

LOCAL EXISTENCE AND GLOBAL NONEXISTENCE RESULTS FOR THE INTEGRO-DIFFERENTIAL DIFFUSION SYSTEM WITH NONLOCAL NONLINEARITY

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ABSTRACT. In the present paper initial problem for the integro-differential diffusion system with nonlocal nonlinear source is considered. The results on existence of local mild solutions and nonexistence of global weak solution to the nonlinear integro-differential diffusion system are presented.

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1. INTRODUCTION AND STATEMENT OF THE PROBLEM

The main goal of the present paper is to obtain results on local existence and global non-existence for the integro-differential diffusion system

$$\begin{cases} u_t(x, t) - D_{0|t}^{1-\alpha} \Delta u(x, t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} |v|^{p-1} v(s) ds, \\ v_t(x, t) - D_{0|t}^{1-\beta} \Delta v(x, t) = \frac{1}{\Gamma(1-\delta)} \int_0^t (t-s)^{-\delta} |u|^{q-1} u(s) ds, \end{cases} \quad (1.1)$$

for $(x, t) \in \mathbb{R}^N \times (0, T) = \Omega_T$, subject to the initial conditions

$$u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, \quad x \in \mathbb{R}^N, \quad (1.2)$$

where $\alpha, \beta, \gamma, \delta \in (0, 1)$, $p > 1$, $q > 1$, $D_{0|t}^\mu$ is the left-handed Riemann-Liouville fractional derivative of order $\mu \in (0, 1)$, and Γ is the gamma function of Euler.

In the first instance, the nonlinear memory term can be considered as an approximation of the classical semi-linear parabolic system (see [9] and the references

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therein)

$$\begin{cases} u_t - \Delta u = v^p, & \text{for } x \in \mathbb{R}^N, t > 0, \\ v_t - \Delta v = u^q, & \text{for } x \in \mathbb{R}^N, t > 0, \end{cases} \quad (1.3)$$

where $N \geq 1, p > 1, q > 1$ and

$$u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, \quad x \in \mathbb{R}^N,$$

since the limit

$$\lim_{\gamma \rightarrow 1} \frac{1}{\Gamma(1-\gamma)} s_+^{-\gamma} = \delta(s),$$

exists in the distribution sense, where δ is the Dirac distribution.

In [6], [7], [8], [22], it is shown that the critical exponent for the case $pq > 1$ of problem (1.3) is determined by

$$\frac{\gamma + 1}{pq - 1} \geq \frac{N}{2},$$

where $\gamma = \max\{p, q\}$.

Recently, Kirane et al. in [17] considered the system of non-linear parabolic equations with non-local diffusions

$$\begin{cases} u_t - aD_{0|t}^{1-\alpha} \Delta u = v^p, & \text{for } x \in \mathbb{R}^N, t > 0, \\ v_t - bD_{0|t}^{1-\beta} \Delta v = u^q, & \text{for } x \in \mathbb{R}^N, t > 0, \end{cases}$$

where $a, b > 0$ and the exponents $p, q \geq 1$ are real numbers, $D_{0|t}^\mu$ is the left-handed Riemann-Liouville fractional derivative of order $0 < \mu < 1$.

In [6], it is shown that the system does not admit global solutions whenever

$$\frac{N}{2} \leq \frac{\min\{p^2 + q, q(\alpha p^2 + 1), p(q\beta + p), pq(\alpha p + \beta - 1)\}}{p^2(q - 1) + q(p - 1)},$$

for $\alpha p + \beta > 2$, or

$$\frac{N}{2} \leq \frac{\min\{q^2 + p, p(\beta q^2 + 1), q(p\alpha + q), pq(\beta q + \alpha - 1)\}}{q^2(p - 1) + p(q - 1)},$$

for $\beta q + \alpha > 2$.

In the case $\alpha = \beta = 1$, problem (3.1)-(1.2) coincides with the Cauchy problem for the semi-linear parabolic system with a nonlinear memory, which was considered by Kirane et al. in [10]

$$\begin{cases} u_t(x, t) - \Delta u(x, t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} |v|^{p-1} v(s) ds, & x \in \mathbb{R}^N, t > 0, \\ v_t(x, t) - \Delta v(x, t) = \frac{1}{\Gamma(1-\delta)} \int_0^t (t-s)^{-\delta} |u|^{q-1} u(s) ds, & x \in \mathbb{R}^N, t > 0, \end{cases} \quad (1.4)$$

supplemented with the initial conditions

$$u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, \quad x \in \mathbb{R}^N, \quad (1.5)$$

where $u_0(x), v_0(x) \in C_0(\mathbb{R}^N)$, $\gamma, \delta \in (0, 1)$ and Γ is the Euler gamma function.

It was shown that it

$$\frac{N}{2} \leq \max \left\{ \frac{(2-\delta)p + (1-\gamma)pq + 1}{pq-1}, \frac{(2-\gamma)q + (1-\delta)pq + 1}{pq-1} \right\},$$

then the mild solution (u, v) of (1.4)-(1.5) blows up in a finite time.

Later on, Fino and Kirane [9] studied the spatio-temporally nonlinear parabolic equation

$$\begin{cases} u_t(x, t) + (-\Delta)^{\frac{\beta}{2}} u(x, t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} |u|^{p-1} u(s) ds & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \quad (1.6)$$

They showed that

(i) : For $u_0 \in C_0(\mathbb{R}^N)$, $u_0 \geq 0$, $u_0 \not\equiv 0$, if

$$p \leq 1 + \frac{\beta(2-\gamma)}{(N-\beta+\beta\gamma)_+} \text{ or } p < \frac{1}{\gamma},$$

then all solutions of problem (1.6) blows up in finite time.

(ii) : For $u_0 \in C_0(\mathbb{R}^N) \cap L^{p_{sc}}(\mathbb{R}^N)$, where $p_{sc} = N(p-1)/\beta(2-\gamma)$, if

$$p > \max \left\{ 1 + \frac{\beta(2-\gamma)}{(N-\beta+\beta\gamma)_+}; \frac{1}{\gamma} \right\},$$

and $\|u_0\|_{L^{p_{sc}}(\mathbb{R}^N)}$ is sufficiently small, then u exists globally.

In the case of a single equation

$$u_t(x, t) - \Delta u(x, t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} |u|^{p-1} u(s) ds \quad x \in \mathbb{R}^N, t > 0, \quad (1.7)$$

Cazenave et al. [4] addressed (with suitable change of variables) the local existence, global existence and blow-up questions while in the paper of Fino and Kirane [9], one can find the blow-up rate of solutions and the necessary conditions for local or global existence.

It was proven in [4], for $u \in C([0, T_{\max}), C_0(\mathbb{R}^N))$, $0 \leq \gamma \leq 1$, $p > 1$ with

$$p_\gamma = 1 + \frac{4-2\gamma}{(N-2+2\gamma)_+},$$

and

$$p_* = \max \left\{ \frac{1}{\gamma}, p_\gamma \right\} \in (0, \infty),$$

that

(i) : If $p \leq p_*$, and $u_0 \in C_0(\mathbb{R}^N)$, $u_0 \geq 0$, $u_0 \not\equiv 0$, then u blows up in finite time.

