Galerkin methods for fractional-stochastic systems

M.A. Akinlar[1], J.F. Gómez-Aguilar[2,*], Fatih Tasci[3]

[1] Bandirma Onyedi Eylul University,
Faculty of Engineering and Natural Sciences,
Engineering Sciences Department, 10200
Bandirma, Balikesir, Turkey

[2]CONACyT-Tecnológico Nacional de México/CENIDET,
Interior Internado Palmira S/N, Col. Palmira,
C.P. 62490, Cuernavaca, Morelos, México.
[*]E-mail: Corresponding author: jose.ga@cenidet.tecnm.mx

[3]Yildiz Technical University, Department of Mathematical Engineering
Istanbul, Turkey
Email: jose.ga@cenidet.tecnm.mx

Abstract

Applicability of undetermined coefficients methods to several fractional-stochastic models is investigated. These models are mostly generated by fractional-order derivative operators and include a fractional white noise term. Application of a polynomial chaos algorithm to stochastic Lotka-Volterra and Benney systems are also investigated. Fractional-stochastic equations considered in this paper are totally original systems which may serve as models for many scientific and engineering phenomena. It is pointed out that Galerkin type methods employed in this paper may be efficiently applied to fractional-order systems having uncertainty or a noise term.

Keywords
Fractional calculus; Stochastic models; Galerkin methods; Noise term.

1 Introduction

Fractional-stochastic differential equations, [4–26], are highly employed tools in modeling of many different phenomena in scientific, engineering and financial applications of contemporary mathematics. An equation involving some terms of fractional-order derivative/integral operators, a noise (written as a derivative of Wiener process), a fractional-Brownian motion and some deterministic operators is said to be a fractional-stochastic differential equation. These types equations may include some additional terms such as Jump or Levy type noise. Riemann-Liouville, Caputo, Grünwald–Letnikov, Atangana-Baleanu are some of the well-known [1–3] fractional-order operators having nonsingular kernel with nonlocal derivative or integral property which makes these operators more efficient than deterministic maps. Although exact solutions of fractional-stochastic differential equations are obtained mostly via Ito's formula, these solutions are obtained only in some special cases. Therefore, numerical solution techniques including Euler's type, finite-differences, undetermined coefficients (Galerkin, Least-square, Collocation, Wavelets, Finite-Elements) are some of highly useful and significant solution techniques in computational mathematics.
Main purpose of the present research work is to study applicability of a few finite-element, polynomial chaos and Galerkin methods to some original differential equations. The fractional-stochastic system of equations considered in this paper consist of deterministic derivative operators, fractional-order derivative operator in Caputo and Riemann-Liouville (RL) sense, fractional-Brownian motion, standard Brownian motion, and fractional white noise. Each equation and system considered in this paper has not been studied in the history of research so far. From this point of view, the present paper is an original and new contribution to science. Investigating applicability different type of undetermined coefficients methods to fractional-stochastic models makes this research work not only original but also significantly useful for researchers in computational science.

2 Fundamentals of fractional Operators

Fractional-order calculus (or differential equations) generated by fractional-order operators (derivative and integral operators) found a significant place in applied and

computational mathematics in recent years. They were employed in many different scientific work at mathematics, physics, economics and engineering. Fractional-order operators take into account historical effects and have non-local computational ability which makes these operators more powerful and desirable in the modeling applications.

Let $g : (0, \infty) \mapsto \mathbb{R}$ be a function. Fractional-order integral operator of order β for g is defined by

$$J^{\beta} g(s) = \frac{1}{\Gamma(\beta)} \int_0^t (s - \sigma)^{\beta - 1} g(\sigma) d\sigma,$$

inhere $\Gamma(\cdot)$ denotes Gamma function.
For $(m \in \mathbb{N})$, Caputo-type derivative operator of order β, $m - 1 < \beta < m$, of $g(s)$ is given by

$$D^{\beta} g(s) = J^{m-\beta} \frac{d^m}{dt^m} g(s).$$

The RL type integral, $I_x^{\alpha} f(x)$ and derivative $R_x^{\alpha} f(x)$ of a function, say $f(x)$, of order α is:

$$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - \rho)^{\alpha - 1} f(\rho) d\rho, \quad x \in [a, b].$$

It is clear that

$$I_x^{\beta} \left(I_x^{\alpha} f(x) \right) = I_x^{\beta + \alpha} f(x) \text{ and } \frac{d}{dx} I_x^{1+\alpha} = I_x^{\alpha} f(x).$$

$$R_x^{\alpha} f(x) = \frac{1}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_a^x \frac{f(\rho) d\rho}{(x - \rho)^{\alpha + 1 - n}} \quad \text{for} \quad n - 1 < \alpha \leq n.$$

In particular, for $n = 2$, i.e. for $1 < \alpha \leq 2$, we have that

$$R_x^{\alpha} f(x) = \frac{1}{\Gamma(2 - \alpha)} \frac{\partial^2}{\partial x^2} \int_a^x \frac{f(\rho) d\rho}{(x - \rho)^{\alpha - 1}}.$$

3 Galerkin Method

Galerkin method is one of the special types of methods of undetermined coefficients. Next, we briefly overview underlying ideas of Galerkin method.

Consider the differential equation (which may be deterministic, fractional, stochastic or hybrid)

$$Fu(x) = k(x) \tag{1}$$

where F is a derivative operator, functions $u(x)$ is known as unknown and $k(x)$ is a given functions. Write $u(x)$ as

$$u(x) = \sum_{i=1}^{n} d_i \varphi_i(x) \tag{2}$$

in which $\varphi_i(x)$ is known as coordinate function and the coefficients, d_i are constants or functions to be determined. Let us write (2) in (1) and define the so called residual term $R(x)$ as

$$R(x) := Fu(x) - k(x) \tag{3}$$

and define

$$I_j(d_1, d_2, \ldots, d_n) := \int_a^b W_j(x) R(x) dx = 0. \tag{4}$$

Choosing weight function $W_j(x)$ in (4) as $W_j(x) := \varphi_i(x)$, resulting method is known as the Galerkin method.

