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On the Cauchy problem for semi-linear σ-evolution equations with time-dependent damping

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Abstract

In this paper, we would like to consider the Cauchy problem for semi-linear σ-evolution equations with time-dependent damping for any $\sigma \geq 1$. Motivated strongly by the classification of damping terms in the paper[34], the first main goal of the present work is to make some generalizations from $\sigma = 1$ to $\sigma > 1$ and simultaneously to investigate decay estimates for solutions to the corresponding linear equations in the so-called effective damping cases. For the next main goals, we are going not only to prove the global well-posedness property of small data solutions but also to indicate blow-up results for solutions to the semi-linear problem. In this concern, the novelty which should be recognized is that the application of a modified test function combined with a judicious choice of test functions gives blow-up phenomena and upper bound estimates for lifespan in both the subcritical case and the critical case, where σ is assumed to be any fractional number. Finally, lower bound estimates for lifespan in some spatial dimensions are also established to find out their sharp results.

KEYWORDS:

σ-evolution equation; WKB-analysis; Global existence of small data solution; Critical exponent; Lifespan estimates

1 | INTRODUCTION

1.1 | Background of the damped wave equations

Let us sketch out some historical background of the damped wave equations with constant and time-dependent coefficients. We start with the following Cauchy problem for the semi-linear classical damped wave equation:

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^p, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

with $p > 1$. At first, recalling the two pioneering papers[27,28] since 1976, which devotes to the study of decay estimates for solutions to the corresponding linear equation of (1), one recognizes that the author also established the asymptotic behavior of solutions to (1) thanks to the presence of the damping term u_t. By denoting the so-called *Fujita exponent*

$$p_{\text{Fuj}}(n) := 1 + \frac{2}{n},$$

which is the critical exponent of the corresponding semi-linear heat equations (see[12] and references therein), the authors in[31] proved a global (in time) existence result for energy solutions to (1) by assuming compactly supported small, data when $p > p_{\text{Fuj}}(n)$ and $p \leq \frac{n}{n-2}$ if $n \geq 3$. Moreover, the authors also established a blow-up result for $1 < p < p_{\text{Fuj}}(n)$ by assuming that the small initial data satisfy some integral sign conditions. Later in[35], the author showed that the critical case $p = p_{\text{Fuj}}(n)$ belongs to the blow-up region by applying the so-called *test function method*, which was originally developed in[1]. This method bases on a contradiction argument and yields sharp exponents for models with a parabolic like decay for solutions. When the blow-up phenomenon in finite time occurs, the sharp lifespan estimates, i.e. the maximal existence time of solutions, for (1) in all spatial dimensions have been investigated in numerous papers[15,19,22,26], namely,

$$\mathrm{LifeSpan}(u) \sim \begin{cases} C\varepsilon^{-\frac{2(p-1)}{2-n(p-1)}} & \text{if } 1 < p < p_{\text{Fuj}}(n), \\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{if } p = p_{\text{Fuj}}(n). \end{cases}$$

Here the positive constant ε presents the size of initial data and $C = C(n, p, u_0, u_1)$ is a positive constant independent of ε. For this reason, we can say that in some sense the study of (1) seems to be completed in 2019.

A further problem of interest is the Cauchy problem for the linear wave equation with time-dependent dissipation

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = 0, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x),\ \ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{2}$$

The term $b(t)u_t$ is called the damping term, which prevents the motion of the wave and reduces its energy, in addition, the coefficient $b = b(t)$ represents the strength of the damping. Asymptotic behavior of solutions and their wave energy change according to the positive coefficient $b = b(t)$ in the damping term. In[33,34], the author proposed a classification of the time-dependent dissipation terms in the following ways:

- Scattering producing to the free wave equation,
- Non-effective dissipation,
- Effective dissipation,
- Over-damping producing.

