mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}$ known as the L^2-norm, and any $\mathbf{a} = (a_1, a_2, \cdots, a_N), \mathbf{b} = (b_1, b_2, \cdots, b_N) \in \mathbb{R}^N$ equipped with the 2-norm $\|\cdot\| = \sqrt{(\cdot)(\cdot)^{\mathrm{T}}}$, where T denotes the transpose operator. The function $\hat{f}(\mathbf{w})$ denotes the N-dimensional Fourier transform (FT) of $f(\mathbf{x})$,[6]

$$\hat{f}(\mathbf{w}) = \int_{\mathbb{R}^N} f(\mathbf{x}) e^{-2\pi i \mathbf{x}\mathbf{w}^{\mathrm{T}}} \mathrm{d}\mathbf{x}, \tag{2}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_N)$, $\mathbf{w} = (\omega_1, \omega_2, \cdots, \omega_N)$, and $\mathbf{x}\mathbf{w}^{\mathrm{T}} = \sum_{k=1}^{N} x_k \omega_k$. This version of uncertainty principle states that a multivariable square integrable function cannot be sharply localized in both the time domain and frequency domain. Given this result, it is theoretically important and practically useful to study its extension to the time-frequency domain.

1.1 | Overview and main result

In this paper we will focus on an interesting extension of the classical N-dimensional Heisenberg's uncertainty principle to the fractional Fourier transform (FRFT),[7] which generalizes the FT by embedding another degree of freedom associated with rotational angle α. Our main goal is to show that the classical result can be extended to time-frequency domain characterized by two N-dimensional FRFTs.

Let us begin by recalling some background and notation on the N-dimensional FRFT.[8]

Definition 1. The N-dimensional FRFT of a function $f(\mathbf{x}) \in L^2(\mathbb{R}^N)$ with the rotational angle α is defined as

$$\mathcal{F}^{\alpha}[f](\mathbf{u}) = \widehat{f}_{\alpha}(\mathbf{u}) = \begin{cases} \int_{\mathbb{R}^N} f(\mathbf{x}) K_{\alpha}(\mathbf{x}, \mathbf{u}) \mathrm{d}\mathbf{x}, & \alpha \neq n\pi \\ f(\mathbf{u}), & \alpha = 2n\pi \\ f(-\mathbf{u}), & \alpha = (2n+1)\pi \end{cases}, n \in \mathbb{Z}, \tag{3}$$

where $\mathbf{u} = (u_1, u_2, \cdots, u_N)$ and the kernel is

$$K_{\alpha}(\mathbf{x}, \mathbf{u}) = (1 - i \cot \alpha)^{\frac{N}{2}} e^{\pi i \left(\|\mathbf{x}\|^2 + \|\mathbf{u}\|^2\right) \cot \alpha - 2\pi i \mathbf{x}\mathbf{u}^{\mathrm{T}} \csc \alpha}. \tag{4}$$

The corresponding inverse formula is given by $f(\mathbf{x}) = \mathcal{F}^{-\alpha}\left[\widehat{f}_{\alpha}\right](\mathbf{x})$.

As it is seen, the N-dimensional FRFT of $\alpha = \left(2n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}$ reduces to the N-dimensional FT. The fractional part of FRFT comes from the fact that another degree of freedom was added to the FT by introducing the parameter α which can be $\alpha \neq \left(n + \frac{1}{2}\right)\pi, n\pi, n \in \mathbb{Z}$. It is such a parameter that enables the FRFT to have flexibility to be used in scenarios that the FT is not applicable (non-stationary signal and image processing, time-frequency analysis, optical system analysis, etc.). There has been particular interest in FRFT's uncertainty principle since it can provide theoretical basis for many realistic applications, such as the effective bandwidth estimation and the quadratic phase system analysis. To be specific, the uncertainty principle in the FRFT domain was first investigated by Ozaktas et al..[7] Shinde et al.[9] proposed a stronger result on the uncertainty product in two FRFT domains for real functions, and then Dang et al.[10] extended this result to complex functions. In addition, Xu et al.[11,12] discussed some extensions of Heisenberg's uncertainty principle on the FRFT, including the FRFT-based logarithmic, entropic and Rényi entropic uncertainty principles. All of these results are dealing with single variable functions. However, in the literature, there are only a few scattered results on the high-dimensional case, see[3,13] for related results. The one proposed in[13] for the N-dimensional FRFT is essentially the classical N-dimensional Heisenberg's uncertainty principle, and therefore, its lower bound is not the tightest. As for the latest one given by[3] for two N-dimensional FRFTs, its lower bound works only for real functions. The main contribution of this paper is to introduce a sharper lower bound on the uncertainty product for multivariable complex functions in two N-dimensional FRFT domains.

We shall also need necessary background and notation on moments and spreads in time, frequency and FRFT domains, and the covariance and absolute covariance in order to give our main result.

Definition 2. Let $\widehat{f}(\mathbf{w})$ be the N-dimensional FT of $f(\mathbf{x}) = \lambda(\mathbf{x}) e^{2\pi i \varphi(\mathbf{x})} \in L^2(\mathbb{R}^N)$, and $\widehat{f}_{\alpha}(\mathbf{u})$ be the N-dimensional FRFT of $f(\mathbf{x})$ with the rotational angle α. Assume that for any $1 \leq k \leq N$ the classical partial derivative $\frac{\partial \varphi}{\partial x_k}$ exists at any point $\mathbf{x} \in \mathbb{R}^N$, and $\mathbf{x} f(\mathbf{x}), \mathbf{w} \widehat{f}(\mathbf{w}) \in L^2(\mathbb{R}^N)$. It is then well-defined that

(i) The spread in the time domain:

$$\triangle \mathbf{x}^2 = \int_{\mathbb{R}^N} \left\| \mathbf{x} - \mathbf{x}^0 \right\|^2 |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x}, \tag{5}$$

where the moment vector in the time domain is

$$\mathbf{x}^0 = (x_1^0, x_2^0, \cdots, x_N^0), x_k^0 = \int_{\mathbb{R}^N} x_k |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} / \|f\|_2^2. \tag{6}$$

(ii) The spread in the frequency domain:

$$\triangle \mathbf{w}^2 = \int_{\mathbb{R}^N} \left\| \mathbf{w} - \mathbf{w}^0 \right\|^2 \left| \widehat{f}(\mathbf{w}) \right|^2 \mathrm{d}\mathbf{w}, \tag{7}$$

where the moment vector in the frequency domain is

$$\mathbf{w}^0 = (\omega_1^0, \omega_2^0, \cdots, \omega_N^0), \omega_k^0 = \int_{\mathbb{R}^N} \omega_k \left| \widehat{f}(\mathbf{w}) \right|^2 \mathrm{d}\mathbf{w} / \|f\|_2^2. \tag{8}$$

(iii) The spread in the FRFT domain:

$$\triangle\, \mathbf{u}_\alpha^2 = \int_{\mathbb{R}^N} \left\| \mathbf{u} - \mathbf{u}^{\alpha,0} \right\|^2 \left| \widehat{f}_\alpha(\mathbf{u}) \right|^2 \mathrm{d}\mathbf{u}, \tag{9}$$

where the moment vector in the FRFT domain is

$$\mathbf{u}^{\alpha,0} = (u_1^{\alpha,0}, u_2^{\alpha,0}, \cdots, u_N^{\alpha,0}), u_k^{\alpha,0} = \int_{\mathbb{R}^N} u_k \left| \widehat{f}_\alpha(\mathbf{u}) \right|^2 \mathrm{d}\mathbf{u} / \|f\|_2^2. \tag{10}$$

(iv) The covariance and absolute covariance:

$$\mathrm{Cov}_{\mathbf{x},\mathbf{w}} = \int_{\mathbb{R}^N} \left(\mathbf{x} - \mathbf{x}^0 \right) \left(\nabla_{\mathbf{x}} \varphi - \mathbf{w}^0 \right)^{\mathrm{T}} \lambda^2(\mathbf{x}) \mathrm{d}\mathbf{x} \tag{11}$$

and

$$\mathrm{COV}_{\mathbf{x},\mathbf{w}} = \int_{\mathbb{R}^N} \left| \mathbf{x} - \mathbf{x}^0 \right| \left| \nabla_{\mathbf{x}} \varphi - \mathbf{w}^0 \right|^{\mathrm{T}} \lambda^2(\mathbf{x}) \mathrm{d}\mathbf{x}, \tag{12}$$

where $\nabla_{\mathbf{x}} \varphi = \left(\frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}, \cdots, \frac{\partial \varphi}{\partial x_N} \right)$ denotes the gradient vector of φ. Here an absolute operator is applied to vectors and we mean an element-wise absolute value. It should also be noted that there is an inequality $\mathrm{COV}_{\mathbf{x},\mathbf{w}} \geq \mathrm{Cov}_{\mathbf{x},\mathbf{w}}$.

Our main result is the following. This result presents an uncertainty principle associated with complex functions' uncertainty product in two N-dimensional FRFT domains.

Theorem 1. Let $\widehat{f}(\mathbf{w})$ be the N-dimensional FT of $f(\mathbf{x}) = \lambda(\mathbf{x}) e^{2\pi i \varphi(\mathbf{x})} \in L^2(\mathbb{R}^N)$, and $\widehat{f}_\alpha(\mathbf{u}), \widehat{f}_\beta(\mathbf{u})$ be the N-dimensional FRFTs of $f(\mathbf{x})$ with rotational angles α, β respectively. Assume that for any $1 \leq k \leq N$ the classical partial derivatives $\frac{\partial \lambda}{\partial x_k}, \frac{\partial \varphi}{\partial x_k}, \frac{\partial f}{\partial x_k}$ exist at any point $\mathbf{x} \in \mathbb{R}^N$, and $\mathbf{x} f(\mathbf{x}), \mathbf{w} \widehat{f}(\mathbf{w}) \in L^2(\mathbb{R}^N)$. Then,

$$\begin{aligned} \triangle\, \mathbf{u}_\alpha^2 \triangle\, \mathbf{u}_\beta^2 \geq & \left(\frac{N^2}{16\pi^2} \|f\|_2^4 + \mathrm{COV}_{\mathbf{x},\mathbf{w}}^2 - \mathrm{Cov}_{\mathbf{x},\mathbf{w}}^2 \right) \sin^2(\alpha - \beta) \\ & + \left[\cos\alpha \cos\beta \triangle\, \mathbf{x}^2 + \sin\alpha \sin\beta \triangle\, \mathbf{w}^2 + \sin(\alpha + \beta) \mathrm{Cov}_{\mathbf{x},\mathbf{w}} \right]^2, \end{aligned} \tag{13}$$

where $\triangle\, \mathbf{x}^2, \triangle\, \mathbf{w}^2, \triangle\, \mathbf{u}_\alpha^2, \triangle\, \mathbf{u}_\beta^2, \mathrm{Cov}_{\mathbf{x},\mathbf{w}}, \mathrm{COV}_{\mathbf{x},\mathbf{w}}$ are defined as shown in Definition 2. If $\nabla_{\mathbf{x}} \varphi$ is continuous and λ is non-zero almost everywhere, then the equality holds if and only if $f(\mathbf{x})$ is a chirp function with the form

$$f(\mathbf{x}) = e^{-\frac{1}{2\zeta} \left\| \mathbf{x} - \mathbf{x}^0 \right\|^2 + d} e^{2\pi i \left[\frac{1}{2\varepsilon} \sum_{m=1}^{N} \eta(x_m) \left(x_m - x_m^0 \right)^2 + \mathbf{w}^0 \mathbf{x}^{\mathrm{T}} + d^{\eta(x_1), \eta(x_2), \cdots, \eta(x_N)} \right]} \tag{14}$$

for some $\zeta, \varepsilon > 0$ and $d, d^{\eta(x_1), \eta(x_2), \cdots, \eta(x_N)} \in \mathbb{R}$, where

$$\eta(x_m) = \begin{cases} 1, & m \in \mathbf{k}_{j_1} \\ -1, & m \in \mathbf{k}_{j_2} \\ \mathrm{sgn}\left(x_m - x_m^0 \right), & m \in \mathbf{k}_{j_3} \\ -\mathrm{sgn}\left(x_m - x_m^0 \right), & m \in \mathbf{k}_{j_4} \end{cases}, \tag{15}$$