4 Stochastic processes and fractional Brownian motion

A Wiener process or standard Brownian motion, eg. see [27–34] for details and features, is a stochastic process $(W_x)_{x \geq 0}$ defined on a probability space $(\mathcal{X}, \mathcal{F}, \mathcal{P})$ with:

i. $W_0 = 0$,

ii. the function $x \mapsto W_x$ is a continuous function with probability 1,

iii. the increments $W_{x+n} - W_x$ have a normal distribution, $\mathcal{N}(0, n)$.

Independent increments mean $W_x - W_l$ and $W_k - W_m$ are independent random variables for $0 \leq l \leq x \leq m \leq k$. $W(x)$ satisfies that:

$$\begin{aligned}
\frac{dW(x)}{dx} &\approx \frac{\widehat{dW(x)}}{dx} = \sum_{k=1}^{n} \rho_k \delta_k(x), \\
\rho_k &= \frac{1}{\sigma} \int_I \delta_k(x) dW(x) \in \mathcal{N}(0, \frac{1}{\sigma}), \text{ that is a standard normal random variable,} \\
dW(x) &:= dW(x_{i+1}) - dW(x_i), \text{ where} \\
I &= \{I_k | I_k := [x_{k-1}, x_k), \ x_k - x_{k-1} = \sigma, \ x_0 = 0, \ x_n = 1\}
\end{aligned}$$

For a bounded function, f, it is not hard to see that

$$E\left(\int_I f(dW(x) - \widehat{dW(x)}) \right) = 0. \tag{5}$$

Fractional-Brownian motion (fBm) is also a highly useful and significant tool in modeling of many different phenomena in science, engineering and finance. Its long range dependence, correlated time increments and having Hurst parameter properties makes it quite useful tool for scientists. Interested reader may see definition and interesting properties of fBm, for example; in [46]. There are many different types of fBm in the literature. We consider

$$\begin{aligned}
B^H(s, x) &= \sum_{k=1}^{\infty} \sqrt{\mu_k} e_k(x) \alpha_k(s), \quad \sum_i \mu_i < \infty, \\
e_k(x) &= \sqrt{2} sin(k\pi x), \ k = 1, 2, \ldots, \\
B^H(s_n, x) - B^H(s_{n-1}, x) &= \sum_{k=1}^{\infty} \sqrt{\mu_k} e_k(x) (\alpha_k(s_n) - \alpha_k(s_{n-1})), \\
\alpha_k(s_n) - \alpha_k(s_{n-1}) &= \sqrt{\Delta_t}\, \zeta_{k,n}, \quad \zeta \in \mathcal{N}(0, 1).
\end{aligned}$$

In the next section, we introduce and study on the first fractional-stochastic differential model.

5 A PDE with fBm

In this section, we concern with numerical solutions of a fractional-stochastic partial differential equation (PDE) that includes second order time derivative, fractional white noise, a Levy noise term and some other derivatives of an unknown function of s and x. We study on an original differential equation to which methods of finite elements have not been applied in the literature. The purpose of studying such an equation is because these types of equations are so very useful in the modeling of many different phenomena in science and engineering and also because of the originality of equation. We are interested in numerical solutions of the equation

$$u_{ss}(s,x) - u_{xx}(s,x) - u(s,x) + dB^H(s) - \frac{\partial}{\partial x}\left[f(u(s,x))\frac{\partial u(s,x)}{\partial x}\right] = g(s,x) \tag{6}$$

where $(s,x) \in \Omega := [0,1] \times [0,1]$ and $u(s,x) = 0$ in $\partial\Omega$.

Weak form of (6):

$$\begin{gathered}\langle u_{ss}(s,x), w\rangle - \langle u_{xx}(s,x), w\rangle - \langle u(s,x), w\rangle + \left\langle dB^H(s,x), w\right\rangle \\ \left\langle -\frac{\partial}{\partial x}\left[f(u(s,x))\frac{\partial u(s,x)}{\partial x}\right], w\right\rangle = \langle g(s,x), w\rangle\end{gathered} \tag{7}$$

Next, we solve (6) with a finite element method and eventually obtain an element matrix equation which is solved with a finite difference and an implicit numerical method. Now, let us firstly write a variational formulation of (6). In order to achieve this; firstly, let us express the unknown function $u(s,x)$ as

$$u(s,x) := \sum_{i=1}^{M} a_i(s)\varphi_i(x) \tag{8}$$

where $\varphi_i(x)$ is a coordinate function.
Following the methodology of variational formulation, we write (8) in the equation (6) and then multiply each term in (6) with $\varphi_j(x)$ and integrate the resulting function

on the boundaries of x. As a result, we obtain that

$$\sum_{i=1}^{M} \ddot{a}_i(s) \int_0^1 \varphi_i(x)\varphi_j(x)dx - \sum_{i=1}^{M} a_i(s) \int_0^1 \frac{d^2\varphi_i(x)}{dx^2}\varphi_j(x)dx -$$

$$\sum_{i=1}^{M} a_i(s) \int_0^1 \varphi_i(x)\varphi_j(x)dx + \int_0^1 \varphi_j(x)dB^H(s,x)dx \tag{9}$$

$$\int_0^1 \frac{\partial}{\partial x}\left[f\left(\sum_{i=1}^{M} a_i(s)\varphi_i(x)\right)\sum_{i=1}^{M} a_i(s)\frac{d\varphi_i(x)}{dx}\right]\varphi_j(x)dx = \int_0^1 g(s,x)\varphi_j(x)dx$$