If the solution behaves asymptotically like that of the wave equation, then the solution scatters to that of the free wave equation as $t \to \infty$. This case is called a *scattering producing* case. If the $L^p - L^q$ estimates, with $1 \le p \le q \le \infty$, for the solution to (2) are closely related to those for the solution to the free wave equation, then the damping term is called *non-effective*. If the solution to (2) has the same decay behavior as that to the corresponding parabolic Cauchy problem

$$\begin{cases} v_t = \dfrac{1}{b(t)}\Delta v, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $v_0 = v_0(x)$ depending on u_0, u_1 is suitably chosen, then the damping term is called *effective*. Finally, if the energy of solutions has no any decay estimate, then the damping term is called *over-damping producing*.

Next, we consider the following Cauchy problem for semi-linear time-dependent damped wave equation:

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = |u|^p, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x),\ \ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{3}$$

In[24], the authors proved that the critical exponent for solutions to (3) with the special effective damping term

$$b(t)u_t = \mu(1+t)^{-\beta}u_t$$

for a constant $\mu > 0$ and $\beta \in (-1, 1)$ remains the same as that to (1), provided that the small initial data have compact support. This means that the authors have obtained a blow-up result if $1 < p \le p_{\text{Fuj}}(n)$ and a global (in time) existence one if $p > p_{\text{Fuj}}(n)$. Later, a global (in time) existence result for (3) was extended in[6] to more general cases of $b(t)$ satisfying a monotonicity condition and a polynomial-like behavior. Moreover, the authors relaxed the assumption of compactly supported data by considering exponentially weighted energy spaces. In particular, the global (in time) existence holds for $p > p_{\text{Fuj}}(n)$ and $p \le \frac{n}{n-2}$ with $n \ge 3$, where the initial data are assumed to be small in exponentially weighted energy spaces. The authors in[5] treated both the subcritical case $1 < p < p_{\text{Fuj}}(n)$ and the critical case $p = p_{\text{Fuj}}(n)$ by developing a *modified test function method*. They showed that there is no global (in time) existence of small data solutions under a suitable sign assumption of the initial data Regarding the so-called scale-invariant damping, i.e. $\beta = 1$, on the one hand we want to underline that a blow-up result was given in[20] when $1 < p < p_S(n+\mu)$ and $\mu \in \big(0, \frac{n^2+n+2}{n+2}\big)$. Here $p_S(n)$ stands for the well-known Strauss exponent, which is the positive root of the quadratic equation $(n-1)p^2 - (n+1)p - 2 = 0$. On the other hand, the authors in[2,7] succeeded in both proving global existence of solutions and determining the critical exponent in the very special situation of $\mu = 2$. For the so-called scattering damping $\beta > 1$, we refer the interested readers to[25] for a blow-up result, provided that the condition $1 < p < p_S(n)$ holds. Speaking more about the critical exponent when $\beta < -1$, one should recognize that this situation is completely different from these previous ones. More precisely, the authors in[23] indicated that the small data solution always exists globally for any $p > 1$, i.e., the critical exponent really disappears in the mentioned case. Among other things, sharp lifespan estimates for solutions to (3) in the critical case $p = p_{\rm Fuj}(n)$ were reported in[18], where $b(t) = (1+t)^{-\beta}$ with $\beta \in [-1, 1)$. Quite recently, the authors in[21] investigated blow-up results together with sharp lifespan estimates for (3) in both the subcritical case and the critical case when more general damping coefficients are considered including

$$b(t) = 1 + t, \quad b(t) = (1+t)\big(1 + \log(1+t)\big) \quad \text{and so on.}$$

1.2 | Main purpose of this paper

This paper is concerned with studying the following semi-linear σ-evolution equations with general time-dependent damping in the whole space:

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^\sigma u(t,x) + b(t)u_t(t,x) = |u(t,x)|^p, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{4}$$

where $\sigma > 1$ is assumed to be any fractional number and $p > 1$. In the present manuscript, we suppose that the dissipation term $b(t)u_t$ satisfies the so-called *effective* assumptions in the following definition according to the classification given in[34].