and where

$$\mathbf{k}_{j_1} = \left\{ k_{11}, k_{12}, \cdots, k_{1j_1} \right\} = \left\{ 1 \leq k \leq N | \frac{\partial \varphi}{\partial x_k} = \frac{1}{\varepsilon} \left(x_k - x_k^0 \right) + \omega_k^0 \right\}, \tag{16}$$

$$\mathbf{k}_{j_2} = \left\{ k_{21}, k_{22}, \cdots, k_{2j_2} \right\} = \left\{ 1 \leq k \leq N | \frac{\partial \varphi}{\partial x_k} = -\frac{1}{\varepsilon} \left(x_k - x_k^0 \right) + \omega_k^0 \right\}, \tag{17}$$

$$\mathbf{k}_{j_3} = \left\{ k_{31}, k_{32}, \cdots, k_{3j_3} \right\} = \left\{ 1 \leq k \leq N | \frac{\partial \varphi}{\partial x_k} = \left\{ \begin{array}{ll} \frac{1}{\varepsilon} \left(x_k - x_k^0 \right) + \omega_k^0, & x_k \geq x_k^0 \\ -\frac{1}{\varepsilon} \left(x_k - x_k^0 \right) + \omega_k^0, & x_k < x_k^0 \end{array} \right\} \right\} \tag{18}$$

and

$$\mathbf{k}_{j_4} = \left\{ k_{41}, k_{42}, \cdots, k_{4j_4} \right\} = \left\{ 1 \leq k \leq N | \frac{\partial \varphi}{\partial x_k} = \left\{ \begin{array}{ll} -\frac{1}{\varepsilon} \left(x_k - x_k^0 \right) + \omega_k^0, & x_k \geq x_k^0 \\ \frac{1}{\varepsilon} \left(x_k - x_k^0 \right) + \omega_k^0, & x_k < x_k^0 \end{array} \right\} \right\} \tag{19}$$

satisfying $\bigcup_{p=1}^{4} \mathbf{k}_{j_p} = \{1, 2, \cdots, N\}$ and $\mathbf{k}_{j_p} \bigcap \mathbf{k}_{j_q} = \emptyset$ for $p \neq q$.

Inequality (13) of Theorem 1 gives a lower bound on the uncertainty product for multivariable complex functions in two N-dimensional FRFT domains. As it is seen, this result includes particular cases some well-known uncertainty inequalities, such as:

(i) For $N = 1$, it becomes the uncertainty inequality for one-dimensional FRFT introduced by Dang et al..[10]

(ii) For $\beta = m\pi, m \in \mathbb{Z}$, it becomes

$$\triangle\, \mathbf{x}^2 \triangle\, \mathbf{u}_\alpha^2 \geq \left(\frac{N^2}{16\pi^2}\|f\|_2^4 + \mathrm{COV}_{\mathbf{x},\mathbf{w}}^2 - \mathrm{Cov}_{\mathbf{x},\mathbf{w}}^2\right)\sin^2\alpha + \left[\cos\alpha \triangle\, \mathbf{x}^2 + \sin\alpha \mathrm{Cov}_{\mathbf{x},\mathbf{w}}\right]^2, \tag{20}$$

which improves the uncertainty inequality for the N-dimensional FRFT proposed in[13], i.e.,

$$\triangle\, \mathbf{x}^2 \triangle\, \mathbf{u}_\alpha^2 \geq \frac{N^2}{16\pi^2}\|f\|_2^4 \sin^2\alpha \tag{21}$$

through providing a tighter lower bound.

(iii) For $\mathrm{Cov}_{\mathbf{x},\mathbf{w}} = 0$ (e.g., real functions satisfying $\nabla_{\mathbf{x}}\varphi \equiv \mathbf{0}$), it becomes

$$\triangle\, \mathbf{u}_\alpha^2 \triangle\, \mathbf{u}_\beta^2 \geq \left(\frac{N^2}{16\pi^2}\|f\|_2^4 + \mathrm{COV}_{\mathbf{x},\mathbf{w}}^2\right)\sin^2(\alpha-\beta) + \left[\cos\alpha\cos\beta \triangle\, \mathbf{x}^2 + \sin\alpha\sin\beta \triangle\, \mathbf{w}^2\right]^2, \tag{22}$$

which improves the uncertainty inequality for two N-dimensional FRFTs given by[3], i.e.,

$$\triangle\, \mathbf{u}_\alpha^2 \triangle\, \mathbf{u}_\beta^2 \geq \frac{N^2}{16\pi^2}\|f\|_2^4 \sin^2(\alpha-\beta) + \left[\cos\alpha\cos\beta \triangle\, \mathbf{x}^2 + \sin\alpha\sin\beta \triangle\, \mathbf{w}^2\right]^2 \tag{23}$$

through providing a tighter lower bound.

Moreover, Theorem 1 of $\alpha = n\pi, \beta = \left(m + \frac{1}{2}\right)\pi, n, m \in \mathbb{Z}$ reduces to an uncertainty principle for N-dimensional FT given by the following corollary.

Corollary 1. Let $\widehat{f}(\mathbf{w})$ be the N-dimensional FT of $f(\mathbf{x}) = \lambda(\mathbf{x})e^{2\pi i\varphi(\mathbf{x})} \in L^2(\mathbb{R}^N)$. Assume that for any $1 \leq k \leq N$ the classical partial derivatives $\frac{\partial\lambda}{\partial x_k}, \frac{\partial\varphi}{\partial x_k}, \frac{\partial f}{\partial x_k}$ exist at any point $\mathbf{x} \in \mathbb{R}^N$, and $\mathbf{x}f(\mathbf{x}), \mathbf{w}\widehat{f}(\mathbf{w}) \in L^2(\mathbb{R}^N)$. Then,

$$\int_{\mathbb{R}^N} \left\|\mathbf{x}-\mathbf{x}^0\right\|^2 |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \int_{\mathbb{R}^N} \left\|\mathbf{w}-\mathbf{w}^0\right\|^2 \left|\widehat{f}(\mathbf{w})\right|^2 \mathrm{d}\mathbf{w} \geq \frac{N^2}{16\pi^2}\|f\|_2^4 + \left[\int_{\mathbb{R}^N} \left|\mathbf{x}-\mathbf{x}^0\right| \left|\nabla_{\mathbf{x}}\varphi - \mathbf{w}^0\right|^{\mathrm{T}} \lambda^2(\mathbf{x})\mathrm{d}\mathbf{x}\right]^2, \tag{24}$$

where $\mathbf{x}^0, \mathbf{w}^0$ are the moment vectors in time and frequency domains respectively, and $\nabla_{\mathbf{x}}\varphi$ is the gradient vector of φ. If $\nabla_{\mathbf{x}}\varphi$ is continuous and λ is non-zero almost everywhere, then the equality holds if and only if $f(\mathbf{x})$ is a chirp function with the form (14).

Inequality (24) of Corollary 1 gives a lower bound on the product of a multivariable complex function's spread in time domain and that in frequency domain. In reality, the proof of Theorem 1 requires a main preparatory lemma proving that the moment vectors $\mathbf{x}^0, \mathbf{w}^0$ found in inequality (24) can be replaced by arbitrary $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$. Thus this lemma can be stated as follows.

Lemma 1. Let $\widehat{f}(\mathbf{w})$ be the N-dimensional FT of $f(\mathbf{x}) = \lambda(\mathbf{x})e^{2\pi i\varphi(\mathbf{x})} \in L^2(\mathbb{R}^N)$, and $\mathbf{a} = (a_1, a_2, \cdots, a_N)$, $\mathbf{b} = (b_1, b_2, \cdots, b_N) \in \mathbb{R}^N$. Assume that for any $1 \leq k \leq N$ the classical partial derivatives $\frac{\partial\lambda}{\partial x_k}, \frac{\partial\varphi}{\partial x_k}, \frac{\partial f}{\partial x_k}$ exist at any point $\mathbf{x} \in \mathbb{R}^N$, and $\mathbf{x}f(\mathbf{x}), \mathbf{w}\widehat{f}(\mathbf{w}) \in L^2(\mathbb{R}^N)$. Then,

$$\int_{\mathbb{R}^N} \|\mathbf{x}-\mathbf{a}\|^2 |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \int_{\mathbb{R}^N} \|\mathbf{w}-\mathbf{b}\|^2 \left|\widehat{f}(\mathbf{w})\right|^2 \mathrm{d}\mathbf{w} \geq \frac{N^2}{16\pi^2}\|f\|_2^4 + \left[\int_{\mathbb{R}^N} |\mathbf{x}-\mathbf{a}| \left|\nabla_{\mathbf{x}}\varphi - \mathbf{b}\right|^{\mathrm{T}} \lambda^2(\mathbf{x})\mathrm{d}\mathbf{x}\right]^2, \tag{25}$$

where $\nabla_{\mathbf{x}}\varphi$ is the gradient vector of φ. If $\nabla_{\mathbf{x}}\varphi$ is continuous and λ is non-zero almost everywhere, then the equality holds if and only if $f(\mathbf{x})$ is a chirp function with the form

$$f(\mathbf{x}) = e^{-\frac{1}{2\zeta}\|\mathbf{x}-\mathbf{a}\|^2 + d} e^{2\pi i\left[\frac{1}{2\epsilon}\sum_{m=1}^{N}\eta(x_m)\left(x_m-a_m\right)^2 + \mathbf{b}\mathbf{x}^{\mathrm{T}} + d^{\eta(x_1),\eta(x_2),\cdots,\eta(x_N)}\right]} \tag{26}$$

for some $\zeta, \varepsilon > 0$ and $d, d^{\eta(x_1),\eta(x_2),\cdots,\eta(x_N)} \in \mathbb{R}$, where

$$\eta(x_m) = \begin{cases} 1, & m \in \mathbf{k}_{j_1} \\ -1, & m \in \mathbf{k}_{j_2} \\ \operatorname{sgn}\left(x_m - a_m\right), & m \in \mathbf{k}_{j_3} \\ -\operatorname{sgn}\left(x_m - a_m\right), & m \in \mathbf{k}_{j_4} \end{cases}, \tag{27}$$

and where

$$\mathbf{k}_{j_1} = \left\{k_{11}, k_{12}, \cdots, k_{1j_1}\right\} = \left\{1 \le k \le N | \frac{\partial\varphi}{\partial x_k} = \frac{1}{\varepsilon}\left(x_k - a_k\right) + b_k\right\}, \tag{28}$$

$$\mathbf{k}_{j_2} = \left\{k_{21}, k_{22}, \cdots, k_{2j_2}\right\} = \left\{1 \le k \le N | \frac{\partial\varphi}{\partial x_k} = -\frac{1}{\varepsilon}\left(x_k - a_k\right) + b_k\right\}, \tag{29}$$

$$\mathbf{k}_{j_3} = \left\{k_{31}, k_{32}, \cdots, k_{3j_3}\right\} = \left\{1 \le k \le N | \frac{\partial\varphi}{\partial x_k} = \left\{ \begin{array}{l} \frac{1}{\varepsilon}\left(x_k - a_k\right) + b_k, \quad x_k \ge a_k \\ -\frac{1}{\varepsilon}\left(x_k - a_k\right) + b_k, \ x_k < a_k \end{array} \right\}\right. \tag{30}$$

and

$$\mathbf{k}_{j_4} = \left\{k_{41}, k_{42}, \cdots, k_{4j_4}\right\} = \left\{1 \le k \le N | \frac{\partial\varphi}{\partial x_k} = \left\{ \begin{array}{l} -\frac{1}{\varepsilon}\left(x_k - a_k\right) + b_k, \ x_k \ge a_k \\ \frac{1}{\varepsilon}\left(x_k - a_k\right) + b_k, \quad x_k < a_k \end{array} \right\}\right. \tag{31}$$

satisfying $\bigcup_{p=1}^{4} \mathbf{k}_{j_p} = \{1, 2, \cdots, N\}$ and $\mathbf{k}_{j_p} \bigcap \mathbf{k}_{j_q} = \emptyset$ for $p \neq q$.

Inequality (25) of Lemma 1 is a sharper N-dimensional Heisenberg's uncertainty inequality which improves the classical result (1) through providing a tighter lower bound.

The remainder of this paper is structured as follows. Section 2 contains the proof of our main preparatory result, Lemma 1. Section 3 contains the proof of our main result, Theorem 1. To be specific, Section 3.1 proves an important relation between spreads of a multivariable complex function in time, frequency and FRFT domains, and Section 3.2 combines Lemma 1 with this relation to prove Theorem 1. Section 4 presents example and experimental results. Potential applications are in Section 5, and the conclusions follow in Section 6.