Our goal is to express the equation (9) in terms of some matrices. In order to achieve this, let us define the following matrices:
Define $K_{M\times M}$ with entries K_{ij} as

$$K_{ij} = \int_0^1 \varphi_i(x)\varphi_j(x)dx. \tag{10}$$

Hence, the first integral in (9) can be written

$$K^T\ddot{a} \tag{11}$$

where a is a $M \times 1$ vector with entries a_i, $i = 1, 2, \ldots, N$. Using the boundary conditions and integration by parts method, the second integral in (9) is written as

$$N^T a \tag{12}$$

where $N_{M\times M}$ is a matrix.
Now let us write the last two terms in (9) as follows:

$$\sum_{i=1}^{M} a_i(s) \int_0^1 f\left(\sum_{i=1}^{M} a_i(s)\varphi_i(x)\right)\frac{d\varphi_i(x)}{dx}\frac{d\varphi_j(x)}{dx}dx = \int_0^1 g(s,x)\varphi_j(x)dx \tag{13}$$

It is clear that we have a nonlinear systems of equations by the unknown $a_i(s)$ in the equation (13). Therefore, we can write (13) in terms of some matrices as

$$L(a)a = c \tag{14}$$

where $L_{M\times M}$ is a matrix with:

$$L_{ij} = \int_0^1 \left[f\left(\sum_{i=1}^{M} a_i(s)\varphi_i(x) \right) \frac{d\varphi_i(x)}{dx} \frac{d\varphi_j(x)}{dx} \right] dx$$

and c is a $M \times 1$ matrix with entries $c_i = \int_0^1 g(s,x)\varphi_j(x)dx$.
By using the notations for fBm described in the previous section, we express the fourth term, $\int_0^1 dB^H(s,x)\varphi_j(x)dx$, in (9) as:

$$\int_0^1 dB^H(s,x)\varphi_j(x)dx = \sum_{k=1}^{M} \sqrt{\mu_k}\alpha_k(s) \int_0^1 e_k(x)\varphi_j(x)dx,$$

is written as $M \times 1$ vector, say d.
Therefore, sorting these matrices out properly in the (9), we obtain the matrix equation:

$$K^T\ddot{a} - \left(N^T + K^T\right) a - L(a)a = c - d \tag{15}$$

Now, using the finite difference approximation for the time derivative, we can write (15) as

$$K^T \left[\frac{a_{k+1} - 2a_k + a_{k-1}}{\Delta t^2} \right] - N^T a_k - K^T a_k - L(a_k)a_k = c - d$$

from which we obtain that

$$a_{k+1} = \left[(K^T)^{-1} \left(N^T a_k + K^T a_k + L(a_k)a_k + c - d \right) \right] \Delta t^2 + 2a_k - a_{k-1}$$

In the next part, we study on a numerical solutions of fractional-stochastic differential equation where we employ fractional derivative operator in the Riemann-Liouville sense.

6 A fractional PDE with a noise term

Consider

$$R_x^{1+\alpha}u(x) - u_{xxx}(x) + dW(x) = g(x),\ x \in [0,1] \tag{16}$$

with $u(0) := 0,$ and $u(1) := k \in \mathbf{R} - 0$, where $R_x^{1+\alpha}u(x)$ is the Riemann-Liouville type derivative of $u(x)$. Let us apply the Galerkin method to Eq.(16) by following the methodology of Galaerkin method described above:

$$\int_0^1 R_x^{1+\alpha}u(x)\varphi_j(x)dx - \int_0^1 u_{xxx}(x)\varphi_j(x)dx + \int_0^1 dW(x)\varphi_j(x)dx = \int_0^1 g(x)\varphi_j(x)dx \tag{17}$$

where $\varphi_j(x)$ is nodal based function defined on Ω.
Defining

$$u = \sum_{k=0}^{M} c_k\varphi_k(x) \tag{18}$$

Let

$$\Delta := \frac{1}{n}$$

$\varphi_j(x)$ is an appropriate base function. Using integration by part in the second term in (17), we get:

$$\sum_{k=0}^{M} c_k \int_0^1 R_x^{\alpha}\varphi_k(x)\varphi_j(x)dx + \sum_{k=0}^{M} c_k \int_0^1 \frac{d^2\varphi_k(x)}{dx^2}\frac{d\varphi_j(x)}{dx}dx +$$

$$\sum_{k=}^{M}\int_0^1 \rho_k\delta_k(x)\varphi_k(x)dx = \int_0^1 g(x)\varphi_j(x)dx$$

Discretization of this equation is given as:

$$\tfrac{1}{\Delta}\left[\sum_{k=0}^{M} c_k \int_{x_{l-1}}^{x_l} R_x^{\alpha}\varphi_k(x)dx - \sum_{k=0}^{M} c_k \int_{x_l}^{x_{l+1}} R_x^{\alpha}\varphi_j(x)dx\right] +$$

$$\sum_{k=0}^{M} c_k \int_0^1 \frac{d^2\varphi_k(x)}{dx^2}\frac{d\varphi_j(x)}{dx}dx + \sum_{k=1}^{M}\int_0^1 \rho_k\delta_k(x)\varphi_k(x)dx = \int_0^1 g(x)\varphi_j(x)dx$$

We can express this equation in terms of matrices or as a matrix equation as follows:

$$\left(F^T - L^T\right) c + M - G = 0,$$

where,

$$M = \sum_{k=1}^{M} \int_0^1 \rho_k \delta_k(x) \varphi_k(x) dx$$

$$L_{ij} = \int_0^1 \frac{d^2\varphi_k(x)}{dx^2} \frac{d\varphi_j(x)}{dx} dx$$

$$G_j = \int_0^1 g(x)\varphi_j(x) dx$$

In the next section, we consider a system of fractional order differential equation to which we will investigate applicability of a Galerkin method. These equations are kind of deterministic fractional differential equations since we do not add a noise term because we already investigated the behavior of noise in the fractional-order differential equations in the previous examples.