(ii) : If $p > p_*$, and $u_0 \in L^{q_{sc}}(\mathbb{R}^N)$ (where $q_{sc} = N(p-1)/(4-2\gamma)$) and $\|u_0\|_{L^{p_{sc}}(\mathbb{R}^N)}$ is sufficiently small, then u exists globally.

We also note that integro-differential diffusion equations of type (3.1) were studied in [2, 5, 11, 12, 20, 16, 19, 23].

The method used to prove the blow-up theorem is the test function method of Mitidieri and Pohozaev [21], Kirane et al. [14], [18].

2. PRELIMINARIES

Definition 2.1. [15] The left and right Riemann-Liouville fractional integrals $I_{0|t}^\alpha f(t)$ and $I_{t|T}^\alpha f(t)$ of order $\alpha \in \mathbb{R}$ ($\alpha > 0$), for all $f(t) \in L^q(0, T)$ ($1 \leq q \leq \infty$), we defined as

$$I_{0|t}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

and

$$I_{t|T}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} f(s) ds,$$

respectively.

Definition 2.2. [15] If $f(t) \in C([0, T])$ the left-handed and right-handed Riemann-Liouville fractional derivatives $D_{0|t}^\alpha f(t)$ and $D_{t|T}^\alpha f(t)$ of order $\alpha \in (0, 1)$ are defined by

$$D_{0|t}^\alpha f(t) = DI_{0|t}^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} D \int_0^t (t-s)^{-\alpha} f(s) ds,$$

and

$$D_{t|T}^\alpha f(t) = -DI_{t|T}^{1-\alpha} f(t) = -\frac{1}{\Gamma(1-\alpha)} D \int_t^T (s-t)^{-\alpha} f(s) ds,$$

for all $f(t) \in [0, T]$, where $D := \frac{d}{dt}$ is the usual time derivative.

Definition 2.3. [15] The left and right Caputo fractional derivatives of order $\alpha \in \mathbb{R}$ ($0 < \alpha < 1$), for an absolute continuous function f are defined, respectively, by

$$\mathcal{D}_{0|t}^\alpha f(t) = I_{0|t}^{1-\alpha} \frac{d}{dt} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds, \quad \forall t \in (0, T],$$

and

$$\mathcal{D}_{t|T}^\alpha f(t) = -I_{t|T}^{1-\alpha} \frac{d}{dt} f(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} f'(s) ds, \quad \forall t \in [0, T].$$

Property 2.4. Let $\alpha > 0, p \geq 1, q \geq 1$ and $1/p + 1/q \leq 1 + \alpha$. If $f(t) \in L^p(0, T), g(t) \in L^q(0, T)$, then we have the formula of integration by parts (see [15], Lemma 2.7. page 76)

$$\int_0^T I_{0|t}^\alpha [f(t)] g(t) dt = \int_0^T f(t) I_{t|T}^\alpha [g(t)] dt.$$

Property 2.5. For any $f(t) \in AC^2[0; T]$, we have (see (2.2.30) in [15])

$$-D \cdot D_{t|T}^\alpha f(t) = D_{t|T}^{1+\alpha} f(t),$$

where

$$AC^2[0; T] := \{f(t) : [0; T] \rightarrow \mathbb{R} \text{ such that } Df(t) \in AC[0; T]\}.$$

Property 2.6. If $f(t) \in L^q(0, T)$ ($1 \leq q \leq \infty$), then the following equality (see [15], Lemma 2.4 p.74)

$$D_{0|t}^\alpha I_{0|t}^\alpha f(t) = f(t),$$

holds almost everywhere on $[0, T]$.

Later on, we will use the following results (see [13]).

If $\psi_1(t) = (1 - \frac{t}{T})^\sigma, t \geq 0, T > 0, \sigma > 1$, then

$$D_{t|T}^{1-\alpha} \psi_1(t) = \frac{(\alpha + \sigma)\Gamma(\sigma + 1)}{\Gamma(1 + \alpha + \sigma)} T^{\alpha-1} \left(1 - \frac{t}{T}\right)^{\sigma+\alpha-1}, \quad (2.1)$$

$$D_{t|T}^{2-\alpha} \psi_1(t) = \frac{(\alpha + \sigma - 1)(\alpha + \sigma)\Gamma(\sigma + 1)}{\Gamma(1 - \alpha + \sigma)} T^{\alpha-2} \left(1 - \frac{t}{T}\right)^{\sigma+\alpha-2}, \quad (2.2)$$

$$\mathcal{D}_{t|T}^{1-\alpha} D_{t|T}^{1-\gamma} \psi_1(t) = \frac{(\sigma + \gamma - 1)\Gamma(\sigma + \gamma - 1)}{\Gamma(\alpha + \sigma + \gamma - 1)} T^{\alpha+\gamma-2} \left(1 - \frac{t}{T}\right)^{\sigma+\alpha+\gamma-2}, \quad (2.3)$$

for all $\alpha \in (0, 1)$; so

$$\left(D_{t|T}^{1-\alpha} \psi_1\right)(T) = 0, \quad \left(D_{t|T}^{1-\alpha} \psi_1\right)(0) = CT^{\alpha-1}, \quad (2.4)$$

where $C = \frac{(\alpha + \sigma)\Gamma(\sigma + 1)}{\Gamma(1 + \alpha + \sigma)}$.

3. LOCAL EXISTENCE

Definition 3.1. (Mild solution). Let $u_0, v_0 \in C_0(\mathbb{R}^N), T > 0$ and $p, q > 1$.

We say that $u, v \in C_0(\mathbb{R}^N, C([0, T]) \times C_0(\mathbb{R}^N, C([0, T]))$ is a mild solution of the system (3.1)-(1.2), if u and v satisfy the following integral equations [see [3], Th. 2.5]

$$\begin{cases} u(x, t) = \int_{\mathbb{R}^N} G(x - y, t) u_0(y) dy + \int_0^t \int_{\mathbb{R}^N} G(x - y, t - \tau) I_{0|s}^{1-\gamma}(|v|^{p-1}v) dy d\tau, \\ v(x, t) = \int_{\mathbb{R}^N} G(x - y, t) v_0(y) dy + \int_0^t \int_{\mathbb{R}^N} G(x - y, t - \tau) I_{0|s}^{1-\delta}(|u|^{q-1}u) dy d\tau, \end{cases} \quad (3.1)$$

for $t \in [0, T]$, where

$$G(x, t) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-i\langle x, \xi \rangle} E_{\alpha, 1}(-\xi^2 t^\alpha) d\xi.$$

Theorem 3.2. (Local existence). Given $u_0, v_0 \in C_0(\mathbb{R}^N)$ and $p, q > 1$, there exist a maximal time $T > 0$ and a unique mild solution $(u, v) \in C_0(\mathbb{R}^N, C[0, T]) \times C_0(\mathbb{R}^N, C[0, T])$ to the system (3.1)-(1.2).