In the sequel, we denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^N$ the Cartesian product of N real number collections, by $\mathbb{Z}$ the set of integers, by T the transpose operator, and by — the complex conjugate operator. The 2-norm operator for vectors and L^2-norm operator for functions denote $\|\cdot\| = \sqrt{(\cdot)(\cdot)^{\mathrm{T}}}$ and $\|\cdot\|_2 = \left(\int_{\mathbb{R}^N} |\cdot(\mathbf{x})|^2 \mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}$, respectively. The function has a complex form $f(\mathbf{x}) = \lambda(\mathbf{x}) e^{2\pi i \varphi(\mathbf{x})}$, unless we emphasize that it is real-valued. The notation $\vartriangle \mathbf{x}^2$, $\vartriangle \mathbf{w}^2$ and $\vartriangle \mathbf{u}_\alpha^2, \vartriangle \mathbf{u}_\beta^2$ denote spreads in time, frequency and FRFT domains respectively, the notation $\mathrm{Cov}_{\mathbf{x},\mathbf{w}}$ and $\mathrm{COV}_{\mathbf{x},\mathbf{w}}$ denote the covariance and absolute covariance respectively, the notation $\mathbf{x}^0$, $\mathbf{w}^0$ and $\mathbf{u}^{\alpha,0}, \mathbf{u}^{\beta,0}$ denote moment vectors in time, frequency and FRFT domains respectively, and the notation $\nabla_{\mathbf{x}}\varphi$ denotes the gradient vector of φ. When an absolute operator is applied to vectors and we mean an element-wise absolute value.

2 | PROOF OF THE MAIN LEMMA

This section gives the proof of our main preparatory result, Lemma 1 which is crucially needed in the proof of our main theorem.

Lemma 1 presents a stronger Heisenberg's uncertainty principle for N-dimensional FT, as we discussed in Section 1.1. The proof of such an N-dimensional FT type of uncertainty principle involving the absolute covariance requires an additional preparatory result, Lemma 2. We first state and prove this preparatory result, and then use it to prove Lemma 1 at the end of this section.

Let us begin proofs by collecting Parseval's relations in N-dimensional FT and FRFT domains.[6,7]

(**Parseval's Relation.**) Let $\widehat{f}(\mathbf{w}), \widehat{g}(\mathbf{w})$ be the N-dimensional FTs of $f(\mathbf{x}), g(\mathbf{x}) \in L^2(\mathbb{R}^N)$ respectively, and $\widehat{f}_\alpha(\mathbf{u}), \widehat{g}_\alpha(\mathbf{u})$ be the N-dimensional FRFTs of $f(\mathbf{x}), g(\mathbf{x})$ with the rotational angle α respectively, then

$$\int_{\mathbb{R}^N} |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^N} \left|\widehat{f}(\mathbf{w})\right|^2 \mathrm{d}\mathbf{w} = \int_{\mathbb{R}^N} \left|\widehat{f}_\alpha(\mathbf{u})\right|^2 \mathrm{d}\mathbf{u} \tag{32}$$

and

$$\int_{\mathbb{R}^N} f(\mathbf{x})\overline{g(\mathbf{x})}\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^N} \widehat{f}(\mathbf{w})\overline{\widehat{g}(\mathbf{w})}\mathrm{d}\mathbf{w} = \int_{\mathbb{R}^N} \widehat{f}_\alpha(\mathbf{u})\overline{\widehat{g}_\alpha(\mathbf{u})}\mathrm{d}\mathbf{u}. \tag{33}$$

Here is the additional preparatory lemma.

Lemma 2. Let $\widehat{f}(\mathbf{w})$ be the N-dimensional FT of $f(\mathbf{x}) = \lambda(\mathbf{x})e^{2\pi i\varphi(\mathbf{x})} \in L^2(\mathbb{R}^N)$, $b \in \mathbb{R}$, and $1 \le k \le N$. Assume that the classical partial derivatives $\frac{\partial\lambda}{\partial x_k}$, $\frac{\partial\varphi}{\partial x_k}$, $\frac{\partial f}{\partial x_k}$ exist at any point $\mathbf{x} \in \mathbb{R}^N$, and $\omega_k\widehat{f}(\mathbf{w}) \in L^2(\mathbb{R}^N)$. Then,

$$\int_{\mathbb{R}^N} (\omega_k - b)^2 \left|\widehat{f}(\mathbf{w})\right|^2 \mathrm{d}\mathbf{w} = \frac{1}{4\pi^2}\int_{\mathbb{R}^N} \left(\frac{\partial\lambda}{\partial x_k}\right)^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^N} \left(\frac{\partial\varphi}{\partial x_k} - b\right)^2 \lambda^2(\mathbf{x})\mathrm{d}\mathbf{x}. \tag{34}$$

Proof. Using (32) of Parseval's relation in N-dimensional FT domain yields

$$\int_{\mathbb{R}^N} (\omega_k - b)^2 \left|\widehat{f}(\mathbf{w})\right|^2 \mathrm{d}\mathbf{w} = \int_{\mathbb{R}^N} \omega_k^2 \left|\widehat{f}(\mathbf{w})\right|^2 \mathrm{d}\mathbf{w} + b^2\int_{\mathbb{R}^N} |f(\mathbf{x})|^2\mathrm{d}\mathbf{x} - 2b\int_{\mathbb{R}^N} \omega_k \left|\widehat{f}(\mathbf{w})\right|^2 \mathrm{d}\mathbf{w}. \tag{35}$$

Since functions $\frac{1}{2\pi i}\frac{\partial f}{\partial x_k}$ and $\omega_k\widehat{f}(\mathbf{w})$ compose an N-dimensional FT pair, using (32) and (33) of Parseval's relation in N-dimensional FT domain, the above equation becomes

$$\begin{aligned}
\int_{\mathbb{R}^N} (\omega_k - b)^2 \left|\widehat{f}(\mathbf{w})\right|^2 \mathrm{d}\mathbf{w} &= \int_{\mathbb{R}^N} \left|\frac{1}{2\pi i}\frac{\partial f}{\partial x_k}\right|^2 \mathrm{d}\mathbf{x} + b^2\int_{\mathbb{R}^N} |f(\mathbf{x})|^2\mathrm{d}\mathbf{x} - 2b\int_{\mathbb{R}^N} \frac{1}{2\pi i}\frac{\partial f}{\partial x_k}\overline{f(\mathbf{x})}\mathrm{d}\mathbf{x} \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^N} \left[\left(\frac{\partial\lambda}{\partial x_k}\right)^2 + 4\pi^2\left(\frac{\partial\varphi}{\partial x_k}\right)^2 \lambda^2(\mathbf{x})\right] \mathrm{d}\mathbf{x} + b^2\int_{\mathbb{R}^N} \lambda^2(\mathbf{x})\mathrm{d}\mathbf{x} \\
&\quad + \frac{bi}{\pi}\int_{\mathbb{R}^N} \frac{\partial\lambda}{\partial x_k}\lambda(\mathbf{x})\mathrm{d}\mathbf{x} - 2b\int_{\mathbb{R}^N} \frac{\partial\varphi}{\partial x_k}\lambda^2(\mathbf{x})\mathrm{d}\mathbf{x} \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^N} \left(\frac{\partial\lambda}{\partial x_k}\right)^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^N} \left(\frac{\partial\varphi}{\partial x_k} - b\right)^2 \lambda^2(\mathbf{x})\mathrm{d}\mathbf{x},
\end{aligned} \tag{36}$$

which gives the required result (34). □

We are now ready to prove Lemma 1.

Proof of Lemma 1. It follows from (34) of Lemma 2 that for any $a, b \in \mathbb{R}$

$$\int_{\mathbb{R}^N} (x_k - a)^2|f(\mathbf{x})|^2\mathrm{d}\mathbf{x}\int_{\mathbb{R}^N} (\omega_k - b)^2 \left|\widehat{f}(\mathbf{w})\right|^2 \mathrm{d}\mathbf{w} = \frac{1}{4\pi^2}I_1 + I_2, \tag{37}$$

where

$$I_1 = \int_{\mathbb{R}^N} (x_k - a)^2|f(\mathbf{x})|^2\mathrm{d}\mathbf{x}\int_{\mathbb{R}^N} \left(\frac{\partial\lambda}{\partial x_k}\right)^2 \mathrm{d}\mathbf{x} \tag{38}$$

and

$$I_2 = \int_{\mathbb{R}^N} (x_k - a)^2|f(\mathbf{x})|^2\mathrm{d}\mathbf{x}\int_{\mathbb{R}^N} \left(\frac{\partial\varphi}{\partial x_k} - b\right)^2 \lambda^2(\mathbf{x})\mathrm{d}\mathbf{x}. \tag{39}$$

Because of the smoothness and integrability assumptions of $\lambda^2(\mathbf{x})$ and $x_k\lambda^2(\mathbf{x})$, using the Cauchy-Schwarz inequality[14] yields

$$I_1 \ge \left[\int_{\mathbb{R}^N} (x_k - a)\lambda(\mathbf{x})\frac{\partial\lambda}{\partial x_k}\mathrm{d}\mathbf{x}\right]^2 = \left[\frac{1}{2}\int_{\mathbb{R}^N} |f(\mathbf{x})|^2\mathrm{d}\mathbf{x}\right]^2 = \frac{\|f\|_2^4}{4} \tag{40}$$

and

$$I_2 \ge \left[\int_{\mathbb{R}^N} \left|(x_k - a)\left(\frac{\partial\varphi}{\partial x_k} - b\right)\right| \lambda^2(\mathbf{x})\mathrm{d}\mathbf{x}\right]^2. \tag{41}$$

With (37), (40) and (41), there is

$$\int_{\mathbb{R}^N}(x_k-a)^2|f(\mathbf{x})|^2\mathrm{d}\mathbf{x}\int_{\mathbb{R}^N}(\omega_k-b)^2\left|\widehat{f}(\mathbf{w})\right|^2\mathrm{d}\mathbf{w}\geq\frac{\|f\|_2^4}{16\pi^2}+\left[\int_{\mathbb{R}^N}\left|(x_k-a)\left(\frac{\partial\varphi}{\partial x_k}-b\right)\right|\lambda^2(\mathbf{x})\mathrm{d}\mathbf{x}\right]^2. \tag{42}$$

It follows from the Cauchy-Schwarz inequality [14,15] that for any $\mathbf{a}=(a_1,a_2,\cdots,a_N),\mathbf{b}=(b_1,b_2,\cdots,b_N)\in\mathbb{R}^N$

$$\begin{aligned}\int_{\mathbb{R}^N}\|\mathbf{x}-\mathbf{a}\|^2|f(\mathbf{x})|^2\mathrm{d}\mathbf{x}\int_{\mathbb{R}^N}\|\mathbf{w}-\mathbf{b}\|^2\left|\widehat{f}(\mathbf{w})\right|^2\mathrm{d}\mathbf{w}&=\sum_{k=1}^N\int_{\mathbb{R}^N}\left(x_k-a_k\right)^2|f(\mathbf{x})|^2\mathrm{d}\mathbf{x}\sum_{k=1}^N\int_{\mathbb{R}^N}\left(\omega_k-b_k\right)^2\left|\widehat{f}(\mathbf{w})\right|^2\mathrm{d}\mathbf{w}\\&\geq\left[\sum_{k=1}^N\left(\int_{\mathbb{R}^N}\left(x_k-a_k\right)^2|f(\mathbf{x})|^2\mathrm{d}\mathbf{x}\int_{\mathbb{R}^N}\left(\omega_k-b_k\right)^2\left|\widehat{f}(\mathbf{w})\right|^2\mathrm{d}\mathbf{w}\right)^{\frac{1}{2}}\right]^2.\end{aligned} \tag{43}$$

Using (42) yields

$$\begin{aligned}\int_{\mathbb{R}^N}\|\mathbf{x}-\mathbf{a}\|^2|f(\mathbf{x})|^2\mathrm{d}\mathbf{x}\int_{\mathbb{R}^N}\|\mathbf{w}-\mathbf{b}\|^2\left|\widehat{f}(\mathbf{w})\right|^2\mathrm{d}\mathbf{w}&\geq\left[\sum_{k=1}^N\left(\frac{\|f\|_2^4}{16\pi^2}+\left(\int_{\mathbb{R}^N}\left|\left(x_k-a_k\right)\left(\frac{\partial\varphi}{\partial x_k}-b_k\right)\right|\lambda^2(\mathbf{x})\mathrm{d}\mathbf{x}\right)^2\right)^{\frac{1}{2}}\right]^2\\&=\frac{\|f\|_2^4}{16\pi^2}\left[\sum_{k=1}^N\left(1+\left(\frac{4\pi}{\|f\|_2^2}\int_{\mathbb{R}^N}\left|\left(x_k-a_k\right)\left(\frac{\partial\varphi}{\partial x_k}-b_k\right)\right|\lambda^2(\mathbf{x})\mathrm{d}\mathbf{x}\right)^2\right)^{\frac{1}{2}}\right]^2\\&\geq\frac{\|f\|_2^4}{16\pi^2}\left[N^2+\left(\sum_{k=1}^N\frac{4\pi}{\|f\|_2^2}\int_{\mathbb{R}^N}\left|\left(x_k-a_k\right)\left(\frac{\partial\varphi}{\partial x_k}-b_k\right)\right|\lambda^2(\mathbf{x})\mathrm{d}\mathbf{x}\right)^2\right]\\&=\frac{N^2}{16\pi^2}\|f\|_2^4+\left[\int_{\mathbb{R}^N}|\mathbf{x}-\mathbf{a}|\left|\nabla_{\mathbf{x}}\varphi-\mathbf{b}\right|^{\mathrm{T}}\lambda^2(\mathbf{x})\mathrm{d}\mathbf{x}\right]^2,\end{aligned} \tag{44}$$

which gives the required result (25).