7 A Fractional-Stochastic System of PDEs

Consider

$$\begin{aligned} D_s^\alpha u(s,x) &= \beta u_{xx}(s,x) + v_x(s,x) + h(s,x) \qquad (19) \\ D_s^\alpha v(s,x) &= v_{xx}(s,x) + \tau u_x(s,x) \end{aligned}$$

where α, β, τ are positive constant parameters, $x \in [0, A]$, $A > 0$, $0 \leq t$, with

$$\begin{aligned} u(0,x) = u(s,0) = v(s,0) &= v(0,x) = u(s,A) = v(s,A) = 0, \ \ x \in [0,A] \\ \frac{\partial u(s,x)}{\partial x} &= \frac{\partial v(s,x)}{\partial x} = 0, \ \text{ at } x = 0 \ \text{ and } x = \alpha, \ \text{ and } \ t \geq 0, \end{aligned}$$

$f(s,x)$ is continuous. The fractional-order derivative operator employed in the equations in this system is used in the Caputo sense.
Problems modeled by these types of systems of partial differential equations raise in physics, chemistry, and engineering. The reason that we chose to study this system of equations is because it is an original differential equation and many experience in science are modeled via systems of differential equations.
The FEM is a type of weighted residual technique that converts the system into a linear system of finite element matrix equations. These matrix equations are solved by computationally highly effective techniques.
We solve this system of fractional order differential equations (19) by means of a finite-element method described shortly as:

Define:

$$\begin{aligned} u(s,x) &\approx \hat{u}(s,x) = N(x)U^e(s) \\ v(s,x) &\approx \hat{v}(s,x) = N(x)V^e(s) \end{aligned} \tag{20}$$

(21)

where $N(x)$ is a row vector consisting of so-called basis functions on a space element (interval) with length L. Let us note that the basis functions that we employ in this study acts as the weight functions in the weighted residual method. Hence, in this part, we use a finite element method which is a type of weighted residual technique. Having noted these, it is clear that the approximated system of equations is written as:

$$D_s^{\alpha}\hat{u}(s,x) = \beta\hat{u}_{xx}(s,x) + \hat{v}_x(s,x) + h(s,x) \tag{22}$$

$$D_t s^{\alpha}\hat{v}(s,x) = \hat{v}_{xx}(s,x) + \tau\hat{u}_x(s,x) \tag{23}$$

We study one-space element, Γ_e, which is an interval in our work since we study on a one-dimensional space (interval). Bearing this in mind, we can write the first equation as:

$$D_s^{\alpha}\hat{u}(s,x) = \beta\hat{u}_{xx}(s,x) + \hat{v}_x(s,x) + h(s,x)$$

system (22) in terms of the notations in (20) as:

$$\int_{\Gamma_e} N^T(x)\Big[N(x)D_s^\alpha U^e(s) - \beta\frac{d}{dx}\left(\frac{dN(x)}{dx}U^e(s)\right) + \frac{dN(x)}{dx}V^e(s) + h(s,x)\Big]d\Gamma_e = \{0\},$$

By means of the method of integration by parts, we have that

$$\begin{aligned}\int_{\Gamma_e}\Big[N^T(x)N(x)D_s^\alpha U^e(s) - \beta\frac{dN(x)}{dx}\frac{dN^T(x)}{dx}U^e(s) + \\ N^T(x)\frac{dN(x)}{dx}V^e(s) + N^T(x)h(s,x)\Big]d\Gamma_e - \Big[\beta N^T(x)\frac{N(x)}{dx}U^e(s)\Big]_{\partial\Gamma} = \{0\},\end{aligned} \tag{24}$$

by using the fact $N(x) \equiv 0$ on $\partial\Gamma$, we get

$$\int_{\Gamma_e}\Big[N^T(x)N(x)D_s^\alpha U^e(s) - \beta\frac{dN(x)}{dx}\frac{dN^T(x)}{dx}U^e(s) + N^T(x)\frac{dN(x)}{dx}V^e(s) - N^T(x)h(s,x)\Big]d\Gamma_e = \{0\}, \tag{25}$$

We write (25) in terms of matrices as:

$$[A(x)]^e D_s^\alpha U^e(s) + ([B(x)]^e)\,U^e(s) + ([C(x)]^e)\,V^e(s) = H^e(s,x) \tag{26}$$

where

$$[A(x)]^e = \int_{\Gamma_e} N^T(x)N(x)d\Gamma_e$$

$$[B(x)]^e = -\int_{\Gamma_e}\beta\frac{dN^T(x)}{dx}\frac{dN(x)}{dx}d\Gamma_e$$

$$[C(x)]^e = \int_{\Gamma_e} N^T(x)\frac{dN(x)}{dx}d\Gamma_e$$

$$H^e(s,x) = N^T(x)h(s,x)$$

Now, in the similar manner, we write the second equation

$$D_s^{\alpha} v(s,x) \quad = \quad v_{xx}(s,x) + \tau u_x(s,x) \tag{27}$$

as:

$$\int\limits_{\Gamma_e} N^T(x)\Big[N(x)D_s^{\alpha}V^e(s) - \tfrac{d}{dx}\left(\tfrac{dN(x)}{dx}V^e(s)\right) + \tau\tfrac{dN(x)}{dx}U^e(s)\Big]d\Gamma_e = \{0\} \tag{28}$$