Proof. For arbitrary $T > 0$, we define the Banach space

$$B_T = \{(u, v) \in C_0(\mathbb{R}^N, C[0, T]) \times C_0(\mathbb{R}^N, C[0, T])\};$$

$$\|(u, v)\|_{B_T} \leq 2(\|u_0\|_{L^\infty(\mathbb{R}^N)} + \|v_0\|_{L^\infty(\mathbb{R}^N)}), \quad (3.2)$$

where $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\mathbb{R}^N)}$ and $\|\cdot\|_{B_T}$ is the norm of B_T defined by

$$\|(u, v)\|_{B_T} = \|u\|_1 + \|v\|_1 = \|u\|_{L^\infty(\mathbb{R}^N, L^\infty[0, T])} + \|v\|_{L^\infty(\mathbb{R}^N, L^\infty[0, T])}.$$

Next, for every $(u, v) \in B_T$, we introduce the map Ψ defined on B_T by

$$\Psi(u, v) := (\Psi_1(u, v), \Psi_2(u, v)),$$

where

$$\begin{aligned} \Psi_1(u, v) &= \int_{\mathbb{R}^N} G(x - y, t) u_0(y) dy \\ &\quad + \int_0^t \int_{\mathbb{R}^N} G(x - y, t - \tau) I_{0|s}^{1-\gamma}(|v|^{p-1}v) dy d\tau, \quad t \in [0, T], \end{aligned}$$

and

$$\begin{aligned} \Psi_2(u, v) &= \int_{\mathbb{R}^N} G(x - y, t) v_0(y) dy \\ &\quad + \int_0^t \int_{\mathbb{R}^N} G(x - y, t - \tau) I_{0|s}^{1-\delta}(|u|^{q-1}u) dy d\tau, \quad t \in [0, T]. \end{aligned}$$

We will prove the local existence by the Banach fixed point theorem.

• Let $\Psi : B_T \rightarrow B_T$ and $(u, v) \in B_T$, according to Lemma 2.2 in [3], we obtain

$$\begin{aligned} \|\Psi(u, v)\|_{B_T} &\leq \|u_0\|_\infty + \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_0^s (s-\tau)^{-\gamma} \|v(\tau)\|_\infty^p d\tau ds \right\|_{L^\infty[0, T]} \\ &\quad + \|v_0\|_\infty + \frac{1}{\Gamma(1-\delta)} \left\| \int_0^t \int_0^s (s-\tau)^{-\delta} \|u(\tau)\|_\infty^q d\tau ds \right\|_{L^\infty[0, T]} \\ &\leq \|u_0\|_\infty + \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_\tau^t (s-\tau)^{-\gamma} \|v(\tau)\|_\infty^p ds d\tau \right\|_{L^\infty[0, T]} \\ &\quad + \|v_0\|_\infty + \frac{1}{\Gamma(1-\delta)} \left\| \int_0^t \int_\tau^t (s-\tau)^{-\delta} \|u(\tau)\|_\infty^q ds d\tau \right\|_{L^\infty[0, T]} \\ &\leq \|u_0\|_\infty + C_1 T^{2-\gamma} \|v\|_1^p + \|v_0\|_\infty + C_2 T^{2-\delta} \|u\|_1^q, \end{aligned}$$

where

$$C_1 := \frac{1}{(1-\gamma)(2-\gamma)\Gamma(1-\gamma)} = \frac{1}{\Gamma(3-\gamma)},$$

$$C_2 := \frac{1}{(1-\delta)(2-\delta)\Gamma(1-\delta)} = \frac{1}{\Gamma(3-\delta)}.$$

As $(u, v) \in B_T$, we get

$$\begin{aligned} \|\Psi(u, v)\|_{B_T} &\leq \|u_0\|_\infty + C_1 T^{2-\gamma} \|v\|_1^p + \|v_0\|_\infty + C_2 T^{2-\delta} \|u\|_1^q \\ &\leq \|u_0\|_\infty + \|v_0\|_\infty + \max\{C_1 T^{2-\gamma} \|v\|_1^{p-1}; C_2 T^{2-\delta} \|u\|_1^{q-1}\} (\|v\|_1 + \|u\|_1) \\ &\leq (\|u_0\|_\infty + \|v_0\|_\infty) + 2T(u_0, v_0)(\|u_0\|_\infty + \|v_0\|_\infty), \end{aligned}$$

where

$$T(u_0, v_0) = \max\{C_1 T^{2-\gamma} 2^{p-1} (\|u_0\|_\infty + \|v_0\|_\infty)^{p-1}; C_2 T^{2-\delta} 2^{q-1} (\|u_0\|_\infty + \|v_0\|_\infty)^{q-1}\}.$$

If we choose T small enough such that

$$2T(u_0, v_0) \leq 1, \quad (3.3)$$

we conclude that $\|\Psi(u)\|_1 \leq 2(\|u_0\|_\infty + \|v_0\|_\infty)$, and hence $\Psi(u, v) \in B_T$.

• Let Ψ is a contraction map: For $(u, v), (\tilde{u}, \tilde{v}) \in B_T$, we have the estimate

$$\begin{aligned} \|\Psi(u, v) - \Psi(\tilde{u}, \tilde{v})\|_{B_T} &\leq \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_0^s (s-\tau)^{-\gamma} \left\| |v|^{p-1}v(\tau) - |\tilde{v}|^{p-1}\tilde{v}(\tau) \right\|_\infty d\tau ds \right\|_{L^\infty[0,T]} \\ &\quad + \frac{1}{\Gamma(1-\delta)} \left\| \int_0^t \int_0^s (s-\tau)^{-\delta} \left\| |u|^{q-1}u(\tau) - |\tilde{u}|^{q-1}\tilde{u}(\tau) \right\|_\infty d\tau ds \right\|_{L^\infty[0,T]} \\ &= \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t \int_\tau^t (s-\tau)^{-\gamma} \left\| |v|^{p-1}v(\tau) - |\tilde{v}|^{p-1}\tilde{v}(\tau) \right\|_\infty ds d\tau \right\|_{L^\infty[0,T]} \\ &\quad + \frac{1}{\Gamma(1-\delta)} \left\| \int_0^t \int_\tau^t (s-\tau)^{-\delta} \left\| |u|^{q-1}u(\tau) - |\tilde{u}|^{q-1}\tilde{u}(\tau) \right\|_\infty ds d\tau \right\|_{L^\infty[0,T]} \\ &= C_1 T^{2-\gamma} \| |v|^{p-1}v - |\tilde{v}|^{p-1}\tilde{v} \|_1 + C_2 T^{2-\delta} \| |u|^{q-1}u - |\tilde{u}|^{q-1}\tilde{u} \|_1. \end{aligned}$$

Now, by the same computations as above, we obtain

$$\begin{aligned} \|\Psi(u, v) - \Psi(\tilde{u}, \tilde{v})\|_{B_T} &\leq C_1 T^{2-\gamma} \| |v|^{p-1}v - |\tilde{v}|^{p-1}\tilde{v} \|_1 \\ &\quad + C_2 T^{2-\delta} \| |u|^{q-1}u - |\tilde{u}|^{q-1}\tilde{u} \|_1 \\ &\leq C(p) C_1 T^{2-\gamma} (\|v^{p-1}\|_1 + \|\tilde{v}^{p-1}\|_1) \|v - \tilde{v}\|_1 \\ &\quad + C(q) C_2 T^{2-\delta} (\|u^{q-1}\|_1 + \|\tilde{u}^{q-1}\|_1) \|u - \tilde{u}\|_1 \\ &\leq 2C(p, q) T(u_0, v_0) \| (u, v) - (\tilde{u}, \tilde{v}) \| \\ &\leq \frac{1}{2} \| (u, v) - (\tilde{u}, \tilde{v}) \|, \end{aligned}$$

thanks to the following inequality

$$\| |u|^{p-1}u - |v|^{p-1}v \| \leq C(p) |u - v| (|u|^{p-1} + |v|^{p-1}), \quad (3.4)$$

T is chosen such that

$$\max\{2C(p, q), 1\} T(u_0, v_0) \leq \frac{1}{2}. \quad (3.5)$$

According to the Banach fixed point theorem, the system (3.1)-(1.2) admits a unique mild solution $(u, v) \in B_T$.