Next we deduce the conditions under which the equality holds in (25).

The inequality (40) brings in conditions obeyed by the amplitude function $\lambda(\mathbf{x})$. The equality in (40) is attained if and only if there exists a positive number ζ such that

$$(x_k-a)\lambda(\mathbf{x})=\zeta\frac{\partial\lambda}{\partial x_k} \tag{45}$$

or

$$-(x_k-a)\lambda(\mathbf{x})=\zeta\frac{\partial\lambda}{\partial x_k}. \tag{46}$$

The first case shall not happen because it could result in a function $\lambda(\mathbf{x})\notin L^2(\mathbb{R}^N)$. [16] Then, $-\left(x_k-a_k\right)\lambda(\mathbf{x})=\zeta_k\frac{\partial\lambda}{\partial x_k}$ holds for all $1\leq k\leq N$ as the first equality in (44) holds. Solving the system of partial differential equations gives

$$\lambda(\mathbf{x})=e^{\sum\limits_{k=1}^N-\frac{1}{2\zeta_k}\left(x_k-a_k\right)^2+d}. \tag{47}$$

The inequality (41) brings in conditions obeyed by the phase function $\varphi(\mathbf{x})$. The equality in (41) is attained if and only if there exists a positive number ε such that

$$\left|(x_k-a)\lambda(\mathbf{x})\right|=\varepsilon\left|\left(\frac{\partial\varphi}{\partial x_k}-b\right)\lambda(\mathbf{x})\right|, \tag{48}$$

or equivalently,

$$\left|x_k-a\right|=\varepsilon\left|\frac{\partial\varphi}{\partial x_k}-b\right| \tag{49}$$

because of the almost everywhere non-zero of λ and the continuity assumption of $\frac{\partial\varphi}{\partial x_k}$. Then, $\left|x_k - a_k\right| = \varepsilon_k \left|\frac{\partial\varphi}{\partial x_k} - b_k\right|$ holds for all $1 \le k \le N$ as the first equality in (44) holds. As it is seen, there can be altogether four cases:[16]

$$\frac{\partial\varphi}{\partial x_k} = \frac{1}{\varepsilon_k}\left(x_k - a_k\right) + b_k, \tag{50}$$

$$\frac{\partial\varphi}{\partial x_k} = -\frac{1}{\varepsilon_k}\left(x_k - a_k\right) + b_k, \tag{51}$$

$$\frac{\partial\varphi}{\partial x_k} = \begin{cases} \frac{1}{\varepsilon_k}\left(x_k - a_k\right) + b_k, & x_k \ge a_k \\ -\frac{1}{\varepsilon_k}\left(x_k - a_k\right) + b_k, & x_k < a_k \end{cases} \tag{52}$$

and

$$\frac{\partial\varphi}{\partial x_k} = \begin{cases} -\frac{1}{\varepsilon_k}\left(x_k - a_k\right) + b_k, & x_k \ge a_k \\ \frac{1}{\varepsilon_k}\left(x_k - a_k\right) + b_k, & x_k < a_k \end{cases}, \tag{53}$$

from which the set of $\{1, 2, \cdots, N\}$ can be partitioned into the following four components:

$$\mathbf{k}_{j_1} = \left\{k_{11}, k_{12}, \cdots, k_{1j_1}\right\} = \left\{1 \le k \le N \,|\, \frac{\partial\varphi}{\partial x_k} = \frac{1}{\varepsilon_k}\left(x_k - a_k\right) + b_k\right\}, \tag{54}$$

$$\mathbf{k}_{j_2} = \left\{k_{21}, k_{22}, \cdots, k_{2j_2}\right\} = \left\{1 \le k \le N \,|\, \frac{\partial\varphi}{\partial x_k} = -\frac{1}{\varepsilon_k}\left(x_k - a_k\right) + b_k\right\}, \tag{55}$$

$$\mathbf{k}_{j_3} = \left\{k_{31}, k_{32}, \cdots, k_{3j_3}\right\} = \left\{1 \le k \le N \,|\, \frac{\partial\varphi}{\partial x_k} = \begin{cases} \frac{1}{\varepsilon_k}\left(x_k - a_k\right) + b_k, & x_k \ge a_k \\ -\frac{1}{\varepsilon_k}\left(x_k - a_k\right) + b_k, & x_k < a_k \end{cases}\right\} \tag{56}$$

and

$$\mathbf{k}_{j_4} = \left\{k_{41}, k_{42}, \cdots, k_{4j_4}\right\} = \left\{1 \le k \le N \,|\, \frac{\partial\varphi}{\partial x_k} = \begin{cases} -\frac{1}{\varepsilon_k}\left(x_k - a_k\right) + b_k, & x_k \ge a_k \\ \frac{1}{\varepsilon_k}\left(x_k - a_k\right) + b_k, & x_k < a_k \end{cases}\right\}. \tag{57}$$

Solving the system of partial differential equations yields

$$\varphi(\mathbf{x}) = \sum_{m=1}^{N} \frac{1}{2\varepsilon_m}\eta(x_m)\left(x_m - a_m\right)^2 + \mathbf{b}\mathbf{x}^{\mathrm{T}} + d^{\eta(x_1),\eta(x_2),\cdots,\eta(x_N)}, \tag{58}$$

where

$$\eta(x_m) = \begin{cases} 1, & m \in \mathbf{k}_{j_1} \\ -1, & m \in \mathbf{k}_{j_2} \\ \operatorname{sgn}\left(x_m - a_m\right), & m \in \mathbf{k}_{j_3} \\ -\operatorname{sgn}\left(x_m - a_m\right), & m \in \mathbf{k}_{j_4} \end{cases}. \tag{59}$$

The inequality (43) brings in conditions obeyed by the parameters ζ_k and ε_k, $k = 1, 2, \cdots, N$. The equality in (43) is attained if and only if the ratio

$$\frac{\int_{\mathbb{R}^N}\left(x_k - a_k\right)^2 |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x}}{\int_{\mathbb{R}^N}\left(\omega_k - b_k\right)^2 \left|\hat{f}(\mathbf{w})\right|^2 \mathrm{d}\mathbf{w}} \tag{60}$$

is a constant independent of k. Using (34), (47) and (58), it follows that

$$\frac{1}{4\pi^2\zeta_k^2} + \frac{1}{\varepsilon_k^2} = \frac{1}{4\pi^2\zeta_l^2} + \frac{1}{\varepsilon_l^2} \text{ for } k, l = 1, 2, \cdots, N. \tag{61}$$

The second inequality in (44) brings in conditions obeyed by the parameters ζ_k and ε_k, $k = 1, 2, \cdots, N$. It follows from (47) and (58) that

$$\|f\|_2^2 = e^{2d}\prod_{k=1}^{N}(\pi\zeta_k)^{\frac{1}{2}} \tag{62}$$

and

$$\int_{\mathbb{R}^N} \left| (x_k - a_k) \left(\frac{\partial \varphi}{\partial x_k} - b_k \right) \right| \lambda^2(\mathbf{x}) \mathrm{d}\mathbf{x} = e^{2d} \frac{\zeta_k}{2\epsilon_k} \prod_{k=1}^{N} (\pi \zeta_k)^{\frac{1}{2}}, \tag{63}$$

and then

$$\frac{4\pi}{\|f\|_2^2} \int_{\mathbb{R}^N} \left| (x_k - a_k) \left(\frac{\partial \varphi}{\partial x_k} - b_k \right) \right| \lambda^2(\mathbf{x}) \mathrm{d}\mathbf{x} = 2\pi \frac{\zeta_k}{\epsilon_k}. \tag{64}$$

Thus, the second equality in (44) is attained if and only if the ratio

$$\frac{\left[1 + \left(2\pi \frac{\zeta_k}{\epsilon_k} \right)^2 \right]^{\frac{1}{2}} - 1}{\left[1 + \left(2\pi \frac{\zeta_k}{\epsilon_k} \right)^2 \right]^{\frac{1}{2}} + 1} \tag{65}$$

is a constant independent of k. It follows that

$$\frac{\zeta_k}{\epsilon_k} = \frac{\zeta_l}{\epsilon_l} \text{ for } k, l = 1, 2, \cdots, N. \tag{66}$$

With (61) and (66), it concludes that ζ_k and ϵ_k are constants independent of k, and denoted respectively by

$$\zeta_k = \zeta, k = 1, 2, \cdots, N \tag{67}$$

and

$$\epsilon_k = \epsilon, k = 1, 2, \cdots, N. \tag{68}$$

Then, the amplitude function (47) and the phase function (58) turn into

$$\lambda(\mathbf{x}) = e^{-\frac{1}{2\zeta}\|\mathbf{x}-\mathbf{a}\|^2 + d} \tag{69}$$

and

$$\varphi(\mathbf{x}) = \frac{1}{2\epsilon} \sum_{m=1}^{N} \eta(x_m) \left(x_m - a_m \right)^2 + \mathbf{b}\mathbf{x}^{\mathrm{T}} + d^{\eta(x_1), \eta(x_2), \cdots, \eta(x_N)} \tag{70}$$

respectively, giving rise to the required result (26). □

3 | PROOF OF THE MAIN THEOREM

This section gives the proof of our main result, Theorem 1. In Section 3.1, we prove a technical lemma on the relationship between spreads of multivariable complex functions in time, frequency and FRFT domains. Then, in Section 3.2 we combine this lemma with Lemma 1 to prove Theorem 1.

3.1 | Relation between a multivariable complex function's spreads in time, frequency and FRFT domains

The spreads of a multivariable function $f(\mathbf{x}) \in L^2(\mathbb{R}^N)$ in time, frequency and FRFT domains stand for its duration, bandwidth and FRFT-bandwidth, which are defined respectively as shown in (5), (7) and (9) of Definition 2. For a specific complex function $f(\mathbf{x}) = \lambda(\mathbf{x}) e^{2\pi i \varphi(\mathbf{x})}$, the relation between these spreads is given below.

Lemma 3. Let $\widehat{f}(\mathbf{w})$ be the N-dimensional FT of $f(\mathbf{x}) = \lambda(\mathbf{x}) e^{2\pi i \varphi(\mathbf{x})} \in L^2(\mathbb{R}^N)$, and $\widehat{f}_\alpha(\mathbf{u})$ be the N-dimensional FRFT of $f(\mathbf{x})$ with the rotational angle α. Assume that for any $1 \le k \le N$ the classical partial derivatives $\frac{\partial \lambda}{\partial x_k}, \frac{\partial \varphi}{\partial x_k}, \frac{\partial f}{\partial x_k}$ exist at any point $\mathbf{x} \in \mathbb{R}^N$, and $\mathbf{x} f(\mathbf{x}), \mathbf{w} \widehat{f}(\mathbf{w}) \in L^2(\mathbb{R}^N)$. Then,

$$\triangle\, \mathbf{u}_\alpha^2 = \cos^2 \alpha \,\triangle\, \mathbf{x}^2 + \sin^2 \alpha \,\triangle\, \mathbf{w}^2 + 2 \sin \alpha \cos \alpha \mathrm{Cov}_{\mathbf{x},\mathbf{w}}, \tag{71}$$

where $\triangle\, \mathbf{x}^2, \triangle\, \mathbf{w}^2, \triangle\, \mathbf{u}_\alpha^2, \mathrm{Cov}_{\mathbf{x},\mathbf{w}}$ are defined as shown in Definition 2.