Definition 1 (Effective dissipation)**.** If the strictly positive function $b = b(t)$ satisfies

(B1) $b \in C^3([0,\infty))$,

(B2) $b'(t)$ does not change its sign and $tb(t) \to \infty$ as $t \to \infty$,

(B3) $\dfrac{|b^{(k)}(t)|}{b(t)} \lesssim \dfrac{1}{(1+t)^k}$ for $k = 1, 2$,

(B4) $\dfrac{1}{b(t)} \notin L^1\big([0,\infty)\big)$,

(B5) $\big((1+t)^2 b(t)\big)^{-1} \in L^1\big([0,\infty)\big)$.

then the damping term $b(t)u_t$ is called effective. These assumptions will be helpful in Sections 2-4. Additionally, we would like to propose one more assumption for $b = b(t)$, which plays a significant role in the blow-up result and the estimates for lifespan in Sections 5 and 6 later, as follows:

$$\textbf{(B-L)} \quad \mathbb{B}_\infty := \limsup_{t\to\infty} \left|\frac{b'(t)}{b^2(t)}\right| < 1.$$

Example 1.1. Let us give several typical examples of $b = b(t)$ enjoying the conditions **(B1)-(B5)** of Definition 1:

- $b(t) = (1+t)^{-\beta}$ with $\beta \in [-1, 1)$,
- $b(t) = \mu(1+t)^{-\beta}\big(\log(e+t)\big)^\gamma$ with $\mu > 0$, $\beta \in (-1, 1)$ and $\gamma \neq 0$.

Moreover, the following functions fulfill the assumption (**B-L**):

- $b(t) = \mu(1+t)^{-\beta}$ with $\mu > 0$ and $\beta \in [-1, 1)$.
- $b(t) = \mu(1+t)^{-1}$ with $\mu > 1$.
- $b(t) = \prod_{k=1}^m \ell_k(t)$ with $m \geq 2$, $\ell_1(t) = 1 + t$ and $\ell_{k+1} = 1 + \log\big(\ell_k(t)\big)$ for $k = 1, 2, \cdots, m-1$.

To the best of authors' knowledge, it seems that there are not so many papers in terms of the study of (4) for any fractional number $\sigma \geq 1$ at present. Let us mention briefly several recent contributions related to (4), namely, the following linear model is of our attention:

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^\sigma u(t,x) + b(t)(-\Delta)^\delta u_t(t,x) = 0, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{5}$$

with $\sigma \geq 1$ and $\delta \in (0,\sigma)$. The authors in [14,16,17] succeeded in deriving some $L^p - L^q$ estimates for the energies of higher order by applying WKB analysis associated with the theory of modified Bessel functions and Faà di Bruno's formula when the dissipation coefficient $b = b(t)$ is considered to be monotonous functions. Coming back the special case of $b(t) = \mu(1+t)^{-\beta}$ with a constant $\mu > 0$ and $\beta \in (-1,1)$, one can see that a classification between effective damping and noneffective damping, which strongly depends on parameters σ, δ and β, is introduced in [3]. Their main work is to study the asymptotic profile of solutions to (5) and simultaneously to clarify that a diffusion phenomenon occurs in the situation of effective one. For this purpose, the point worth noticing is that the case of $\delta = 0$ is not treated completely because the presence of the structural damping $(-\Delta)^\delta u_t$ with $\delta > 0$ generates a smoothing effect, which disappears for the classical damping u_t (see more [4] when $b(t) = 1$). Among other things, we recognize that the information about lifespan of solutions to (4) has not ever been appeared in these papers even if $\delta > 0$.