We can reiterate the process until we reach a maximal time T_{\max} for which we have the alternatives.

4. GLOBAL NONEXISTENCE

Definition 4.1. (Weak solution).

Let $u_0, v_0 \in L_{loc}^\infty(\mathbb{R}^N)$, and $T > 0$. Then, we say that

$$(u, v) \in L_{loc}^\infty(\mathbb{R}^N, L^q(0, T)) \times L_{loc}^\infty(\mathbb{R}^N, L^q(0, T))$$

is a weak solution of the system (3.1)-(1.2) if

$$\begin{aligned} \int_{\mathbb{R}^N} u_0 \varphi(x, 0) dx + \int_0^T \int_{\mathbb{R}^N} I_{0|t}^{1-\gamma}(v^p) \varphi dx dt \\ = - \int_0^T \int_{\mathbb{R}^N} u \mathcal{D}_{t|T}^{1-\alpha} \Delta \varphi dx dt - \int_0^T \int_{\mathbb{R}^N} u \varphi_t dx dt, \end{aligned} \quad (4.1)$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} v_0 \psi(x, 0) dx + \int_0^T \int_{\mathbb{R}^N} I_{0|t}^{1-\delta}(u^q) \psi dx dt \\ = - \int_0^T \int_{\mathbb{R}^N} v \mathcal{D}_{t|T}^{1-\beta} \Delta \psi dx dt - \int_0^T \int_{\mathbb{R}^N} v \psi_t dx dt, \end{aligned} \quad (4.2)$$

for any function $\varphi, \psi \in C^2(\mathbb{R}^N, C^1[0, T])$ such that $\varphi(x, T) = \psi(x, T) = 0$.

Lemma 4.2. (*Mild \rightarrow Weak solution*)

Assume $u_0, v_0 \in C_0(\mathbb{R}^N)$, let $u, v \in C_0(\mathbb{R}^N, C[0, T])$ be a mild solution of (3.1)-(1.2), then (u, v) is also a weak solution to the system (3.1)-(1.2).

For the proof of Lemma (4.2), see [[10], Lemma 4.2].

Theorem 4.3. Let $p > 1, q > 1$. Suppose that

$$\frac{N}{2} \leq \max \left\{ \frac{(2-\delta)p + (1-\gamma)pq + 1}{pq-1}, \frac{(2-\gamma)q + (1-\delta)pq + 1}{pq-1} \right\}.$$

Then, the system (3.1)-(1.2) does not admit nontrivial global nonnegative weak solutions.

Proof. The proof is by contradiction. Suppose that (u, v) is a global mild solution to (3.1)-(1.2), then (u, v) is a solution of (3.1)-(1.2) where $u, v \in C_0(\mathbb{R}^N, C[0; T])$, such that $u(t), v(t) > 0$ for all $t \in [0; T]$, $T > 1$.

Let us choose as in [10]

$$\varphi(x, t) = D_{t|T}^{1-\gamma} \tilde{\varphi}(x, t) := D_{t|T}^{1-\gamma} (\varphi_1^l(x) \varphi_2(t)),$$

and

$$\psi(x, t) = D_{t|T}^{1-\delta} \tilde{\varphi}(x, t) := D_{t|T}^{1-\delta} (\varphi_1^l(x) \varphi_2(t)),$$

with

$$\varphi_1(x) := \Phi \left(\frac{|x|}{T^{1/2}} \right),$$

and

$$\varphi_2(t) := \left(1 - \frac{t}{T} \right)^\eta,$$

where $l \geq \frac{pq}{(p-1)(q-1)}, \eta > 1$ and the function Φ is a smooth nonnegative non-increasing function such that

$$\Phi(z) = \begin{cases} 1, & \text{if } 0 \leq z \leq 1, \\ 0, & \text{if } z \geq 2, \end{cases}$$

and $0 \leq \Phi(z) \leq 1$, $z \mid \Phi'(z) \mid \leq C_1$, for all $z > 0$.

Using (2.4), we obtain

$$\int_{\Omega} u_0 D_{t|T}^{1-\gamma} \tilde{\varphi}(x, 0) + \int_{\Omega_T} I_{0|t}^{1-\gamma}(v^p) D_{t|T}^{1-\gamma} \tilde{\varphi} = - \int_{\Omega_T} u \Delta \mathcal{D}_{t|T}^{1-\alpha} D_{t|T}^{1-\gamma} \tilde{\varphi} - \int_{\Omega_T} u D D_{t|T}^{1-\gamma} \tilde{\varphi}, \quad (4.3)$$

and

$$\int_{\Omega} v_0 D_{t|T}^{1-\delta} \tilde{\varphi}(x, 0) + \int_{\Omega_T} I_{0|t}^{1-\delta}(u^q) D_{t|T}^{1-\delta} \tilde{\varphi} = - \int_{\Omega_T} v \Delta \mathcal{D}_{t|T}^{1-\beta} D_{t|T}^{1-\delta} \tilde{\varphi} - \int_{\Omega_T} v D D_{t|T}^{1-\delta} \tilde{\varphi}, \quad (4.4)$$

where

$$\Omega_T = [0, T] \times \Omega \quad \text{for} \quad \Omega = \{x \in \mathbb{R}^N; |x| \leq 2T^{1/2}\}, \quad \int_{\Omega_T} = \int_{\Omega_T} dx dt, \quad \int_{\Omega} = \int_{\Omega} dx.$$

Moreover, using Property (2.4) and (2.4) in the left-hand side of (4.3) and (4.4), and Property (2.5) in the right-hand side, we get

$$CT^{\gamma-1} \int_{\Omega} u_0 \varphi_1^l + \int_{\Omega_T} D_{0|t}^{1-\gamma} [I_{0|t}^{1-\gamma} v^p] \tilde{\varphi} = - \int_{\Omega_T} u \Delta \mathcal{D}_{t|T}^{1-\alpha} D_{t|T}^{1-\gamma} \tilde{\varphi} + \int_{\Omega_T} u D_{t|T}^{2-\gamma} \tilde{\varphi},$$

and

$$CT^{\delta-1} \int_{\Omega} v_0 \varphi_1^l + \int_{\Omega_T} D_{0|t}^{1-\delta} [I_{0|t}^{1-\delta} u^q] \tilde{\varphi} = - \int_{\Omega_T} v \Delta \mathcal{D}_{t|T}^{1-\beta} D_{t|T}^{1-\delta} \tilde{\varphi} + \int_{\Omega_T} v D_{t|T}^{2-\delta} \tilde{\varphi}.$$