We can further restate this equation as:

$$\begin{gathered}\int\limits_{\Gamma_e}\Big[N^T(x)N(x)D_s^{\alpha}V^e(s) - \tfrac{dN^T(x)}{dx}\tfrac{dN(x)}{dx}V^e(s) + \\ \tau N^T(x)\tfrac{dN(x)}{dx}U^e(s)\Big]d\Gamma_e - \Big[N^T(x)\tfrac{dN(x)}{dx}V^e(s)\Big]_{\partial\Gamma_e} = \{0\}\end{gathered} \tag{29}$$

by the boundary conditions, we get

$$\begin{gathered}\int\limits_{\Gamma_e}\Big[N^T(x)N(x)D_s^{\alpha}V^e(s) - \tfrac{dN^T(x)}{dx}\tfrac{dN(x)}{dx}V^e(s) + \\ \tau N^T(x)\tfrac{dN(x)}{dx}U^e(s)\Big]d\Gamma_e = \{0\}\end{gathered} \tag{30}$$

Similarly, we have:

$$[A(x)]^e D_s^{\alpha}V^e(s) + \left(\frac{1}{\beta}[B(x)]^e V^e(s) = [C(x)]^e U^e(s)\right) \tag{31}$$

Now, simply by localizing the element matrices, we can rewrite the last two equations as:

$$[A(x)]^e D_s^{\alpha}U^e(s) + \left([B(x)]^e\right)U^e(s) + \left([C(x)]^e\right)V^e(s) = H^e(s,x) \tag{32}$$

$$[A(x)]^e D_s^{\alpha}V^e(s) + \left(\frac{1}{\beta}[B(x)]^e V^e(s) = [C(x)]^e U^e(s)\right) \tag{33}$$

In the next section, we apply a polynomial chaos method to a nonlinear stochastic differential equation called Banney equation and a nonlinear system of stochastic ordinary differential equations called Lotka-Volterra system.

8 Polynomial Chaos Method

Stochastic differential equations hold a significant place in engineering and economics due to their power in the correct modeling of systems including uncertainties, randomness and noise. In this section, we study applicability of polynomial chaos (PC) method to some stochastic nonlinear predator-prey models and partial differential equations. In particular, we concern with approximate solutions of stochastic Lotka-Volterra and Benney equations. Polynomial chaos technique [37, 43, 44] is a special case of undetermined coefficients or Galerkin methods. We solve stochastic systems approximately and present simulations to see the effects of uncertainties in the systems. This work is the first work to illustrate the applicability of polynomial chaos approach to the Lotka-Volterra system and Benney equation in the history of research.

The Lotka-Volterra (LV) equations, see e.g. [35, 38, 40–42] are some couple of first-order, non-linear system of differential equations mostly employed in mathematical biology, in particular, in the modeling of predator-prey model dynamics. We consider the Lotka-Volterra equations as stochastic (or random) processes. We introduce a new approach based on polynomial chaos (PC) expansions for numerical solutions of Lotka-Volterra (LV) equations containing some uncertainty appearing at the system. PC technique is one of the highly efficient techniques which perturbs the stochastic differential equation into a deterministic system of equations. In the literature, there have been several different approaches for the solutions of stochastic LV equations. For instance, at [39] Klebaner et al. consider the a stochastic LV model using the semi-martingale approach. They set up a a large deviation principle and obtain a bound for asymptotic of time to extinction of prey population.

Next, we overview polynomial chaos expansion method and introduce stochastic Lotka-Volterra models and Benney equations. We further deal with numerical solutions of these systems via application of PC method to these systems.

8.1 Stochastic Galerkin Method

Let us suppose that $(\Omega, \mathcal{A}, \mathcal{P})$ represent a probability space and consider a general time dependent stochastic partial differential equation with random coefficients as:

$$\begin{aligned} \frac{\partial u}{\partial s} + \Gamma(s,x,\nu;u) &= f, \quad (s,x) \in \mathcal{D} \times [0,T_f] \\ \mathcal{C}(s,x,\nu;u) &= g, \quad (s,x) \in \partial\mathcal{D} \times [0,T_f] \\ \mathcal{K}(0,x,\nu;u) &= h, \quad x \in \mathcal{D}, \end{aligned}$$

where coordinates are $s-$time, $x-$space and $\nu-$random or uncertain. The solution of this system might be written as

$$\begin{aligned} u(s,x,\nu) : \mathcal{D} \times [0,T_f] \times \Omega \to \mathbb{R} \\ \mathcal{D} \subset \mathbb{R}^D, \;\; D = 1,2,3. \end{aligned}$$

In order to solve this system, we must calculate $u(s,x,\nu)$. First, let us assume that

$$a(x,y) \equiv a(x, y_1(\nu), \ldots, y_d(\nu)).$$

The solution is then,

$$u(s,x,y_1(\nu),\ldots,y_d(\nu)) : \mathcal{D} \times [0,T] \times \prod_{i=1}^{d} \Gamma_i \to \mathbb{R}$$

$$y_i : \Omega \to \Gamma_i$$

Methods of undetermined coefficients including Galerkin, finite-elements, wavelets, collocation, least-squares are some popular methods in the literature to compute the $u(s,x,\nu)$. We are concerning with the so-called stochastic Galerkin method that can be shortly described as follows:
Given the random space

$$\mathcal{S} \equiv L_2(\mathcal{L}, \mathcal{P}_y),$$

i. Build an approximation space

$$\mathcal{S}_p \equiv Span\{\psi_0(y), \psi_1(y), \ldots, \psi_p(y)\} \subset \mathcal{S}$$

ii. Represent the uncertainties as

$$\hat{a}(x,y) = \sum_i a_i(x)\psi_i(y)$$

iii. Approximate the solution

$$\hat{u}(x,t,y) = \sum_i u_i(x,t)\psi_i(y)$$

- Compute the coefficients $u_i(x,t)$ via Galerkin projection.