Our main goal in this paper is on the one hand to prove global (in time) existence of small data Sobolev solutions to the Cauchy problem (4) by using some achieved estimates for solutions from the corresponding linear equation. The crux of getting such estimates with the fractional power operator comes from WKB analysis associated with diagonalization procedures effectively. On the other hand, we would like to indicate blow-up phenomenon and sharp lifespan estimates for solutions to (4) as well, where $\sigma > 1$ in (4) is assumed to be any fractional number. More precisely, to deal with the fractional Laplacian $(-\Delta)^\sigma$, well-known as a non-local operator, a modified test function method with a judicious choice of test functions will be applied to give blow-up results in both the subcritical exponent and the critical exponent. Speaking about sharp estimates for lifespan of solutions, one of new contributions in this paper, we not only catch upper bound ones by utilizing a suitable test function method linked to nonlinear differential inequalities but also establish lower bound ones by constructing polynomial type time-weighted Sobolev spaces. Throughout this work, we can understand clearly how the dissipation coefficient $b(t)$ and the fractional power σ influence on the above-mentioned results.

Notations

- We write $f \lesssim g$ when there exists a constant $C > 0$ such that $f \leq Cg$, and $f \approx g$ when $g \lesssim f \lesssim g$.
- As usual, the spaces H^a and $\dot{H}^a$ with $a \geq 0$ stand for Bessel and Riesz potential spaces based on the L^2 spaces. Here $\big\langle D\big\rangle^a$ and $|D|^a$ denote the pseudo-differential operator with symbol $\big\langle \xi\big\rangle^a$ and the fractional Laplace operator with symbol $|\xi|^a$, respectively.
- For a given number $s \in \mathbb{R}$, we denote $[s]^+ := \max\{s,0\}$ and $[s] := \max\big\{k \in \mathbb{Z} : k \leq s\big\}$.
- For later convenience, with $s \in [0,t]$ we denote by $\mathscr{B}(s,t)$ the primitive of $\frac{1}{b(\tau)}$ which vanishes at $t = s$, namely,

$$\mathscr{B}(s,t) := \int_s^t \frac{1}{b(\tau)} d\tau.$$

Main results

Let us now state the main results which will be proved in the present paper. To get started, we obtain the global existence of small data solutions to (4) from the Sobolev space.

Theorem 1 (Global existence). Let us assume that the conditions **(B1)-(B5)** are satisfied. Let $\sigma > 1$ and $(u_0,u_1) \in \mathcal{D}^\alpha_m := \big(H^\alpha \cap L^m\big) \times \big(L^2 \cap L^m\big)$ with $m \in [1,2)$ and $\alpha \in (0,\sigma]$. Moreover, we suppose that the exponent p satisfies

$$p > 1 + \frac{2m\sigma}{n} \quad \text{and} \quad \begin{cases} \frac{2}{m} \leq p & \text{if} \quad n \leq 2\alpha, \\ \frac{2}{m} \leq p \leq \frac{n}{n-2\alpha} & \text{if} \quad 2\alpha < n \leq \frac{4\alpha}{2-m}. \end{cases}$$

Then, there exists a sufficiently small constant $\varepsilon > 0$ such that for any data $(u_0,u_1) \in \mathcal{D}^\alpha_m$ satisfying the assumption

$$\|(u_0,u_1)\|_{\mathcal{D}^\alpha_m} := \|u_0\|_{H^\alpha} + \|u_0\|_{L^m} + \|u_1\|_{L^2} + \|u_1\|_{L^m} \leq \varepsilon,$$

there is a uniquely determined global (in time) Sobolev solution

$$u \in C\big([0,\infty), H^\alpha\big)$$

to (4). Furthermore, the solution satisfies the following decay estimates:

$$\|u(t,\cdot)\|_{L^2} \lesssim \big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)} \|(u_0,u_1)\|_{\mathcal{D}^\alpha_m},$$

$$\|\,|D|^\alpha u(t,\cdot)\|_{L^2} \lesssim \big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\alpha}{2\sigma}} \|(u_0,u_1)\|_{\mathcal{D}^\alpha_m}.$$

The next main result is concerned with indicating the sharpness of the exponent p to (4).