Furthermore, from Property (2.6) we obtain

$$\int_{\Omega_T} v^p \tilde{\varphi} + CT^{\gamma-1} \int_{\Omega} u_0 \varphi_1^l = - \int_{\Omega_T} u \Delta \mathcal{D}_{t|T}^{1-\alpha} D_{t|T}^{1-\gamma} \tilde{\varphi} + \int_{\Omega_T} u D_{t|T}^{2-\gamma} \tilde{\varphi},$$

and

$$\int_{\Omega_T} u^q \tilde{\varphi} + CT^{\delta-1} \int_{\Omega} v_0 \varphi_1^l(x) = - \int_{\Omega_T} v \Delta \mathcal{D}_{t|T}^{1-\beta} D_{t|T}^{1-\delta} \tilde{\varphi} + \int_{\Omega_T} v D_{t|T}^{2-\delta} \tilde{\varphi}.$$

The inequality $(-\Delta)(\varphi_1^l) \leq l\varphi_1^{l-1}(-\Delta)\varphi_1$ allows us to write:

$$\begin{aligned} \int_{\Omega_T} v^p \tilde{\varphi} + CT^{\gamma-1} \int_{\Omega} u_0 \varphi_1^l &\leq C \int_{\Omega_T} u \varphi_1^{l-1} |(-\Delta)\varphi_1 \mathcal{D}_{t|T}^{1-\alpha} D_{t|T}^{1-\gamma} \varphi_2| \\ &\quad + \int_{\Omega_T} u \varphi_1^l D_{t|T}^{2-\gamma} \varphi_2 \\ &\leq C \int_{\Omega_T} u \tilde{\varphi}^{1/q} \tilde{\varphi}^{-1/q} \varphi_1^{l-1} |(-\Delta)\varphi_1 \mathcal{D}_{t|T}^{1-\alpha} D_{t|T}^{1-\gamma} \varphi_2| \\ &\quad + \int_{\Omega_T} u \tilde{\varphi}^{1/q} \tilde{\varphi}^{-1/q} \varphi_1^l D_{t|T}^{2-\gamma} \varphi_2, \end{aligned} \quad (4.5)$$

and

$$\begin{aligned}
\int_{\Omega_T} u^q \tilde{\varphi} + CT^{\delta-1} \int_{\Omega} v_0 \varphi_1^l &\leq C \int_{\Omega_T} v \varphi_1^{l-1} |(-\Delta) \varphi_1 \mathcal{D}_{t|T}^{1-\beta} D_{t|T}^{1-\delta} \varphi_2| \\
&\quad + \int_{\Omega_T} v \varphi_1^l D_{t|T}^{2-\delta} \varphi_2 \\
&\leq C \int_{\Omega_T} v \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} \varphi_1^{l-1} |(-\Delta) \varphi_1 \mathcal{D}_{t|T}^{1-\beta} D_{t|T}^{1-\delta} \varphi_2| \\
&\quad + \int_{\Omega_T} v \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} \varphi_1^l D_{t|T}^{2-\delta} \varphi_2.
\end{aligned} \tag{4.6}$$

Hence, as $u_0, v_0 \geq 0$, using Hölder's inequality, we obtain

$$\int_{\Omega_T} v^p(x, t) \tilde{\varphi}(x, t) \leq \left(\int_{\Omega_T} u^q(x, t) \tilde{\varphi}(x, t) \right)^{1/q} \cdot \mathcal{A}, \tag{4.7}$$

and

$$\int_{\Omega_T} u^q(x, t) \tilde{\varphi}(x, t) \leq \left(\int_{\Omega_T} v^p(x, t) \tilde{\varphi}(x, t) \right)^{1/p} \cdot \mathcal{B}, \tag{4.8}$$

where we have set

$$\begin{aligned}
\mathcal{A} &:= C \left(\int_{\Omega_T} \varphi_1^l \varphi_2^{-\frac{1}{q-1}} |D_{t|T}^{2-\gamma} \varphi_2|^{q'} \right)^{1/q'} \\
&\quad + C \left(\int_{\Omega_T} \varphi_1^{l-q'} \varphi_2^{-\frac{1}{q-1}} |\Delta \varphi_1 \mathcal{D}_{t|T}^{1-\alpha} D_{t|T}^{1-\gamma} \varphi_2|^{q'} \right)^{1/q'},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B} &:= C \left(\int_{\Omega_T} \varphi_1^l \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{2-\delta} \varphi_2|^{p'} \right)^{1/p'} \\
&\quad + C \left(\int_{\Omega_T} \varphi_1^{l-p'} \varphi_2^{-\frac{1}{p-1}} |\Delta \varphi_1 \mathcal{D}_{t|T}^{1-\beta} D_{t|T}^{1-\delta} \varphi_2|^{p'} \right)^{1/p'},
\end{aligned}$$

with $p' := p/(p-1)$ and $q' := q/(q-1)$.

Now, combining the terms in (4.5) and (4.6), we obtain

$$\begin{aligned} \left(\int_{\Omega_T} v^p(x, t) \tilde{\varphi}(x, t) \right)^{1-1/pq} &\leq \mathcal{B}^{1/q} \cdot \mathcal{A}, \\ \left(\int_{\Omega_T} u^q(x, t) \tilde{\varphi}(x, t) \right)^{1-1/pq} &\leq \mathcal{A}^{1/p} \cdot \mathcal{B}. \end{aligned} \quad (4.9)$$

At this stage, we introduce the scaled variables: $\tau = T^{-1}t, \xi = T^{-1/2}x$; then, we have

$$\begin{aligned} \mathcal{A} &:= CT^{[(\gamma-2)q' + (1+\frac{N}{2})]\frac{1}{q'}} + CT^{[(-1+\alpha+\gamma-2)q' + (1+\frac{N}{2})]\frac{1}{q'}}, \\ \mathcal{B} &:= CT^{[(\delta-2)p' + (1+\frac{N}{2})]\frac{1}{p'}} + CT^{[(-1+\beta+\delta-2)p' + \frac{N}{2}]\frac{1}{p'}}. \end{aligned} \quad (4.10)$$

It can easily be seen that

$$\gamma - 2 > -1 + \alpha + \gamma - 2$$

and

$$\delta - 2 > -1 + \beta + \delta - 2,$$

for $0 < \alpha, \beta, \gamma, \delta < 1$.

Using (2.1) and (2.2) in the right-hand side of the terms in (4.7), we obtain the estimates

$$\begin{aligned} \left(\int_{\Omega_T} v^p(x, t) \tilde{\varphi}(x, t) \right)^{1-1/pq} &\leq CT^{\theta_1}, \\ \left(\int_{\Omega_T} u^q(x, t) \tilde{\varphi}(x, t) \right)^{1-1/pq} &\leq CT^{\theta_2}, \end{aligned} \quad (4.11)$$

where

$$\theta_1 := \left((\delta - 2)p' + \left(1 + \frac{N}{2}\right) \right) \frac{1}{p'q} + \left((\gamma - 2)q' + \left(1 + \frac{N}{2}\right) \right) \frac{1}{q'}, \quad (4.12)$$

and

$$\theta_2 := \left((\gamma - 2)q' + \left(1 + \frac{N}{2}\right) \right) \frac{1}{pq'} + \left((\delta - 2)p' + \left(1 + \frac{N}{2}\right) \right) \frac{1}{p'}. \quad (4.13)$$

We have to consider three cases:

- The case $\theta_1 < 0$ (resp., $\theta_2 < 0$): we pass to the limit in (4.8), as $T \rightarrow \infty$; we obtain

$$\begin{aligned} \lim_{T \rightarrow \infty} \int_0^T \int_{|x| \leq 2T^{1/2}} v^p(x, t) \tilde{\varphi}(x, t) dx dt &= 0, \\ \text{resp., } \lim_{T \rightarrow \infty} \int_0^T \int_{|x| \leq 2T^{1/2}} u^q(x, t) \tilde{\varphi}(x, t) dx dt &= 0. \end{aligned}$$

According to the Lebesgue dominated convergence theorem, the continuity in time and space of v (resp., u) and using the fact that $\tilde{\varphi}(x, t) \rightarrow 1$ as $T \rightarrow \infty$, we obtain

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^N} v^p(x, t) dx dt &= 0 \Rightarrow v \equiv 0, \\ \text{resp., } \int_0^T \int_{\mathbb{R}^N} u^q(x, t) dx dt &= 0 \Rightarrow u \equiv 0. \end{aligned}$$

Contradiction.