Any random variable $u(\nu) \in L_2(\Omega, \mathcal{P})$ can be represented as

$$\begin{aligned} u(\nu) &= u_0 H_0 \\ &+ \sum_{i=1}^{\infty} u_i H_1(y_i(\nu)) \\ &+ \sum_{i=1}^{\infty}\sum_{k=1}^{i} u_{ik} H_2(y_i(\nu), y_k(\nu)) \\ &+ \sum_{i=1}^{\infty}\sum_{k=1}^{i}\sum_{j=1}^{k} u_{ikj} H_3(y_i(\nu), y_k(\nu), y_j(\nu)) + \dots \end{aligned}$$

where H_n is multi-dimensional Hermite polynomial.

$$u(y) \equiv u(y_1(\nu), \dots, y_d(\nu)),$$

in other words, finite dimensional uncertainty, the approximation may also be written in a finite-dimensional form. By a simpler notation, we state that

$$u(y(\nu)) = \sum_{i=0}^{\infty} u_i \psi_i(y(\nu)) = \sum_{i=0}^{\infty} u_i \psi_i(y)$$

where $\psi_i(y)$ is a multi-dimensional Hermite polynomial. The basis $\{\psi_i(y)\}_{i=0}^{\infty}$ is a complete basis in $L_2(\mathbb{R}^d, \mathcal{P}_y)$, i.e.,

$$\mathcal{S} \;=\; Span\{\psi_0(y), \psi_1(y), \dots\} = L_2(\mathbb{R}^d, \mathcal{P}_y)$$

A finite order approximation

$$\hat{u}(y) \;=\; \sum_{i=0}^{P} u_i \psi_i(y)$$

corresponds to a p-th (sotal) order approximation of $u(y)$ in

$$\mathcal{S}_p \;=\; Span\{\psi_0(y), \psi_1(y), \ldots, \psi_p(y)\} \subset \mathcal{S}$$

where the generalized Fourier coefficients u_i are obtained by

$$u_i = \langle u\psi_i \rangle / \langle \psi_i^2 \rangle$$

The expansion converges in mean-square sense

$$\lim_{p\to\infty} \langle (u - \hat{u})^2 \rangle = 0$$

For computational purposes, the representation

$$\langle f \rangle \;=\; \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} f(\xi) e^{-\frac{|\xi|^2}{2}} d\xi,$$

By representing the solution $u(s, x, w)$ as

$$u(s, x, \nu) \;=\; \sum_{i=0}^{P} u_i(s, x)\psi_i(\xi)$$

Solution expansion

$$\tilde{u}(x, t, y) \;=\; \sum_{j=0}^{P} u_j(x, t)\psi_j(y)$$

Galerkin projection on $\mathcal{S}_p$:

$$\begin{aligned}
\langle \frac{\partial \tilde{u}}{\partial s} + \Gamma(x, t, \cdot; \tilde{u}) - f, \psi_i \rangle &= 0 \quad i = 0, 1, \ldots, P \\
\langle \mathcal{C}(x, t, \cdot; \hat{u}) - g, \psi_i \rangle &= 0 \quad i = 0, 1, \ldots, P \\
\langle \mathcal{K}(x, 0, \cdot; \hat{u}) - h, \psi_i \rangle &= 0 \quad i = 0, 1, \ldots, P.
\end{aligned}$$

8.2 Solutions of stochastic Lotka-Volterra system

The Lotka-Volterra system:

$$\begin{aligned} u'(s,x,\nu) &= u(s,x,\nu)(a - bv(s,x,\nu)) = u(s,x,\nu)a - bu(s,x,\nu)v(s,x,\nu), \quad (34) \\ v'(s,x,\nu) &= v(s,x,\nu)(-c + du(s,x,\nu)) = -cv(s,x,\nu) + du(s,x,\nu)v(s,x,\nu), \end{aligned}$$

where a, b, c, d are constants.
We have that:

$$u(s,x,\nu) = \sum_{i=0}^{M} u_i(s,x)\psi_i(\xi(\nu)), \quad v(s,x,\nu) = \sum_{j=0}^{M} v_j(s,x)\psi_j(\xi(\nu))$$

$$a(\nu) = \sum_{i=0}^{M} a_i\psi_i(\xi(\nu)), \quad b(\nu) = \sum_{i=0}^{M} b_i\psi_i(\xi(\nu)),$$

$$c(\nu) = \sum_{i=0}^{M} c_i\psi_i(\xi(\nu)), \quad d(\nu) = \sum_{i=0}^{M} d_i\psi_i(\xi(\nu))$$

$$\sum_{i=0}^{M} \frac{\partial u_i}{\partial s}\psi_i = \sum_{i=0}^{M}\sum_{j=0}^{M} u_i(s,x)\psi_i a_j\psi_j - \sum_{j=0}^{M}\sum_{i=0}^{M}\sum_{k=0}^{M} b_j\psi_j u_i(s,x)\psi_i v_k(s,x)\psi_k$$

$$\sum_{j=0}^{M} \frac{\partial v_j}{\partial s}\psi_j = -\sum_{i=0}^{M}\sum_{j=0}^{M} c_i\psi_i v_j(s,x)\psi_j + \sum_{j=0}^{M}\sum_{i=0}^{M}\sum_{k=0}^{M} d_j\psi_j u_i(s,x)\psi_i v_k(s,x)\psi_k$$