Theorem 2 (Blow-up). Let $\sigma > 1$ and $n \geq 1$. Assume that we choose the initial data $u_0 \in L^1$ and $u_1 \in L^1$ satisfying the following relations:

$$\int_{\mathbb{R}^n} \big(u_0(x) + \mathbb{B}_0 u_1(x)\big)dx > 0, \tag{6}$$

where $\mathbb{B}_0 := \int_0^\infty \exp\Big(-\int_0^t b(\tau)d\tau\Big)dt$. Moreover, we suppose that the following conditions hold:

$$\begin{cases} p \leq 1 + \dfrac{2\sigma}{n} & \text{if } \ \sigma \text{ is an integer number,} \\ p < 1 + \dfrac{2\sigma}{n} & \text{if } \ \sigma \text{ is a fractional number.} \end{cases} \tag{7}$$

Then, there is no global (in time) Sobolev solution $u \in C\big([0,\infty), L^2\big)$ to (4).

Remark 1. Obviously, it follows from Theorems 1 and 2 that the critical exponent p_{crit}, which classifies between global (in time) existence of small data solutions and finite time blow-up of (even) small data solutions, is

$$p_{\text{crit}} = 1 + \frac{2\sigma}{n}.$$

The final main result involves the lifespan estimates for solutions to (4). To demonstrate this, let us consider the initial data $\big(\varepsilon u_0(x), \varepsilon u_1(x)\big)$ in place of $\big(u_0(x), u_1(x)\big)$ for (4), where ε is a small positive constant which presents the size of initial data.

Theorem 3 (Upper bound of lifespan). Under the same assumptions as in Theorem 2 together with the condition (7) for the exponent p, there exists a positive constant $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the following upper bound estimates for the lifespan of solutions to (4) hold:

$$\mathscr{B}\big(0, \text{LifeSpan}(u)\big) \leq \begin{cases} C\varepsilon^{-\frac{2\sigma(p-1)}{2\sigma-n(p-1)}} & \text{if } \ p < 1 + \dfrac{2\sigma}{n}, \\ \exp\big(C\varepsilon^{-(p-1)}\big) & \text{if } \ p = 1 + \dfrac{2\sigma}{n}, \end{cases} \tag{8}$$

where C is a positive constant independent of ε.

Theorem 4 (Lower bound of lifespan). Let $\sigma > 1$, $n \leq 4\alpha$ and $(u_0, u_1) \in \big(H^\alpha \cap L^1\big) \times \big(L^2 \cap L^1\big)$ with $\alpha \in (0, \sigma]$. Assume that the exponent $p \geq 2$ fulfills (7) together with the condition $p \leq n/(n-2\alpha)$ if $n > 2\alpha$. Then, there exists a positive constant ε_0 such that for any $\varepsilon \in (0, \varepsilon_0]$ the following lower bound estimates for the lifespan of solutions to (4) hold:

$$\mathscr{B}\big(0, \text{LifeSpan}(u)\big) \geq \begin{cases} c\varepsilon^{-\frac{2\sigma(p-1)}{2\sigma-n(p-1)}} & \text{if } \ p < 1 + \dfrac{2\sigma}{n}, \\ \exp\big(c\varepsilon^{-(p-1)}\big) & \text{if } \ p = 1 + \dfrac{2\sigma}{n}, \end{cases} \tag{9}$$

where $c = c(n, u_0, u_1)$ is a positive constant independent of ε.

Remark 2. Summarizing the derived results (8) and (9) in Theorems 3 and 4 we claim that the sharp lifespan estimates for solutions to the Cauchy problem (4) in both the subcritical case and the critical case are given by the following (implicit) relation:

$$\mathscr{B}\big(0, \text{LifeSpan}(u)\big) \sim \begin{cases} C\varepsilon^{-\frac{2\sigma(p-1)}{2\sigma-n(p-1)}} & \text{if } \ p < 1 + \dfrac{2\sigma}{n}, \\ \exp\big(C\varepsilon^{-(p-1)}\big) & \text{if } \ p = 1 + \dfrac{2\sigma}{n}, \end{cases}$$

with a positive constant C independent of ε.