- The case $\theta_1 = 0$ (resp., $\theta_2 = 0$): using inequality (4.8) as $T \rightarrow \infty$, we infer that

$$\begin{aligned} v &\in L^p((0; \infty); L^p(\mathbb{R}^N)), \\ \text{resp., } u &\in L^q((0; \infty); L^q(\mathbb{R}^N)). \end{aligned} \quad (4.14)$$

Then, we follow the idea of [10]. So, we take

$$\varphi_1(x) := \Phi(|x|/(B^{-1/2}T^{1/2}))$$

instead of the one chosen above, where $1 \leq B < T$ is large enough such that when $T \rightarrow \infty$ we do not have $B \rightarrow \infty$ at the same time.

Then, by repeating the same calculation as above and taking into account the support of $\Delta\varphi_1$ and if we let

$$\Sigma_B := [0, T] \times \{x \in \mathbb{R}^N; |x| \leq 2B^{-1/2}T^{1/2}\},$$

$$\Delta_B := [0, T] \times \{x \in \mathbb{R}^N; B^{-1/2}T^{1/2} \leq |x| \leq 2B^{-1/2}T^{1/2}\}.$$

We obtain as in (4.3)-(4.4)

$$\begin{aligned} \int_{\Sigma_B} v^p \tilde{\varphi} dx dt &\leq C \int_{\Sigma_B} u \tilde{\varphi}^{1/q} \tilde{\varphi}^{-1/q} \varphi_1^l |D_{t|T}^{2-\gamma} \varphi_2| dx dt \\ &+ C \int_{\Delta_B} u \tilde{\varphi}^{1/q} \tilde{\varphi}^{-1/q} \varphi_1^{l-1} |(-\Delta)\varphi_1| \mathcal{D}_{t|T}^{1-\alpha} D_{t|T}^{1-\gamma} \varphi_2| dx dt, \end{aligned} \quad (4.15)$$

and

$$\begin{aligned} \int_{\Sigma_B} u^q \tilde{\varphi} dx dt &\leq C \int_{\Sigma_B} v \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} \varphi_1^l |D_{t|T}^{2-\delta} \varphi_2| dx dt \\ &+ C \int_{\Delta_B} v \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} \varphi_1^{l-1} |(-\Delta)\varphi_1| \mathcal{D}_{t|T}^{1-\beta} D_{t|T}^{1-\delta} \varphi_2| dx dt. \end{aligned} \quad (4.16)$$

On the other hand, as (u, v) is a global solution, then u and v satisfies (4.1)-(4.2) locally and in particular on Δ_B :

$$\begin{aligned} \int_{\Delta_B} v^p \tilde{\varphi} dx dt &\leq C \int_{\Delta_B} u \tilde{\varphi}^{1/q} \tilde{\varphi}^{-1/q} \varphi_1^l |D_{t|T}^{2-\gamma} \varphi_2| dx dt \\ &+ C \int_{\Delta_B} u \tilde{\varphi}^{1/q} \tilde{\varphi}^{-1/q} \varphi_1^{l-1} |(-\Delta)\varphi_1| \mathcal{D}_{t|T}^{1-\alpha} D_{t|T}^{1-\gamma} \varphi_2| dx dt, \end{aligned} \quad (4.17)$$

and

$$\begin{aligned} \int_{\Delta_B} u^q \tilde{\varphi} dx dt &\leq C \int_{\Delta_B} v \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} \varphi_1^l |D_{t|T}^{2-\delta} \varphi_2| dx dt \\ &\quad + C \int_{\Delta_B} u \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} \varphi_1^{l-1} |(-\Delta) \varphi_1| \mathcal{D}_{t|T}^{1-\beta} D_{t|T}^{1-\delta} \varphi_2| dx dt. \end{aligned} \quad (4.18)$$

At this stage, we set

$$\begin{aligned} \mathcal{U}_1 &:= \int_{\Sigma_B} u^q \tilde{\varphi} dx dt, \\ \mathcal{U}_2 &:= \int_{\Delta_B} u^q \tilde{\varphi} dx dt, \end{aligned}$$

and

$$\begin{aligned} \mathcal{V}_1 &:= \int_{\Sigma_B} v^p \tilde{\varphi} dx dt, \\ \mathcal{V}_2 &:= \int_{\Delta_B} v^p \tilde{\varphi} dx dt. \end{aligned}$$

Then, using Hölders inequality in (37)(38) and (39)(40), we obtain

$$\begin{cases} \mathcal{V}_1 \leq \mathcal{U}_1^{1/q} \mathcal{A}_1 + \mathcal{U}_2^{1/q} \mathcal{C}_1, \\ \mathcal{U}_1 \leq \mathcal{V}_1^{1/p} \mathcal{B}_1 + \mathcal{V}_2^{1/q} \mathcal{C}_2, \end{cases} \quad (4.19)$$

and

$$\begin{cases} \mathcal{V}_2 \leq \mathcal{U}_2^{1/q} \mathcal{A}_2 + \mathcal{U}_1^{1/q} \mathcal{C}_1, \\ \mathcal{U}_2 \leq \mathcal{V}_2^{1/p} \mathcal{B}_2 + \mathcal{V}_1^{1/q} \mathcal{C}_2, \end{cases} \quad (4.20)$$

where we have set

$$\begin{aligned} \mathcal{A}_1 &:= C \left(\int_{\Sigma_B} \varphi_1^l \varphi_2^{-\frac{1}{q-1}} |D_{t|T}^{2-\gamma} \varphi_2|^{q'} dx dt \right)^{1/q'}, \\ \mathcal{A}_2 &:= C \left(\int_{\Delta_B} \varphi_1^l \varphi_2^{-\frac{1}{q-1}} |D_{t|T}^{2-\gamma} \varphi_2|^{q'} dx dt \right)^{1/q'}, \\ \mathcal{B}_1 &:= C \left(\int_{\Sigma_B} \varphi_1^l \varphi_2^{-\frac{1}{q-1}} |D_{t|T}^{2-\delta} \varphi_2|^{p'} dx dt \right)^{1/p'}, \\ \mathcal{B}_2 &:= C \left(\int_{\Delta_B} \varphi_1^l \varphi_2^{-\frac{1}{q-1}} |D_{t|T}^{2-\delta} \varphi_2|^{p'} dx dt \right)^{1/p'}, \end{aligned}$$

and

$$\begin{aligned}\mathcal{C}_1 &:= C \left(\int_{\Delta_B} \varphi_1^{l-q'} \varphi^{-\frac{1}{q-1}} |(-\Delta)\varphi_1| \mathcal{D}_{t|T}^{1-\alpha} D_{t|T}^{1-\gamma} \varphi_2^{q'} dx dt \right)^{1/q'}, \\ \mathcal{C}_2 &:= C \left(\int_{\Delta_B} \varphi_1^{l-p'} \varphi^{-\frac{1}{p-1}} |(-\Delta)\varphi_1| \mathcal{D}_{t|T}^{1-\alpha} D_{t|T}^{1-\delta} \varphi_2^{p'} dx dt \right)^{1/p'}.\end{aligned}$$