$$\frac{\partial u_m}{\partial s} = \sum_{i=0}^{M}\sum_{j=0}^{M} \frac{<\psi_m\psi_i\psi_j>}{<\psi_m^2>} u_i(s,x)a_j - \sum_{j=0}^{M}\sum_{i=0}^{M}\sum_{k=0}^{M} \frac{<\psi_m\psi_j\psi_i\psi_k>}{<\psi_m^2>} b_j u_i(s,x)v_k(s,x)$$

$$\frac{\partial v_m}{\partial s} = -\sum_{i=0}^{M}\sum_{j=0}^{M} \frac{<\psi_m\psi_i\psi_j>}{<\psi_m^2>} c_i v_j(s,x) + \sum_{j=0}^{M}\sum_{i=0}^{M}\sum_{k=0}^{M} \frac{<\psi_m\psi_j\psi_i\psi_k>}{<\psi_m^2>} d_j u_i(s,x)v_k(s,x)$$

$$\frac{\partial u_m}{\partial s} \;=\; \frac{1}{<\psi_m^2>}\sum_{i=0}^{M}\sum_{j=0}^{M} u_i(s,x)a_j e_{mij} - \frac{1}{<\psi_m^2>}\sum_{j=0}^{M}\sum_{i=0}^{M}\sum_{k=0}^{M} b_j u_i(s,x)v_k(s,x)e_{mjik}$$

$$\frac{\partial v_m}{\partial s} \;=\; -\frac{1}{<\psi_m^2>}\sum_{i=0}^{M}\sum_{j=0}^{M} c_i v_j(s,x)e_{mij} + \frac{1}{<\psi_m^2>}\sum_{j=0}^{M}\sum_{i=0}^{M}\sum_{k=0}^{M} d_j u_i(s,x)v_k(s,x)e_{mjik}$$

where $m = 0, 1, 2, \ldots, N$, $e_{mij} =< \psi_m\psi_i\psi_j >$, $e_{mjik} =< \psi_m\psi_j\psi_i\psi_k >$.

8.3 Numerical Solution of Stochastic Benney Equation

The Benney equation is defined as [36, 45]:

$$u_s(s,x) + (u^n)_x(s,x) + u_{xx}(s,x) + \mu u_{xxx}(s,x) + u_{xxxx}(s,x) \;=\; 0, \tag{35}$$

where μ is a constant term. We will take $n = 2$ and consider the stochastic Benney equation. Stochastic Benney equation can be expressed as:

$$\begin{gathered} u_s(s,x,\nu) + 2u(s,x,\nu)u_x(s,x,\nu) + u_{xx}(s,x,\nu) + \\ \mu(s,\nu)u_{xxx}(s,x,\nu) + u_{xxxx}(s,x,\nu) \;=\; 0, \end{gathered} \tag{36}$$

with $u(0,x,\nu) \;=\; u^0(x,\nu)$.

$$\begin{gathered} u(s,x,\nu) \;=\; \sum_{i=0}^{M} u_i(s,x)\psi_i(\xi(\nu)), \quad \mu(\nu) \;=\; \sum_{i=0}^{M} \mu_i\psi_i(\xi(\nu)), \\ u^0(x,\nu) \;=\; \sum_{i=0}^{M} u_i^0(x)\psi_i(\xi(\nu)). \end{gathered} \tag{37}$$

Now using (37) at the equation (36) we obtain

$$\sum_{i=0}^{M} \frac{\partial u_i}{\partial s}\psi_i + 2\sum_{i=0}^{M}\sum_{j=0}^{M} u_j\psi_j\frac{\partial u_i}{\partial x}\psi_i + \sum_{i=0}^{M} \frac{\partial^2 u_i}{\partial x^2}\psi_i + \sum_{i=0}^{M}\sum_{j=0}^{M} \mu_j\psi_j\frac{\partial^3 u_i}{\partial x^3}\psi_i + \sum_{i=0}^{M} \frac{\partial^4 u_i}{\partial x^4}\psi_i \;=\; 0,$$

with the initial condition

$$\sum_{i=0}^{M} u_i(0,x)\psi_i \;=\; \sum_{i=0}^{M} u_i^0(x)\psi_i.$$

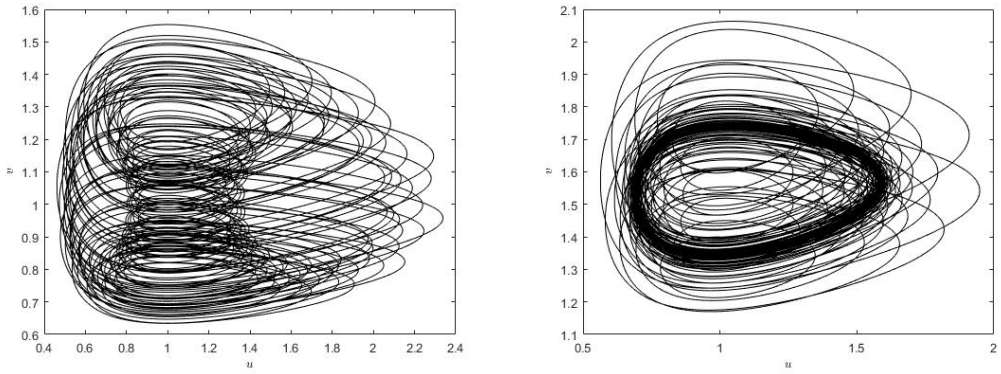
From (38) we get

$$\frac{\partial u_k}{\partial s} + 2\sum_{i=0}^{M}\sum_{j=0}^{M} \frac{<\psi_k\psi_j\psi_i>}{<\psi_k^2>} u_j \frac{\partial u_i}{\partial x} + \frac{\partial^2 u_k}{\partial x^2} + \sum_{i=0}^{M}\sum_{j=0}^{M} \frac{<\psi_k\psi_j\psi_i>}{<\psi_k^2>} \mu_j \frac{\partial^3 u_i}{\partial x^3} + \frac{\partial^4 u_k}{\partial x^4} \;=\; 0. \qquad (38)$$