Example 1.2. Let us give several typical examples of $b = b(t)$ to find out the (explicit) sharp estimates for lifespan as follows:

- If $b(t) = \mu(1+t)^{-\beta}$ with $\mu > 0$ and $\beta \in (-1,1)$, then

$$\text{LifeSpan}(u) \sim \begin{cases} C\varepsilon^{-\frac{2\sigma(p-1)}{2\sigma-n(p-1)}\cdot\frac{1}{1+\beta}} & \text{if } p < 1+\dfrac{2\sigma}{n}, \\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{if } p = 1+\dfrac{2\sigma}{n}. \end{cases}$$

- If $b(t) = \mu(1+t)$ with $\mu > 0$, then

$$\text{LifeSpan}(u) \sim \begin{cases} \exp\left(C\varepsilon^{-\frac{2\sigma(p-1)}{2\sigma-n(p-1)}}\right) & \text{if } p < 1+\dfrac{2\sigma}{n}, \\ \exp\Big(\exp\left(C\varepsilon^{-(p-1)}\right)\Big) & \text{if } p = 1+\dfrac{2\sigma}{n}. \end{cases}$$

- If $b(t) = \prod_{k=1}^{m} \ell_k(t)$ with $m \geq 2$, $\ell_1(t) = 1+t$ and $\ell_{k+1} = 1+\log\left(\ell_k(t)\right)$ for $k = 1, 2, \cdots, m-1$, then

$$\text{LifeSpan}(u) \sim \begin{cases} \exp^{[m]}\left(C\varepsilon^{-\frac{2\sigma(p-1)}{2\sigma-n(p-1)}}\right) & \text{if } p < 1+\dfrac{2\sigma}{n}, \\ \exp^{[m+1]}\left(C\varepsilon^{-(p-1)}\right) & \text{if } p = 1+\dfrac{2\sigma}{n}, \end{cases}$$

where $\exp^{[1]}(t) = \exp(t)$ and $\exp^{[k+1]}(t) = \exp\left(\exp^{[k]}(t)\right)$ for $k = 1, 2, \cdots, m$.

The structure of this paper is organized as follows: In Sections 2 and 3, we devote the study of the corresponding linear equation of (4) to conclude some decay estimates for solutions. Next, we give the proofs of the global (in time) existence of small data Sobolev solutions and the blow-up results for (4) in Sections 4 and 5, respectively. Finally, Section 6 is to present the estimates for upper bound and lower bound of lifespan when the finite time blow-up phenomena of solutions to (4) occur.

2 | THE STUDY OF THE CORRESPONDING LINEAR EQUATION

Our starting point in this paper is to the study the corresponding linear Cauchy problem for (4), namely,

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^\sigma u(t,x) + b(t)u_t(t,x) = 0, & (t,x) \in (0,\infty)\times\mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{10}$$

where $\sigma > 1$ and $n \geq 1$.

We apply the partial Fourier transformation with respect to spatial variables to (10) with $\hat{u} = \hat{u}(t,\xi) = \mathcal{F}_{x\to\xi}\left(u(t,x)\right)(t,\xi)$ to get

$$\begin{cases} \hat{u}_{tt} + |\xi|^{2\sigma}\hat{u} + b(t)\hat{u}_t = 0, & (t,\xi) \in (0,+\infty)\times\mathbb{R}^n, \\ \hat{u}(0,\xi) = \varepsilon\hat{u}_0(\xi), \quad \hat{u}_t(0,\xi) = \varepsilon\hat{u}_1(\xi), & \xi \in \mathbb{R}^n. \end{cases} \tag{11}$$