Proof. It follows from the definition of N-dimensional FRFT in the case of $\alpha = n\pi, n \in \mathbb{Z}$ that $\widehat{f}_\alpha(\mathbf{u}) = f(\mathbf{u})$ or $\widehat{f}_\alpha(\mathbf{u}) = f(-\mathbf{u})$, and then

$$\triangle\, \mathbf{u}_\alpha^2 = \int_{\mathbb{R}^N} \left\|\mathbf{u} - \mathbf{u}^{\alpha,0}\right\|^2 \left|\widehat{f}_\alpha(\mathbf{u})\right|^2 \mathrm{d}\mathbf{u} = \int_{\mathbb{R}^N} \left\|\mathbf{x} - \mathbf{x}^0\right\|^2 |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = \triangle\, \mathbf{x}^2, \tag{72}$$

which gives the required result (71) of $\alpha = n\pi, n \in \mathbb{Z}$. As for the case of $\alpha \neq n\pi, n \in \mathbb{Z}$, using (32) of Parseval's relation in N-dimensional FRFT domain gives for any $1 \leq k \leq N$

$$\int_{\mathbb{R}^N} \left(u_k - u_k^{\alpha,0}\right)^2 \left|\widehat{f}_\alpha(\mathbf{u})\right|^2 \mathrm{d}\mathbf{u} = \int_{\mathbb{R}^N} u_k^2 \left|\widehat{f}_\alpha(\mathbf{u})\right|^2 \mathrm{d}\mathbf{u} - \left(u_k^{\alpha,0}\right)^2 \|f\|_2^2. \tag{73}$$

In view of the N-dimensional FRFT's inverse formula, there is

$$\frac{\sin\alpha}{2\pi i}\frac{\partial f}{\partial x_k} + \cos\alpha x_k f(\mathbf{x}) = \int_{\mathbb{R}^N} u_k \widehat{f}_\alpha(\mathbf{u}) K_{-\alpha}(\mathbf{u},\mathbf{x})\mathrm{d}\mathbf{u}, \tag{74}$$

which indicates that functions $\frac{\sin\alpha}{2\pi i}\frac{\partial f}{\partial x_k} + \cos\alpha x_k f(\mathbf{x})$ and $u_k \widehat{f}_\alpha(\mathbf{u})$ compose an N-dimensional FRFT pair. From (32) and (33) of Parseval's relation in N-dimensional FRFT domain, (73) becomes

$$\int_{\mathbb{R}^N} \left(u_k - u_k^{\alpha,0}\right)^2 \left|\widehat{f}_\alpha(\mathbf{u})\right|^2 \mathrm{d}\mathbf{u} = \int_{\mathbb{R}^N} \left|\frac{\sin\alpha}{2\pi i}\frac{\partial f}{\partial x_k} + \cos\alpha x_k f(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} - \left[\int_{\mathbb{R}^N} \left(\frac{\sin\alpha}{2\pi i}\frac{\partial f}{\partial x_k} + \cos\alpha x_k f(\mathbf{x})\right)\overline{f(\mathbf{x})}\mathrm{d}\mathbf{x}\right]^2 / \|f\|_2^2. \tag{75}$$

Because of (36), the relations

$$\int_{\mathbb{R}^N} \omega_k^2 \left|\widehat{f}(\mathbf{w})\right|^2 \mathrm{d}\mathbf{w} = \frac{1}{4\pi^2}\int_{\mathbb{R}^N} \left[\left(\frac{\partial\lambda}{\partial x_k}\right)^2 + 4\pi^2\left(\frac{\partial\varphi}{\partial x_k}\right)^2 \lambda^2(\mathbf{x})\right] \mathrm{d}\mathbf{x} \tag{76}$$

and

$$\int_{\mathbb{R}^N} \omega_k \left|\widehat{f}(\mathbf{w})\right|^2 \mathrm{d}\mathbf{w} = \int_{\mathbb{R}^N} \frac{\partial\varphi}{\partial x_k}\lambda^2(\mathbf{x})\mathrm{d}\mathbf{x} \tag{77}$$

hold, resulting in

$$\begin{aligned}
\int_{\mathbb{R}^N} \left|\frac{\sin\alpha}{2\pi i}\frac{\partial f}{\partial x_k} + \cos\alpha x_k f(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} \;&=\; \cos^2\alpha \int_{\mathbb{R}^N} x_k^2 |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} + \frac{\sin^2\alpha}{4\pi^2}\int_{\mathbb{R}^N} \left[\left(\frac{\partial\lambda}{\partial x_k}\right)^2 + 4\pi^2\left(\frac{\partial\varphi}{\partial x_k}\right)^2 \lambda^2(\mathbf{x})\right] \mathrm{d}\mathbf{x} \\
&\quad +2\sin\alpha\cos\alpha \int_{\mathbb{R}^N} x_k \frac{\partial\varphi}{\partial x_k}\lambda^2(\mathbf{x})\mathrm{d}\mathbf{x} \\
&= \cos^2\alpha \int_{\mathbb{R}^N} x_k^2 |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} + \sin^2\alpha \int_{\mathbb{R}^N} \omega_k^2 \left|\widehat{f}(\mathbf{w})\right|^2 \mathrm{d}\mathbf{w} \\
&\quad +2\sin\alpha\cos\alpha \int_{\mathbb{R}^N} x_k \frac{\partial\varphi}{\partial x_k}\lambda^2(\mathbf{x})\mathrm{d}\mathbf{x}
\end{aligned} \tag{78}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(\frac{\sin\alpha}{2\pi i}\frac{\partial f}{\partial x_k} + \cos\alpha x_k f(\mathbf{x})\right)\overline{f(\mathbf{x})}\mathrm{d}\mathbf{x} \;&=\; \cos\alpha \int_{\mathbb{R}^N} x_k |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} + \sin\alpha \int_{\mathbb{R}^N} \frac{\partial\varphi}{\partial x_k}\lambda^2(\mathbf{x})\mathrm{d}\mathbf{x} \\
&= \cos\alpha \int_{\mathbb{R}^N} x_k |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} + \sin\alpha \int_{\mathbb{R}^N} \omega_k \left|\widehat{f}(\mathbf{w})\right|^2 \mathrm{d}\mathbf{w} \\
&= \left(\cos\alpha x_k^0 + \sin\alpha\omega_k^0\right) \|f\|_2^2.
\end{aligned} \tag{79}$$

Substituting into (75), and using (32) of Parseval's relation in N-dimensional FT domain and (77) yields

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(u_k - u_k^{\alpha,0}\right)^2 \left|\widehat{f}_\alpha(\mathbf{u})\right|^2 \mathrm{d}\mathbf{u} \;&=\; \cos^2\alpha \int_{\mathbb{R}^N} \left(x_k - x_k^0\right)^2 |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} + \sin^2\alpha \int_{\mathbb{R}^N} \left(\omega_k - \omega_k^0\right)^2 \left|\widehat{f}(\mathbf{w})\right|^2 \mathrm{d}\mathbf{w} \\
&\quad +2\sin\alpha\cos\alpha \int_{\mathbb{R}^N} \left(x_k - x_k^0\right)\left(\frac{\partial\varphi}{\partial x_k} - \omega_k^0\right)\lambda^2(\mathbf{x})\mathrm{d}\mathbf{x},
\end{aligned} \tag{80}$$

and then

$$
\begin{aligned}
\triangle\, \mathbf{u}_{\alpha}^{2} &= \int_{\mathbb{R}^N} \left\| \mathbf{u} - \mathbf{u}^{\alpha,0} \right\|^2 \left| \widehat{f}_{\alpha}(\mathbf{u}) \right|^2 \mathrm{d}\mathbf{u} \\
&= \cos^2\alpha \int_{\mathbb{R}^N} \left\| \mathbf{x} - \mathbf{x}^{0} \right\|^2 |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} + \sin^2\alpha \int_{\mathbb{R}^N} \left\| \mathbf{w} - \mathbf{w}^{0} \right\|^2 \left| \widehat{f}(\mathbf{w}) \right|^2 \mathrm{d}\mathbf{w} + 2\sin\alpha\cos\alpha \int_{\mathbb{R}^N} \left(\mathbf{x} - \mathbf{x}^{0}\right) \left(\nabla_{\mathbf{x}}\varphi - \mathbf{w}^{0}\right)^{\mathrm{T}} \lambda^2(\mathbf{x})\mathrm{d}\mathbf{x} \\
&= \cos^2\alpha \,\triangle\, \mathbf{x}^2 + \sin^2\alpha \,\triangle\, \mathbf{w}^2 + 2\sin\alpha\cos\alpha \mathrm{Cov}_{\mathbf{x},\mathbf{w}}.
\end{aligned} \tag{81}
$$

Combining (72) and (81) gives the required result (71). □

3.2 | Combining Lemma 1 with the preparatory result: proof of the main theorem

In this section, we combine Lemma 3 with Lemma 1 to prove Theorem 1.

Proof of Theorem 1. Using (71) of Lemma 3 gives

$$
\begin{aligned}
\triangle\, \mathbf{u}_{\alpha}^{2} \,\triangle\, \mathbf{u}_{\beta}^{2} &= \left(\cos^2\alpha \,\triangle\, \mathbf{x}^2 + \sin^2\alpha \,\triangle\, \mathbf{w}^2 + 2\sin\alpha\cos\alpha \mathrm{Cov}_{\mathbf{x},\mathbf{w}}\right) \left(\cos^2\beta \,\triangle\, \mathbf{x}^2 + \sin^2\beta \,\triangle\, \mathbf{w}^2 + 2\sin\beta\cos\beta \mathrm{Cov}_{\mathbf{x},\mathbf{w}}\right) \\
&= \left(\triangle\, \mathbf{x}^2 \,\triangle\, \mathbf{w}^2 - \mathrm{Cov}_{\mathbf{x},\mathbf{w}}^2\right) \sin^2(\alpha-\beta) + \left[\cos\alpha\cos\beta \,\triangle\, \mathbf{x}^2 + \sin\alpha\sin\beta \,\triangle\, \mathbf{w}^2 + \sin(\alpha+\beta)\mathrm{Cov}_{\mathbf{x},\mathbf{w}}\right]^2.
\end{aligned} \tag{82}
$$

Setting $\mathbf{a} = \mathbf{x}^0, \mathbf{b} = \mathbf{w}^0$ in (25) of Lemma 1, it follows that (24) holds, i.e.,

$$
\triangle\, \mathbf{x}^2 \,\triangle\, \mathbf{w}^2 \ge \frac{N^2}{16\pi^2} \|f\|_2^4 + \mathrm{COV}_{\mathbf{x},\mathbf{w}}^2. \tag{83}
$$

Combining (82) with (83) yields

$$
\begin{aligned}
\triangle\, \mathbf{u}_{\alpha}^{2} \,\triangle\, \mathbf{u}_{\beta}^{2} \ge& \left(\frac{N^2}{16\pi^2} \|f\|_2^4 + \mathrm{COV}_{\mathbf{x},\mathbf{w}}^2 - \mathrm{Cov}_{\mathbf{x},\mathbf{w}}^2\right) \sin^2(\alpha-\beta) \\
&+ \left[\cos\alpha\cos\beta \,\triangle\, \mathbf{x}^2 + \sin\alpha\sin\beta \,\triangle\, \mathbf{w}^2 + \sin(\alpha+\beta)\mathrm{Cov}_{\mathbf{x},\mathbf{w}}\right]^2,
\end{aligned} \tag{84}
$$

which gives the required result (13). As for the condition that reaches the equality relation, it is none other than the one giving rise to the equality in (83) (i.e., (24)). Thus the chirp function (26) of $\mathbf{a} = \mathbf{x}^0, \mathbf{b} = \mathbf{w}^0$ gives the required result (14). □

4 | EXAMPLE AND NUMERICAL SIMULATION

In this section, we perform a two-dimensional example and simulation to illustrate the correctness of the derived results.