$$\frac{\partial u_k}{\partial s} + \frac{2}{\psi_k^2}\sum_{i=0}^{M}\sum_{j=0}^{M} u_j \frac{\partial u_i}{\partial x} e_{kji} + \frac{\partial^2 u_k}{\partial x^2} + \frac{1}{<\psi_k^2>}\sum_{i=0}^{M}\sum_{j=0}^{M} \mu_j \frac{\partial^3 u_i}{\partial x^3} e_{kji} + \frac{\partial^4 u_k}{\partial x^4} \;=\; 0.$$

9 Numerical Simulations

Now, we showed applicability of polynomial chaos method to a predator-prey model and Benney equations, respectively. Figures 1a-1b show numerical simulations for Eq. (33).



(a) (b)

Figure 1: Numerical simulation for Eq. (33).

Figures 2a-2d show numerical simulations for Eq. (34).

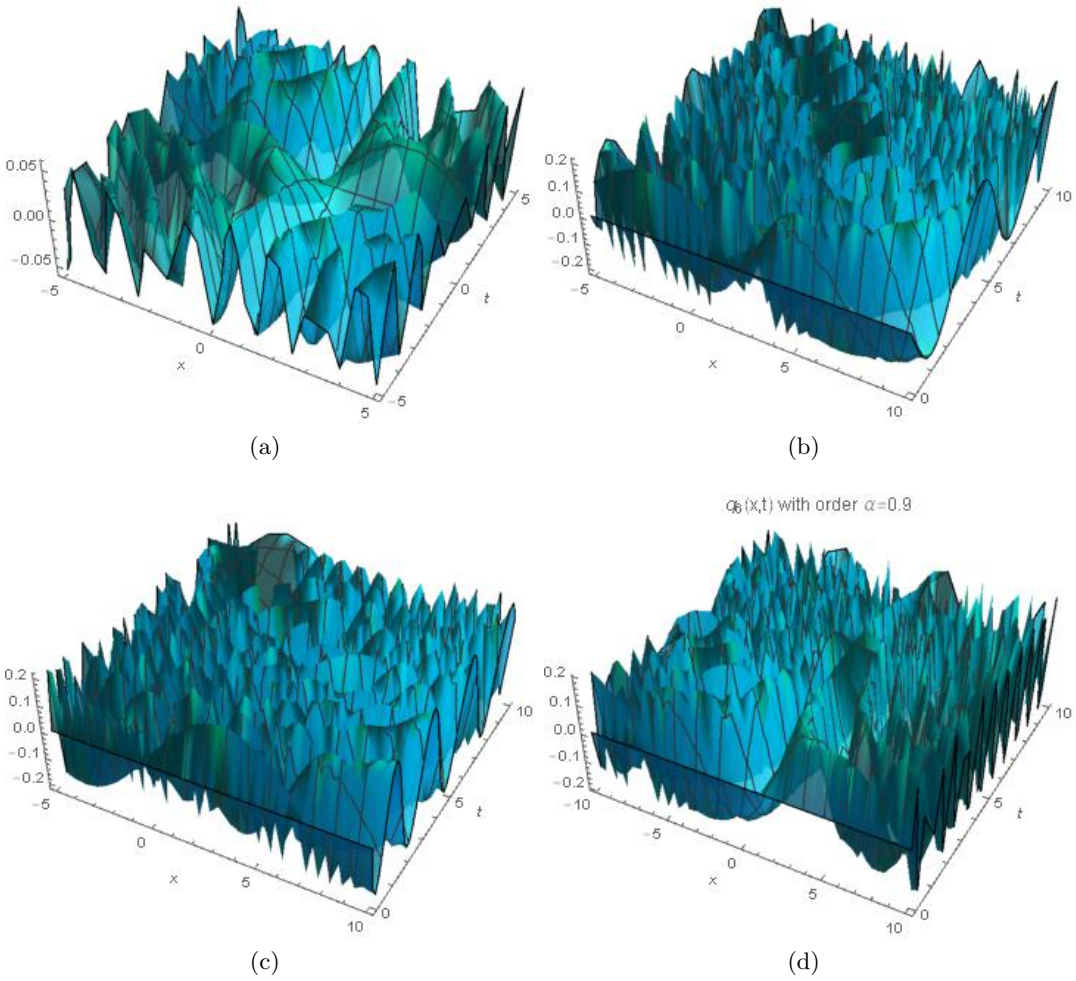


Figure 2: Numerical simulation for Eq. (34).

10 Conclusion and outlook

We studied on applicability of Galerkin type numerical solution methods to the approximations of fractional-stochastic models. We observed effect of Gaussian white noise, fractional-order operators and fractional Brownian motion. We also studied applicability of Polynomial chaos technique to the approximate solutions of stochastic LV and Benney systems. Computational results indicated that applicability of Galerkin type methods may be achieved in a highly efficient and strong manner. In a future extension of this research work, we will be concerning with fractional-stochastic equations involving Levy and Poisson jump uncertainties. We will investigate the combination of Galerkin methods with random walk numerical techniques.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Authors' contributions

M.A. Akinlar: Conceptualization, methodology, validation, investigation, Writing-original draft preparation, writing-review; J.F. Gómez-Aguilar: Conceptualization, methodology, validation, writing-review and editing; Fatih Tasci: Validation, formal analysis, investigation. All authors have read and agreed to the published version of the manuscript.

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