By applying the transformation

$$\hat{u}(t,\xi) = \exp\Big(-\frac{1}{2}\int_0^t b(\tau)d\tau\Big)\hat{v}(t,\xi),$$

one transfers the Cauchy problem (11) into

$$\begin{cases} \hat{v}_{tt} + m(t,\xi)\hat{v} = 0, & (t,\xi) \in [0,\infty)\times\mathbb{R}^n, \\ \hat{v}(0,\xi) = \hat{v}_0(\xi), \quad \hat{v}_t(0,\xi) = \hat{v}_1(\xi), & \xi \in \mathbb{R}^n, \end{cases} \tag{12}$$

where $\hat{v}_0(\xi) = \hat{u}_0(\xi)$ and $\hat{v}_1(\xi) = \frac{b(0)}{2}\hat{u}_0(\xi) + \hat{u}_1(\xi)$ and the coefficient $m = m(t,\xi)$ of the mass term is defined by

$$m(t,\xi) := |\xi|^{2\sigma} - \frac{1}{4}b^2(t) - \frac{1}{2}b'(t). \tag{13}$$

We see that $b'(t)$ is a negligible term in (13), that is, it holds $|b'(t)| = o(b^2(t))$ as $t \to \infty$. Then, the term $|\xi|^{2\sigma} - b^2(t)/4$ can be considered as the principal part of the term $m(t,\xi)$. Hence, we introduce

$$\Gamma := \left\{(t,\xi) \in [0,\infty)\times\mathbb{R}^n \,:\, |\xi|^\sigma = \frac{1}{2}b(t)\right\}$$

which divides the extended phase space into two regions as follows:

$$\Pi_{\text{hyp}} = \left\{ (t,\xi) \in [0,\infty) \times \mathbb{R}^n \,:\, |\xi|^\sigma > \frac{1}{2} b(t) \right\} \quad \text{and} \quad \Pi_{\text{ell}} = \left\{ (t,\xi) \in [0,\infty) \times \mathbb{R}^n \,:\, |\xi|^\sigma < \frac{1}{2} b(t) \right\}.$$

Let us define the auxiliary weight function

$$\langle \xi \rangle_{b(t)} := \sqrt{\left| |\xi|^{2\sigma} - \frac{b^2(t)}{4} \right|}. \tag{14}$$

Remark 3. It holds

$$\partial_t \langle \xi \rangle_{b(t)} = \mp \frac{b(t) b'(t)}{4 \langle \xi \rangle_{b(t)}},$$

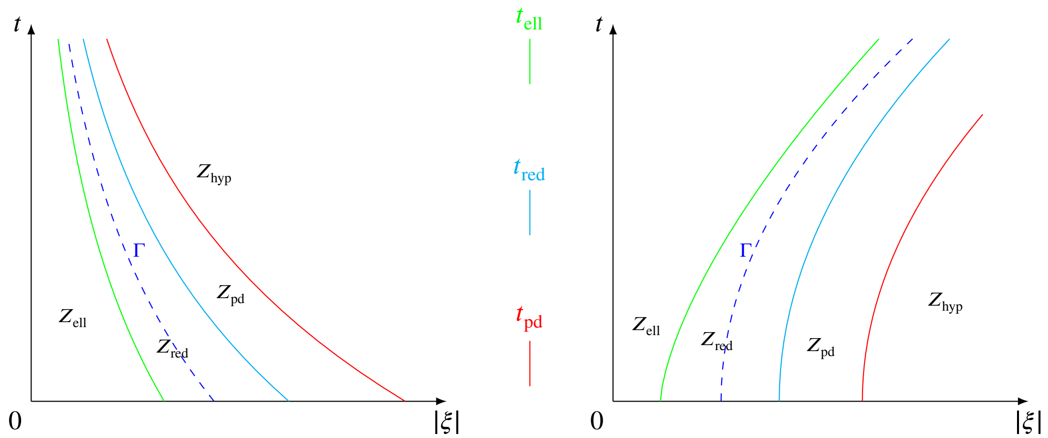
where the upper sign is taken in the hyperbolic region.