Taking $N = 2$ for example, the two-dimensional complex function is chosen as

$$
f(x_1, x_2) = e^{\sum_{k=1}^{2} -\frac{1}{2\zeta_k}\left(x_k - x_k^0\right)^2 + d} e^{2\pi i\left[\frac{1}{2\epsilon_1}\left(x_1 - x_1^0\right)^2 - \frac{1}{2\epsilon_2}\left(x_2 - x_2^0\right)^2 + \sum_{m=1}^{2} \omega_m^0 x_m + d_1\right]} \tag{85}
$$

that is a function of the amplitude form (47) and the phase form (58), where $\zeta_k, \epsilon_k > 0$, $k = 1, 2$, $d, d_1 \in \mathbb{R}$, and $e^{2d} \prod_{k=1}^{2} (\pi\zeta_k)^{\frac{1}{2}} = 1$. Then, it calculates that

$$
\|f\|_2^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2d} e^{\sum_{k=1}^{2} -\frac{1}{\zeta_k}\left(x_k - x_k^0\right)^2} \mathrm{d}x_1 \mathrm{d}x_2 = e^{2d} \prod_{k=1}^{2} \int_{\mathbb{R}} e^{-\frac{1}{\zeta_k}x_k^2} \mathrm{d}x_k = e^{2d} \prod_{k=1}^{2} (\pi\zeta_k)^{\frac{1}{2}} = 1, \tag{86}
$$

$$
\begin{aligned}
\triangle\, \mathbf{x}^2 &= e^{2d} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(x_1 - x_1^0\right)^2 e^{\sum_{k=1}^{2} -\frac{1}{\zeta_k}\left(x_k - x_k^0\right)^2} \mathrm{d}x_1 \mathrm{d}x_2 + e^{2d} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(x_2 - x_2^0\right)^2 e^{\sum_{k=1}^{2} -\frac{1}{\zeta_k}\left(x_k - x_k^0\right)^2} \mathrm{d}x_1 \mathrm{d}x_2 \\
&= e^{2d} \int_{\mathbb{R}} x_1^2 e^{-\frac{1}{\zeta_1}x_1^2} \mathrm{d}x_1 \int_{\mathbb{R}} e^{-\frac{1}{\zeta_2}x_2^2} \mathrm{d}x_2 + e^{2d} \int_{\mathbb{R}} x_2^2 e^{-\frac{1}{\zeta_2}x_2^2} \mathrm{d}x_2 \int_{\mathbb{R}} e^{-\frac{1}{\zeta_1}x_1^2} \mathrm{d}x_1 \\
&= \frac{\zeta_1 + \zeta_2}{2},
\end{aligned} \tag{87}
$$

$$\begin{aligned}
\triangle\, \mathbf{w}^2 &= \sum_{k=1}^{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\omega_k - \omega_k^0\right)^2 \left|\hat{f}(\mathbf{w})\right|^2 \mathrm{d}\mathbf{w} \\
&= \sum_{k=1}^{2} \left[\frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left|\frac{\partial f}{\partial x_k}\right|^2 \mathrm{d}x_1 \mathrm{d}x_2 - \left(\omega_k^0\right)^2 \right] \\
&= \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\frac{1}{\zeta_1^2} \left(x_1 - x_1^0\right)^2 + 4\pi^2 \left(\frac{1}{\varepsilon_1} \left(x_1 - x_1^0\right) + \omega_1^0 \right)^2 \right] e^{\sum_{k=1}^{2} -\frac{1}{\zeta_k}\left(x_k - x_k^0\right)^2 + 2d} \mathrm{d}x_1 \mathrm{d}x_2 - \left(\omega_1^0\right)^2 \\
&\quad + \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\frac{1}{\zeta_2^2} \left(x_2 - x_2^0\right)^2 + 4\pi^2 \left(-\frac{1}{\varepsilon_2} \left(x_2 - x_2^0\right) + \omega_2^0 \right)^2 \right] e^{\sum_{k=1}^{2} -\frac{1}{\zeta_k}\left(x_k - x_k^0\right)^2 + 2d} \mathrm{d}x_1 \mathrm{d}x_2 - \left(\omega_2^0\right)^2 \\
&= \frac{\zeta_1}{2} \left(\frac{1}{4\pi^2 \zeta_1^2} + \frac{1}{\varepsilon_1^2} \right) + \frac{\zeta_2}{2} \left(\frac{1}{4\pi^2 \zeta_2^2} + \frac{1}{\varepsilon_2^2} \right),
\end{aligned} \tag{88}$$

$$\begin{aligned}
\mathrm{Cov}_{\mathbf{x},\mathbf{w}} &= \sum_{k=1}^{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(x_k - x_k^0\right) \left(\frac{\partial \varphi}{\partial x_k} - \omega_k^0 \right) e^{\sum_{m=1}^{2} -\frac{1}{\zeta_m}\left(x_m - x_m^0\right)^2 + 2d} \mathrm{d}x_1 \mathrm{d}x_2 \\
&= \frac{1}{\varepsilon_1} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(x_1 - x_1^0\right)^2 e^{\sum_{m=1}^{2} -\frac{1}{\zeta_m}\left(x_m - x_m^0\right)^2 + 2d} \mathrm{d}x_1 \mathrm{d}x_2 - \frac{1}{\varepsilon_2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(x_2 - x_2^0\right)^2 e^{\sum_{m=1}^{2} -\frac{1}{\zeta_m}\left(x_m - x_m^0\right)^2 + 2d} \mathrm{d}x_1 \mathrm{d}x_2 \\
&= \frac{\zeta_1}{2\varepsilon_1} - \frac{\zeta_2}{2\varepsilon_2},
\end{aligned} \tag{89}$$

$$\begin{aligned}
\mathrm{COV}_{\mathbf{x},\mathbf{w}} &= \sum_{k=1}^{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left|x_k - x_k^0\right| \left| \frac{\partial \varphi}{\partial x_k} - \omega_k^0 \right| e^{\sum_{m=1}^{2} -\frac{1}{\zeta_m}\left(x_m - x_m^0\right)^2 + 2d} \mathrm{d}x_1 \mathrm{d}x_2 \\
&= \sum_{k=1}^{2} \frac{1}{\varepsilon_k} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(x_k - x_k^0\right)^2 e^{\sum_{m=1}^{2} -\frac{1}{\zeta_m}\left(x_m - x_m^0\right)^2 + 2d} \mathrm{d}x_1 \mathrm{d}x_2 \\
&= \frac{\zeta_1}{2\varepsilon_1} + \frac{\zeta_2}{2\varepsilon_2},
\end{aligned} \tag{90}$$

$$\begin{aligned}
\triangle\, \mathbf{u}_\alpha^2 &= \cos^2\alpha \,\triangle\, \mathbf{x}^2 + \sin^2\alpha \,\triangle\, \mathbf{w}^2 + 2\sin\alpha\cos\alpha \mathrm{Cov}_{\mathbf{x},\mathbf{w}} \\
&= \frac{\zeta_1 + \zeta_2}{2} \cos^2\alpha + \left[\frac{\zeta_1}{2} \left(\frac{1}{4\pi^2 \zeta_1^2} + \frac{1}{\varepsilon_1^2} \right) + \frac{\zeta_2}{2} \left(\frac{1}{4\pi^2 \zeta_2^2} + \frac{1}{\varepsilon_2^2} \right) \right] \sin^2\alpha + \left(\frac{\zeta_1}{\varepsilon_1} - \frac{\zeta_2}{\varepsilon_2} \right) \sin\alpha\cos\alpha,
\end{aligned} \tag{91}$$

$$\begin{aligned}
\triangle\, \mathbf{u}_\beta^2 &= \cos^2\beta \,\triangle\, \mathbf{x}^2 + \sin^2\beta \,\triangle\, \mathbf{w}^2 + 2\sin\beta\cos\beta \mathrm{Cov}_{\mathbf{x},\mathbf{w}} \\
&= \frac{\zeta_1 + \zeta_2}{2} \cos^2\beta + \left[\frac{\zeta_1}{2} \left(\frac{1}{4\pi^2 \zeta_1^2} + \frac{1}{\varepsilon_1^2} \right) + \frac{\zeta_2}{2} \left(\frac{1}{4\pi^2 \zeta_2^2} + \frac{1}{\varepsilon_2^2} \right) \right] \sin^2\beta + \left(\frac{\zeta_1}{\varepsilon_1} - \frac{\zeta_2}{\varepsilon_2} \right) \sin\beta\cos\beta.
\end{aligned} \tag{92}$$

From (87) and (88), there is

$$\begin{aligned}
\triangle\, \mathbf{x}^2 \,\triangle\, \mathbf{w}^2 &= \frac{\zeta_1 + \zeta_2}{2} \left[\frac{\zeta_1}{2} \left(\frac{1}{4\pi^2 \zeta_1^2} + \frac{1}{\varepsilon_1^2} \right) + \frac{\zeta_2}{2} \left(\frac{1}{4\pi^2 \zeta_2^2} + \frac{1}{\varepsilon_2^2} \right) \right] \\
&= \frac{1}{16\pi^2} \left(\frac{1}{\zeta_1} + \frac{1}{\zeta_2} \right) (\zeta_1 + \zeta_2) + \frac{\zeta_1^2}{4\varepsilon_1^2} + \frac{\zeta_2^2}{4\varepsilon_2^2} + \left(\frac{1}{4\varepsilon_1^2} + \frac{1}{4\varepsilon_2^2} \right) \zeta_1 \zeta_2.
\end{aligned} \tag{93}$$

Using the fact that the inequalities

$$\frac{\zeta_2}{\zeta_1} + \frac{\zeta_1}{\zeta_2} \geq 2 \tag{94}$$

and

$$\frac{1}{\varepsilon_1^2}+\frac{1}{\varepsilon_2^2}\geq\frac{2}{\varepsilon_1\varepsilon_2} \tag{95}$$

hold, it follows that

$$\triangle\,\mathbf{x}^2\,\triangle\,\mathbf{w}^2\geq\frac{1}{4\pi^2}+\left(\frac{\zeta_1}{2\varepsilon_1}+\frac{\zeta_2}{2\varepsilon_2}\right)^2. \tag{96}$$

It therefore concludes from (90) that

$$\triangle\,\mathbf{x}^2\,\triangle\,\mathbf{w}^2\geq\frac{1}{4\pi^2}+\mathrm{COV}^2_{\mathbf{x},\mathbf{w}}>\frac{1}{4\pi^2}. \tag{97}$$

From (87) and (91), there is

$$\triangle\,\mathbf{x}^2\,\triangle\,\mathbf{u}_\alpha^2=\frac{(\zeta_1+\zeta_2)^2}{4}\cos^2\alpha+\frac{\zeta_1+\zeta_2}{2}\left[\frac{\zeta_1}{2}\left(\frac{1}{4\pi^2\zeta_1^2}+\frac{1}{\varepsilon_1^2}\right)+\frac{\zeta_2}{2}\left(\frac{1}{4\pi^2\zeta_2^2}+\frac{1}{\varepsilon_2^2}\right)\right]\sin^2\alpha+\frac{\zeta_1+\zeta_2}{2}\left(\frac{\zeta_1}{\varepsilon_1}-\frac{\zeta_2}{\varepsilon_2}\right)\sin\alpha\cos\alpha. \tag{98}$$

Using (96) gives

$$\begin{aligned}\triangle\,\mathbf{x}^2\,\triangle\,\mathbf{u}_\alpha^2 &\geq \frac{(\zeta_1+\zeta_2)^2}{4}\cos^2\alpha+\left[\frac{1}{4\pi^2}+\left(\frac{\zeta_1}{2\varepsilon_1}+\frac{\zeta_2}{2\varepsilon_2}\right)^2\right]\sin^2\alpha+\frac{\zeta_1+\zeta_2}{2}\left(\frac{\zeta_1}{\varepsilon_1}-\frac{\zeta_2}{\varepsilon_2}\right)\sin\alpha\cos\alpha\\ &=\left[\frac{1}{4\pi^2}+\left(\frac{\zeta_1}{2\varepsilon_1}+\frac{\zeta_2}{2\varepsilon_2}\right)^2-\left(\frac{\zeta_1}{2\varepsilon_1}-\frac{\zeta_2}{2\varepsilon_2}\right)^2\right]\sin^2\alpha+\left[\frac{\zeta_1+\zeta_2}{2}\cos\alpha+\left(\frac{\zeta_1}{2\varepsilon_1}-\frac{\zeta_2}{2\varepsilon_2}\right)\sin\alpha\right]^2.\end{aligned} \tag{99}$$