Combining the terms in (4.19) and (4.20), we get

$$\mathcal{V}_1 \leq \mathcal{V}_1^{1/pq} \mathcal{B}_1^{1/q} \mathcal{A}_1 + \mathcal{V}_2^{1/pq} \mathcal{C}_2^{1/q} \mathcal{A}_1 + \mathcal{V}_2^{1/pq} \mathcal{B}_2^{1/q} \mathcal{C}_1 + \mathcal{V}_2^{1/pq} \mathcal{C}_2^{1/q} \mathcal{C}_1, \quad (4.21)$$

and

$$\mathcal{U}_1 \leq \mathcal{U}_1^{1/pq} \mathcal{A}_1^{1/p} \mathcal{B}_1 + \mathcal{U}_2^{1/pq} \mathcal{C}_1^{1/p} \mathcal{B}_1 + \mathcal{U}_2^{1/pq} \mathcal{A}_2^{1/p} \mathcal{C}_2 + \mathcal{U}_2^{1/pq} \mathcal{C}_1^{1/p} \mathcal{C}_2. \quad (4.22)$$

To estimate the first term in the right-hand side of (4.21) and (4.22), we apply Young's inequality

$$ab \leq \frac{1}{pq} a^{pq} + \frac{pq-1}{pq} b^{\frac{pq}{pq-1}}, \quad p > 1, \quad q > 1,$$

with

$$\begin{aligned}a &= \mathcal{V}_1^{1/pq}, \\ b &= \mathcal{B}_1^{1/q} \mathcal{A}_1,\end{aligned}$$

and

$$\begin{aligned}a &= \mathcal{U}_1^{1/pq}, \\ b &= \mathcal{A}_1^{1/p} \mathcal{B}_1.\end{aligned}$$

This leads us

$$\left(1 - \frac{1}{pq}\right) \mathcal{V}_1 \leq \frac{pq-1}{pq} \mathcal{B}_1^{\frac{p}{pq-1}} \mathcal{A}_1^{\frac{pq}{pq-1}} + \mathcal{V}_2^{1/pq} [\mathcal{C}_2^{1/q} \mathcal{A}_1 + \mathcal{B}_2^{1/q} \mathcal{C}_1 + \mathcal{C}_2^{1/q} \mathcal{C}_1],$$

and

$$\left(1 - \frac{1}{pq}\right) \mathcal{U}_1 \leq \frac{pq-1}{pq} \mathcal{A}_1^{\frac{q}{pq-1}} \mathcal{B}_1^{\frac{pq}{pq-1}} + \mathcal{U}_2^{1/pq} [\mathcal{C}_1^{1/p} \mathcal{B}_1 + \mathcal{A}_2^{1/p} \mathcal{C}_2 + \mathcal{C}_1^{1/p} \mathcal{C}_2].$$

Taking into account the definition of φ and applying the following change of variables $\tau = T^{-1}t, \xi = T^{-1/2}B^{1/2}x$ in the integrals in $\mathcal{A}_i, \mathcal{B}_i$ and $\mathcal{C}_i (i = 1, 2)$, we get

$$\mathcal{V}_1 \leq CT^{\theta_1 \frac{pq}{pq-1}} B^{\eta_1 \frac{pq}{pq-1}} + \mathcal{V}_2^{1/pq} [CT^{\theta_1} B^{\eta_2} + CT^{\theta_1} B^{\eta_3} + CT^{\theta_1} B^{\eta_4}],$$

and

$$\mathcal{U}_1 \leq CT^{\theta_2 \frac{pq}{pq-1}} B^{\mu_1 \frac{pq}{pq-1}} + \mathcal{U}_2^{1/pq} [CT^{\theta_1} B^{\mu_2} + CT^{\theta_1} B^{\mu_3} + CT^{\theta_1} B^{\mu_4}],$$

where

$$\begin{cases} \eta_1 := -\frac{N}{2} \left(\frac{1}{p'q} + \frac{1}{q'} \right), \\ \eta_2 := \frac{1}{q} - \frac{N}{2} \left(\frac{1}{p'q} + \frac{1}{q'} \right), \\ \eta_3 := 1 - \frac{N}{2} \left(\frac{1}{p'q} + \frac{1}{q'} \right), \\ \eta_4 := 1 + \frac{1}{q} - \frac{N}{2} \left(\frac{1}{p'q} + \frac{1}{q'} \right), \end{cases}$$

and

$$\begin{cases} \mu_1 := -\frac{N}{2} \left(\frac{1}{pq'} + \frac{1}{p'} \right), \\ \mu_2 := \frac{1}{p} - \frac{N}{2} \left(\frac{1}{pq'} + \frac{1}{p'} \right), \\ \mu_3 := 1 - \frac{N}{2} \left(\frac{1}{pq'} + \frac{1}{p'} \right), \\ \mu_4 := 1 + \frac{1}{p} - \frac{N}{2} \left(\frac{1}{pq'} + \frac{1}{p'} \right). \end{cases}$$

Let us recall that $\theta_1 = 0$ (resp., $\theta_2 = 0$) implies that

$$\begin{aligned} \mathcal{V}_1 &\leq CB^{\eta_1 \frac{pq}{pq-1}} + \mathcal{V}_2^{1/pq} [CB^{\eta_2} + CB^{\eta_3} + CB^{\eta_4}], \\ \text{resp., } \mathcal{U}_1 &\leq CB^{\mu_1 \frac{pq}{pq-1}} + \mathcal{U}_2^{1/pq} [CB^{\mu_2} + CB^{\mu_3} + CB^{\mu_4}]. \end{aligned} \quad (4.23)$$

Now, as $v \in L^p(0, \infty; L^p(\mathbb{R}^N))$, (resp., $u \in L^q(0, \infty; L^q(\mathbb{R}^N))$), we have

$$\lim_{T \rightarrow \infty} \mathcal{V}_2 = 0 \quad (\text{resp., } \lim_{T \rightarrow \infty} \mathcal{U}_2 = 0).$$

Taking the limit as $T \rightarrow \infty$ in (4.23) and using the Lebesgue dominated convergence theorem, we conclude that

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^N} v^p(x, t) dx dt &\leq CB^{\eta_1 \frac{pq}{pq-1}} \\ \left(\text{resp., } \int_0^T \int_{\mathbb{R}^N} u^q(x, t) dx dt &\leq CB^{\mu_1 \frac{pq}{pq-1}} \right). \end{aligned} \quad (4.24)$$

Finally, as $\eta_1 < 0$ and $\mu_1 < 0$, taking the limit as $B \rightarrow \infty$ in (4.24), and using the continuity of u and v , we conclude that $v \equiv 0$ or $u \equiv 0$, and (4.19)-(4.20) implies that $u \equiv v \equiv 0$, which is a contradiction.