It therefore concludes from (87), (89) and (90) that

$$\triangle\,\mathbf{x}^2\,\triangle\,\mathbf{u}_\alpha^2\geq\left(\frac{1}{4\pi^2}+\mathrm{COV}^2_{\mathbf{x},\mathbf{w}}-\mathrm{Cov}^2_{\mathbf{x},\mathbf{w}}\right)\sin^2\alpha+\left[\cos\alpha\,\triangle\,\mathbf{x}^2+\sin\alpha\mathrm{Cov}_{\mathbf{x},\mathbf{w}}\right]^2>\frac{1}{4\pi^2}\sin^2\alpha. \tag{100}$$

From (91) and (92), there is

$$\begin{aligned}&\triangle\,\mathbf{u}_\alpha^2\,\triangle\,\mathbf{u}_\beta^2\\ =&\left[\frac{\zeta_1+\zeta_2}{2}\cos^2\alpha+\left(\frac{\zeta_1}{2}\left(\frac{1}{4\pi^2\zeta_1^2}+\frac{1}{\varepsilon_1^2}\right)+\frac{\zeta_2}{2}\left(\frac{1}{4\pi^2\zeta_2^2}+\frac{1}{\varepsilon_2^2}\right)\right)\sin^2\alpha+\left(\frac{\zeta_1}{\varepsilon_1}-\frac{\zeta_2}{\varepsilon_2}\right)\sin\alpha\cos\alpha\right]\\ &\times\left[\frac{\zeta_1+\zeta_2}{2}\cos^2\beta+\left(\frac{\zeta_1}{2}\left(\frac{1}{4\pi^2\zeta_1^2}+\frac{1}{\varepsilon_1^2}\right)+\frac{\zeta_2}{2}\left(\frac{1}{4\pi^2\zeta_2^2}+\frac{1}{\varepsilon_2^2}\right)\right)\sin^2\beta+\left(\frac{\zeta_1}{\varepsilon_1}-\frac{\zeta_2}{\varepsilon_2}\right)\sin\beta\cos\beta\right]\\ =&\left[\frac{\zeta_1+\zeta_2}{2}\left(\frac{\zeta_1}{2}\left(\frac{1}{4\pi^2\zeta_1^2}+\frac{1}{\varepsilon_1^2}\right)+\frac{\zeta_2}{2}\left(\frac{1}{4\pi^2\zeta_2^2}+\frac{1}{\varepsilon_2^2}\right)\right)-\left(\frac{\zeta_1}{2\varepsilon_1}-\frac{\zeta_2}{2\varepsilon_2}\right)^2\right]\sin^2(\alpha-\beta)\\ &+\left[\frac{\zeta_1+\zeta_2}{2}\cos\alpha\cos\beta+\left(\frac{\zeta_1}{2}\left(\frac{1}{4\pi^2\zeta_1^2}+\frac{1}{\varepsilon_1^2}\right)+\frac{\zeta_2}{2}\left(\frac{1}{4\pi^2\zeta_2^2}+\frac{1}{\varepsilon_2^2}\right)\right)\sin\alpha\sin\beta+\left(\frac{\zeta_1}{2\varepsilon_1}-\frac{\zeta_2}{2\varepsilon_2}\right)\sin(\alpha+\beta)\right]^2\end{aligned} \tag{101}$$

Using (96) yields

$$\begin{aligned}&\triangle\,\mathbf{u}_\alpha^2\,\triangle\,\mathbf{u}_\beta^2\\ \geq&\left[\frac{1}{4\pi^2}+\left(\frac{\zeta_1}{2\varepsilon_1}+\frac{\zeta_2}{2\varepsilon_2}\right)^2-\left(\frac{\zeta_1}{2\varepsilon_1}-\frac{\zeta_2}{2\varepsilon_2}\right)^2\right]\sin^2(\alpha-\beta)\\ &+\left[\frac{\zeta_1+\zeta_2}{2}\cos\alpha\cos\beta+\left(\frac{\zeta_1}{2}\left(\frac{1}{4\pi^2\zeta_1^2}+\frac{1}{\varepsilon_1^2}\right)+\frac{\zeta_2}{2}\left(\frac{1}{4\pi^2\zeta_2^2}+\frac{1}{\varepsilon_2^2}\right)\right)\sin\alpha\sin\beta+\left(\frac{\zeta_1}{2\varepsilon_1}-\frac{\zeta_2}{2\varepsilon_2}\right)\sin(\alpha+\beta)\right]^2\end{aligned} \tag{102}$$

It therefore concludes from (87)–(90) that

$$\triangle\,\mathbf{u}_\alpha^2\,\triangle\,\mathbf{u}_\beta^2\geq\left(\frac{1}{4\pi^2}+\mathrm{COV}^2_{\mathbf{x},\mathbf{w}}-\mathrm{Cov}^2_{\mathbf{x},\mathbf{w}}\right)\sin^2(\alpha-\beta)+\left[\cos\alpha\cos\beta\,\triangle\,\mathbf{x}^2+\sin\alpha\sin\beta\,\triangle\,\mathbf{w}^2+\sin(\alpha+\beta)\mathrm{Cov}_{\mathbf{x},\mathbf{w}}\right]^2. \tag{103}$$

Particularly, for $\frac{\zeta_1}{\varepsilon_1}=\frac{\zeta_2}{\varepsilon_2}$, i.e., $\mathrm{Cov}_{\mathbf{x},\mathbf{w}}=0$, (103) becomes

$$\begin{aligned}\triangle\,\mathbf{u}_\alpha^2\,\triangle\,\mathbf{u}_\beta^2 &\geq \left(\frac{1}{4\pi^2}+\mathrm{COV}^2_{\mathbf{x},\mathbf{w}}\right)\sin^2(\alpha-\beta)+\left[\cos\alpha\cos\beta\,\triangle\,\mathbf{x}^2+\sin\alpha\sin\beta\,\triangle\,\mathbf{w}^2\right]^2\\ &> \frac{1}{4\pi^2}\sin^2(\alpha-\beta)+\left[\cos\alpha\cos\beta\,\triangle\,\mathbf{x}^2+\sin\alpha\sin\beta\,\triangle\,\mathbf{w}^2\right]^2.\end{aligned} \tag{104}$$

When $\zeta_1 = \zeta_2 = \zeta$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon$, the equality relations in (94) and (95) hold, implying that the equality in (96) is attained. It therefore concludes that

$$\triangle\, \mathbf{x}^2 \triangle\, \mathbf{w}^2 = \frac{1}{4\pi^2} + \mathrm{COV}^2_{\mathbf{x},\mathbf{w}} = \frac{1}{4\pi^2} + \frac{\zeta^2}{\varepsilon^2} > \frac{1}{4\pi^2}, \tag{105}$$

$$\begin{aligned}\triangle\, \mathbf{x}^2 \triangle\, \mathbf{u}^2_\alpha &= \left(\frac{1}{4\pi^2} + \mathrm{COV}^2_{\mathbf{x},\mathbf{w}} - \mathrm{Cov}^2_{\mathbf{x},\mathbf{w}}\right)\sin^2\alpha + \left[\cos\alpha \triangle\, \mathbf{x}^2 + \sin\alpha \mathrm{Cov}_{\mathbf{x},\mathbf{w}}\right]^2 \\ &= \left(\frac{1}{4\pi^2} + \frac{\zeta^2}{\varepsilon^2}\right)\sin^2\alpha + \zeta^2\cos^2\alpha \\ &> \frac{1}{4\pi^2}\sin^2\alpha,\end{aligned} \tag{106}$$

$$\begin{aligned}\triangle\, \mathbf{u}^2_\alpha \triangle\, \mathbf{u}^2_\beta &= \left(\frac{1}{4\pi^2} + \mathrm{COV}^2_{\mathbf{x},\mathbf{w}} - \mathrm{Cov}^2_{\mathbf{x},\mathbf{w}}\right)\sin^2(\alpha-\beta) + \left[\cos\alpha\cos\beta \triangle\, \mathbf{x}^2 + \sin\alpha\sin\beta \triangle\, \mathbf{w}^2 + \sin(\alpha+\beta)\mathrm{Cov}_{\mathbf{x},\mathbf{w}}\right]^2 \\ &= \left(\frac{1}{4\pi^2} + \frac{\zeta^2}{\varepsilon^2}\right)\sin^2(\alpha-\beta) + \left[\zeta\cos\alpha\cos\beta + \zeta\left(\frac{1}{4\pi^2\zeta^2} + \frac{1}{\varepsilon^2}\right)\sin\alpha\sin\beta\right]^2 \\ &> \frac{1}{4\pi^2}\sin^2(\alpha-\beta) + \left[\cos\alpha\cos\beta \triangle\, \mathbf{x}^2 + \sin\alpha\sin\beta \triangle\, \mathbf{w}^2\right]^2 \\ &= \frac{1}{4\pi^2}\sin^2(\alpha-\beta) + \left[\zeta\cos\alpha\cos\beta + \zeta\left(\frac{1}{4\pi^2\zeta^2} + \frac{1}{\varepsilon^2}\right)\sin\alpha\sin\beta\right]^2.\end{aligned} \tag{107}$$

In the example let $\zeta_1 = 1$, $\zeta_2 = \frac{1}{2}$, $\varepsilon_1 = 2$, $\varepsilon_2 = 1$, $\alpha = \frac{2\pi}{3}$ and $\beta = \frac{\pi}{6}$, it calculates that

$$\triangle\, \mathbf{x}^2 \triangle\, \mathbf{w}^2 \approx 0.309746582899407, \tag{108}$$

$$\frac{1}{4\pi^2} + \mathrm{COV}^2_{\mathbf{x},\mathbf{w}} \approx 0.275330295910584, \tag{109}$$

$$\frac{1}{4\pi^2} \approx 0.025330295910584, \tag{110}$$

$$\triangle\, \mathbf{x}^2 \triangle\, \mathbf{u}^2_\alpha \approx 0.372934937174556, \tag{111}$$

$$\left(\frac{1}{4\pi^2} + \mathrm{COV}^2_{\mathbf{x},\mathbf{w}} - \mathrm{Cov}^2_{\mathbf{x},\mathbf{w}}\right)\sin^2\alpha + \left[\cos\alpha \triangle\, \mathbf{x}^2 + \sin\alpha \mathrm{Cov}_{\mathbf{x},\mathbf{w}}\right]^2 \approx 0.347122721932938, \tag{112}$$

$$\frac{1}{4\pi^2}\sin^2\alpha \approx 0.018997721932938, \tag{113}$$

$$\triangle\, \mathbf{u}^2_\alpha \triangle\, \mathbf{u}^2_\beta \approx 0.331041346184749, \tag{114}$$

$$\left(\frac{1}{4\pi^2} + \mathrm{COV}^2_{\mathbf{x},\mathbf{w}} - \mathrm{Cov}^2_{\mathbf{x},\mathbf{w}}\right)\sin^2(\alpha-\beta) + \left[\cos\alpha\cos\beta \triangle\, \mathbf{x}^2 + \sin\alpha\sin\beta \triangle\, \mathbf{w}^2 + \sin(\alpha+\beta)\mathrm{Cov}_{\mathbf{x},\mathbf{w}}\right]^2 \approx 0.296625059195926, \tag{115}$$

$$\frac{1}{4\pi^2}\sin^2(\alpha-\beta) + \left[\cos\alpha\cos\beta \triangle\, \mathbf{x}^2 + \sin\alpha\sin\beta \triangle\, \mathbf{w}^2\right]^2 \approx 0.046625059195926, \tag{116}$$

then it concludes from the view point of numerical simulation that the results (97), (100), (103) and (104) hold.

In the example let $\zeta_1 = \zeta_2 = 1$, $\varepsilon_1 = \varepsilon_2 = 2$, $\alpha = \frac{2\pi}{3}$ and $\beta = \frac{\pi}{6}$, it calculates that

$$\triangle\, \mathbf{x}^2 \triangle\, \mathbf{w}^2 \approx 0.275330295910584, \tag{117}$$

$$\frac{1}{4\pi^2} + \mathrm{COV}^2_{\mathbf{x},\mathbf{w}} \approx 0.275330295910584, \tag{118}$$

$$\frac{1}{4\pi^2} \approx 0.025330295910584, \tag{119}$$

$$\triangle\, \mathbf{x}^2 \triangle\, \mathbf{u}^2_\alpha \approx 0.456497721932938, \tag{120}$$

$$\left(\frac{1}{4\pi^2}+\mathrm{COV}_{\mathbf{x},\mathbf{w}}^2-\mathrm{Cov}_{\mathbf{x},\mathbf{w}}^2\right)\sin^2\alpha+\left[\cos\alpha\vartriangle\mathbf{x}^2+\sin\alpha\mathrm{Cov}_{\mathbf{x},\mathbf{w}}\right]^2\approx 0.456497721932938, \tag{121}$$

$$\frac{1}{4\pi^2}\sin^2\alpha\approx 0.018997721932938, \tag{122}$$

$$\vartriangle\mathbf{u}_\alpha^2\vartriangle\mathbf{u}_\beta^2\approx 0.373795204665280, \tag{123}$$

$$\left(\frac{1}{4\pi^2}+\mathrm{COV}_{\mathbf{x},\mathbf{w}}^2-\mathrm{Cov}_{\mathbf{x},\mathbf{w}}^2\right)\sin^2(\alpha-\beta)+\left[\cos\alpha\cos\beta\vartriangle\mathbf{x}^2+\sin\alpha\sin\beta\vartriangle\mathbf{w}^2+\sin(\alpha+\beta)\mathrm{Cov}_{\mathbf{x},\mathbf{w}}\right]^2\approx 0.373795204665280, \tag{124}$$

$$\frac{1}{4\pi^2}\sin^2(\alpha-\beta)+\left[\cos\alpha\cos\beta\vartriangle\mathbf{x}^2+\sin\alpha\sin\beta\vartriangle\mathbf{w}^2\right]^2\approx 0.123795204665280, \tag{125}$$

then it concludes from the view point of numerical simulation that the results (105), (106) and (107) hold.

5 | POTENTIAL APPLICATIONS

In the classical N-dimensional Heisenberg's uncertainty principle case the largest universal lower bound $\frac{N^2}{16\pi^2}\|f\|_2^4$ for all functions can be reached only if $\mathrm{COV}_{\mathbf{x},\mathbf{w}}=0$. The proposed new corollary provides full characterization of the functions that make the equality relation hold in the uncertainty inequality, giving rise to a tighter lower bound $\frac{N^2}{16\pi^2}\|f\|_2^4+\mathrm{COV}_{\mathbf{x},\mathbf{w}}^2$, which includes particular case the classical one when $\mathrm{COV}_{\mathbf{x},\mathbf{w}}=0$. The philosophy of the N-dimensional FRFT based uncertainty principles is similar to that for the uncertainty principles in the classical setting. In the classical uncertainty principle in two N-dimensional FRFT domains case the largest universal lower bound for all functions is $\frac{N^2}{16\pi^2}\|f\|_2^4\sin^2(\alpha-\beta)$. Our previous work shows that a sharper lower bound can be $\frac{N^2}{16\pi^2}\|f\|_2^4\sin^2(\alpha-\beta)+\left[\cos\alpha\cos\beta\vartriangle\mathbf{x}^2+\sin\alpha\sin\beta\vartriangle\mathbf{w}^2\right]^2$, but this holds only for real functions. In our current work, the proposed new theorem gives a further larger lower bound $\left(\frac{N^2}{16\pi^2}\|f\|_2^4+\mathrm{COV}_{\mathbf{x},\mathbf{w}}^2\right)\sin^2(\alpha-\beta)+\left[\cos\alpha\cos\beta\vartriangle\mathbf{x}^2+\sin\alpha\sin\beta\vartriangle\mathbf{w}^2\right]^2$ for real functions, a special form of the derived universal lower bound $\left(\frac{N^2}{16\pi^2}\|f\|_2^4+\mathrm{COV}_{\mathbf{x},\mathbf{w}}^2-\mathrm{Cov}_{\mathbf{x},\mathbf{w}}^2\right)\sin^2(\alpha-\beta)+\left[\cos\alpha\cos\beta\vartriangle\mathbf{x}^2+\sin\alpha\sin\beta\vartriangle\mathbf{w}^2+\sin(\alpha+\beta)\mathrm{Cov}_{\mathbf{x},\mathbf{w}}\right]^2$ for complex functions. In such a way the new results present stronger uncertainty inequalities that imply the weaker ones, disclosing more information on the uncertainty products to be estimated. Thus, the new uncertainty principles could be able to process whatever practical application problems the old ones might be useful in solving, resulting in better performance.

An alternative mathematical formulation of the classical N-dimensional Heisenberg's uncertainty principle is

$$\frac{N^2}{16\pi^2}\|f\|_2^4=\min\left\{\vartriangle\mathbf{x}^2\vartriangle\mathbf{w}^2:\mathbf{x}f(\mathbf{x}),\mathbf{w}\hat{f}(\mathbf{w})\in L^2(\mathbb{R}^N)\right\}, \tag{126}$$

where the minimum value $\frac{N^2}{16\pi^2}\|f\|_2^4$ of the uncertainty product $\vartriangle\mathbf{x}^2\vartriangle\mathbf{w}^2$ can be reached. For most functions this limit usually cannot be achieved, and the corresponding uncertainty product is actually larger than $\frac{N^2}{16\pi^2}\|f\|_2^4$. Our result indicates that a better estimate is $\frac{N^2}{16\pi^2}\|f\|_2^4+\mathrm{COV}_{\mathbf{x},\mathbf{w}}^2$. Similarly, an alternative mathematical formulation of the classical uncertainty principle for the N-dimensional FRFT is

$$\frac{N^2}{16\pi^2}\|f\|_2^4\sin^2\alpha=\min\left\{\vartriangle\mathbf{x}^2\vartriangle\mathbf{u}_\alpha^2:\mathbf{x}f(\mathbf{x}),\mathbf{w}\hat{f}(\mathbf{w})\in L^2(\mathbb{R}^N)\right\}, \tag{127}$$

where the minimum value $\frac{N^2}{16\pi^2}\|f\|_2^4\sin^2\alpha$ of the uncertainty product $\vartriangle\mathbf{x}^2\vartriangle\mathbf{u}_\alpha^2$ can be reached. Our previous result provides a better estimate which says that the uncertainty product cannot be smaller than $\frac{N^2}{16\pi^2}\|f\|_2^4\sin^2\alpha+\cos^2\alpha\left(\vartriangle\mathbf{x}^2\right)^2$ for real functions. Our current result shows that, because of $\mathrm{COV}_{\mathbf{x},\mathbf{w}}^2\geq\mathrm{Cov}_{\mathbf{x},\mathbf{w}}^2$, a further better estimate is $\left(\frac{N^2}{16\pi^2}\|f\|_2^4+\mathrm{COV}_{\mathbf{x},\mathbf{w}}^2-\mathrm{Cov}_{\mathbf{x},\mathbf{w}}^2\right)\sin^2\alpha+\left[\cos\alpha\vartriangle\mathbf{x}^2+\sin\alpha\mathrm{Cov}_{\mathbf{x},\mathbf{w}}\right]^2$. Uncertainty principles are suitable for the effective estimation of bandwidths. For instance, if $\vartriangle\mathbf{x}^2$ is known, it follows that

$$\vartriangle\mathbf{w}^2\geq\frac{\frac{N^2}{16\pi^2}\|f\|_2^4+\mathrm{COV}_{\mathbf{x},\mathbf{w}}^2}{\vartriangle\mathbf{x}^2}\geq\frac{\frac{N^2}{16\pi^2}\|f\|_2^4}{\vartriangle\mathbf{x}^2} \tag{128}$$

and

$$
\begin{aligned}
\triangle \mathbf{u}_\alpha^2 &\geq \frac{\left(\frac{N^2}{16\pi^2}\|f\|_2^4 + \mathrm{COV}_{\mathbf{x},\mathbf{w}}^2 - \mathrm{Cov}_{\mathbf{x},\mathbf{w}}^2\right)\sin^2\alpha + \left[\cos\alpha \triangle \mathbf{x}^2 + \sin\alpha \mathrm{Cov}_{\mathbf{x},\mathbf{w}}\right]^2}{\triangle \mathbf{x}^2} \\
&\geq \frac{\frac{N^2}{16\pi^2}\|f\|_2^4 \sin^2\alpha + \cos^2\alpha\left(\triangle \mathbf{x}^2\right)^2}{\triangle \mathbf{x}^2} \\
&\geq \frac{\frac{N^2}{16\pi^2}\|f\|_2^4 \sin^2\alpha}{\triangle \mathbf{x}^2}.
\end{aligned} \tag{129}
$$

Note that even the second term of the above inequality chains usually cannot be reached, except it is a chirp function given by (14) of Theorem 1.

The FRFT provides a mathematical model for analyzing and describing optical systems composed of an arbitrary sequence of thin lenses and sections of free space. Uncertainty relations are often used to estimate spreads in transformation domains. Therefore, uncertainty principles related to the spread in the FRFT domain reveal that the immediate application can be found in the discussion of some well-known optical physics phenomenons, such as the Fresnel diffraction and the FRFT system between planar surfaces.[17] We first consider a planar reference plane related to the scale parameter s. Fresnel diffracting it in the order observed at a distance d from the screen can be described by an FRFT with the rotational angle α satisfying $\tan\alpha = \frac{d}{s^2}$. Using (20), there exist an estimate to the spread of the observed field at the distance

$$
\triangle \mathbf{u}_\alpha^2 \geq \frac{1}{1+\left(\frac{s^2}{d}\right)^2} \frac{\frac{N^2}{16\pi^2}\|f\|_2^4 + \mathrm{COV}_{\mathbf{x},\mathbf{w}}^2}{\triangle \mathbf{x}^2} + \frac{1}{1+\left(\frac{d}{s^2}\right)^2} \triangle \mathbf{x}^2 + \frac{2}{\frac{d}{s^2}+\frac{s^2}{d}} \mathrm{Cov}_{\mathbf{x},\mathbf{w}}, \tag{130}
$$

implying that for short d the effective spread $2\sqrt{\triangle \mathbf{u}_\alpha^2}$ is slightly larger than that at the planar reference plane $2\sqrt{\triangle \mathbf{x}^2}$ and for large distances d the spread of the field is almost independent of d and reciprocally proportional to the field spread in the planar reference plane. We then focus on two planar surfaces associated with the scale parameter s. Using a lens to compensate the spherical phase factors at both surfaces yields an FRFT system with the rotational angle α satisfying $\sin\alpha = \frac{d}{s^2}$ and $\tan\left(\frac{\alpha}{2}\right) = \frac{z}{s^2}$, where d and z denote the separation of the lenses and their focal length respectively. Therefore the relation (20) becomes

$$
\triangle \mathbf{u}_\alpha^2 \geq \frac{d^2}{s^4} \frac{\frac{N^2}{16\pi^2}\|f\|_2^4 + \mathrm{COV}_{\mathbf{x},\mathbf{w}}^2}{\triangle \mathbf{x}^2} + \left(\frac{d}{z}-1\right)^2 \triangle \mathbf{x}^2 + 2\frac{d}{s^2}\left(\frac{d}{z}-1\right)\mathrm{Cov}_{\mathbf{x},\mathbf{w}}, \tag{131}
$$

implying that for small d the effective spread $2\sqrt{\triangle \mathbf{u}_\alpha^2}$ is slightly larger than that at the planar surfaces $2\sqrt{\triangle \mathbf{x}^2}$ and for a pair of d and z with similar values the spread of the field is proportional to d or z and reciprocally proportional to the field spread in the planar surfaces.

6 | CONCLUSIONS

Uncertainty principles in two N-dimensional FRFT domains are investigated. The lower bounds obtained are tighter than the existing forms for three categories, those are, N-dimensional FT, N-dimensional FRFT and two N-dimensional FRFTs, in the literature. It turns out that the lower bounds are attainable by a chirp function with Gaussian envelop and quadratic phase. The correctness of the derived results is validated by example and experiment, and the effectiveness is illustrated by applications in the effective estimation of bandwidths in time-frequency analysis and spreads in optical system analysis.

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Conflict of interest

The author declares no potential conflict of interests.

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