• For the case $p < 1/\gamma$ and $q < 1/\beta$, we repeat the same argument as in the case $\theta_1 < 0$ or $\theta_2 < 0$ by choosing this time the test function as follows:

$$\tilde{\varphi}(x, t) = (\varphi_2(x))^l \varphi_2(t),$$

where $\varphi_1(x) = \Phi(|x|/R)$, $\varphi_2(t) = (1 - t/T)^r$, $r > 1$ and $R \in (0, T)$ large enough such that when $T \rightarrow \infty$ we do not have $R \rightarrow \infty$ at the same time; the function Φ is the same as above.

After that, similar as in (4.9), we can get

$$\begin{aligned} \left(\int_{\tilde{\mathcal{C}}_T} v^p(x, t) \tilde{\varphi}(x, t) dx dt \right)^{1-1/pq} &\leq \mathbb{E}^{1/q} \mathbb{D}, \\ \left(\int_{\tilde{\mathcal{C}}_T} u^q(x, t) \tilde{\varphi}(x, t) dx dt \right)^{1-1/pq} &\leq \mathbb{D}^{1/p} \mathbb{E}, \end{aligned} \quad (4.25)$$

where

$$\mathcal{C}_T := [0; T] \times \{x \in \mathbb{R}^N; |x| \leq 2R\},$$

$$\begin{aligned} \mathbb{D} := & C \left(\int_{\mathcal{C}_T} \varphi_1^l \varphi_2^{-\frac{1}{q-1}} |D_{t|T}^{2-\gamma} \varphi_2|^{q'} dx dt \right)^{1/q'} \\ & + C \left(\int_{\mathcal{C}_T} \varphi_1^{l-q'} \varphi_2^{-\frac{1}{q-1}} |\Delta \varphi_1 \mathcal{D}_{t|T}^{1-\alpha} D_{t|T}^{1-\gamma} \varphi_2|^{q'} dx dt \right)^{1/q'}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E} := & C \left(\int_{\mathcal{C}_T} \varphi_1^l \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{2-\delta} \varphi_2|^{p'} dx dt \right)^{1/p'} \\ & + C \left(\int_{\mathcal{C}_T} \varphi_1^{l-p'} \varphi_2^{-\frac{1}{p-1}} |\Delta \varphi_1 \mathcal{D}_{t|T}^{1-\beta} D_{t|T}^{1-\delta} \varphi_2|^{p'} dx dt \right)^{1/p'}. \end{aligned}$$

Then, the scaled variables $\xi = R^{-1}x$, $\tau = T^{-1}t$ and (2.2)-(2.4) allow us to write:

$$\begin{aligned} & \left(\int_{\mathcal{C}_T} v^p(x, t) \tilde{\varphi}(x, t) dx dt \right)^{1-1/pq} \leq C_1(T, R), \\ & \text{and} \\ & \left(\int_{\mathcal{C}_T} u^q(x, t) \tilde{\varphi}(x, t) dx dt \right)^{1-1/pq} \leq C_2(T, R), \end{aligned} \tag{4.26}$$

where

$$C_1(T, R) = CT^{\zeta_1} R^{\rho_1} + CT^{\zeta_2} R^{\rho_2} + CT^{\zeta_3} R^{\rho_3} + CT^{\zeta_4} R^{\rho_4},$$

and

$$C_2(T, R) = CT^{\omega_1} R^{\theta_1} + CT^{\omega_2} R^{\theta_2} + CT^{\omega_3} R^{\theta_3} + CT^{\omega_4} R^{\theta_4},$$

with

$$\begin{cases} \zeta_1 := \frac{1}{q} \left[\frac{1}{p'} + \delta - 2 \right] + \left[\frac{1}{q'} + \gamma - 2 \right]; \\ \zeta_2 := \frac{1}{q} \left[\frac{1}{p'} + \delta - 2 \right] + \left[\frac{1}{q'} + \gamma - 2 + \alpha \right]; \\ \zeta_3 := \frac{1}{q} \left[\frac{1}{p'} + \delta - 2 + \beta \right] + \left[\frac{1}{q'} + \gamma - 2 \right]; \\ \zeta_4 := \frac{1}{q} \left[\frac{1}{p'} + \delta - 2 + \beta \right] + \left[\frac{1}{q'} + \gamma - 2 + \alpha \right], \\ \omega_1 := \frac{1}{p} \left[\gamma - 2 + \frac{1}{q'} \right] + \left[\delta - 2 + \frac{1}{p'} \right]; \\ \omega_2 := \frac{1}{p} \left[\gamma - 2 + \frac{1}{q'} \right] + \left[\alpha + \gamma - 2 + \frac{1}{p'} \right]; \\ \omega_3 := \frac{1}{p} \left[\beta + \delta - 2 + \frac{1}{q'} \right] + \left[\delta - 2 + \frac{1}{p'} \right], \\ \omega_4 := \frac{1}{p} \left[\beta + \delta - 2 + \frac{1}{q'} \right] + \left[\alpha + \gamma - 2 + \frac{1}{p'} \right], \end{cases}$$

and

$$\begin{cases} \rho_1 := \frac{N}{p'q} + \frac{N}{q'}; \\ \rho_2 := \frac{N}{p'q} + \frac{N}{q'} - 2; \\ \rho_3 := \frac{N}{q'} + \frac{1}{q} \left[\frac{N}{p'} - 2 \right]; \\ \rho_4 := \frac{1}{q} \left[\frac{N}{p'} - 2 \right] + \frac{N}{q'} - 2, \end{cases}$$

$$\begin{cases} \theta_1 := \frac{N}{pq'} + \frac{N}{p'}; \\ \theta_2 := \frac{N}{pq'} + \frac{N}{p'} - 2; \\ \theta_3 := \frac{N}{p'} + \frac{1}{p}[\frac{N}{q'} - 2], \\ \theta_4 := \frac{1}{p}[\frac{N}{q'} - 2] + \frac{N}{p'} - 2. \end{cases}$$

Taking the limit as $T \rightarrow \infty$ and using the fact that

$$p < \frac{1}{\delta} \Leftrightarrow \frac{1}{p'} < 1 - \delta$$

and

$$q < \frac{1}{\gamma} \Leftrightarrow \frac{1}{q'} < 1 - \gamma,$$

from (4.26) implies

$$\int_0^\infty \int_{|x| \leq 2R} v^p \tilde{\varphi}(x, t) dx dt = 0$$

and

$$\int_0^\infty \int_{|x| \leq 2R} u^q \tilde{\varphi}(x, t) dx dt = 0.$$

Finally, by letting $R \rightarrow \infty$, we get a contradiction with the fact that $u(x, t) > 0$ and $v(x, t) > 0$ for all $x \in \mathbb{R}^N, t > 0$. This leads to $u(x, t) = v(x, t) = 0$.

The requirement $\theta_1 \leq 0$ in (4.12)

$$\frac{N}{2} \leq \max \left\{ \frac{(2 - \delta)p + (1 - \gamma)pq + 1}{pq - 1}, \frac{(2 - \gamma)q + (1 - \delta)pq + 1}{pq - 1} \right\},$$

provides us with a critical exponent which coincides with the well-known Fujita exponent in case $\delta = 2, \gamma = 1, q = 1$ and $\gamma = 2, \delta = 1, p = 1$, respectively.

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