Now we will divide both regions of the extended phase space into some zones. The zones are defined as follows:

$$\begin{aligned}
Z_{\text{hyp}}(N) &= \left\{ (t,\xi) \in [0,\infty) \times \mathbb{R}^n \,:\, \langle \xi \rangle_{b(t)} \geq N \frac{b(t)}{2} \right\} \cap \Pi_{\text{hyp}}, \\
Z_{\text{pd}}(N,\varepsilon) &= \left\{ (t,\xi) \in [0,\infty) \times \mathbb{R}^n \,:\, \varepsilon \frac{b(t)}{2} \leq \langle \xi \rangle_{b(t)} \leq N \frac{b(t)}{2} \right\} \cap \Pi_{\text{hyp}}, \\
Z_{\text{red}}(\varepsilon) &= \left\{ (t,\xi) \in [0,\infty) \times \mathbb{R}^n \,:\, \langle \xi \rangle_{b(t)} \leq \varepsilon \frac{b(t)}{2} \right\}, \\
Z_{\text{ell}}(\varepsilon, t_0) &= \left\{ (t,\xi) \in [0,\infty) \times \mathbb{R}^n \,:\, \langle \xi \rangle_{b(t)} \geq \varepsilon \frac{b(t)}{2} \right\} \cap \Pi_{\text{ell}} \cap \{ t \geq t_0 \}.
\end{aligned}$$

Here in general, N and t_0 are large positive constants and ε is a small positive constant, which will be chosen later. Let us introduce separating lines between these zones as follows (see Fig.1):

- by $t_{\text{ell}} = t_{\text{ell}}(|\xi|)$, we denote the separating line between the zones $Z_{\text{ell}}(\varepsilon, t_0)$ and $Z_{\text{red}}(\varepsilon)$;
- by $t_{\text{red}} = t_{\text{red}}(|\xi|)$, we denote the separating line between the zones $Z_{\text{red}}(\varepsilon)$ and $Z_{\text{pd}}(N,\varepsilon)$;
- by $t_{\text{pd}} = t_{\text{pd}}(|\xi|)$, we denote the separating line between the zones $Z_{\text{pd}}(N,\varepsilon)$ and $Z_{\text{hyp}}(N)$.



a. The case that $b(t)$ is decreasing   b. The case that $b(t)$ is increasing

Figure 1 Division of extended phase space into zones

In the consideration of the zones, we will omit the details and sketch only the main steps of the estimates for the corresponding fundamental solutions since the desired estimates can be proved in the same way as those in[6,32,34].

2.1 | Considerations in the hyperbolic zone $Z_{\text{hyp}}(N)$

In the hyperbolic zone $Z_{\text{hyp}}(N)$ it holds $\langle\xi\rangle_{b(t)} \sim |\xi|^\sigma$. We consider the micro-energy $V = \big(\langle\xi\rangle_{b(t)}\hat{v}, D_t\hat{v}\big)^{\mathrm{T}}$. Then, it holds

$$D_tV = \begin{pmatrix} 0 & \langle\xi\rangle_{b(t)} \\ \langle\xi\rangle_{b(t)} & 0 \end{pmatrix} V + \begin{pmatrix} \dfrac{D_t\langle\xi\rangle_{b(t)}}{\langle\xi\rangle_{b(t)}} & 0 \\ -\dfrac{b'(t)}{2\langle\xi\rangle_{b(t)}} & 0 \end{pmatrix} V. \tag{15}$$

We are interested in the fundamental solution $E^V_{\text{hyp}} = E^V_{\text{hyp}}(t,s,\xi)$ to the system (15). At first, we consider the first matrix as principal part and the second one as remainder. Then, we carry out two steps of diagonalization procedure to make the remainder integrable over the hyperbolic zone $Z_{\text{hyp}}(N)$. Summarizing, asymptotic behavior of the fundamental solution $E^V_{\text{hyp}} = E^V_{\text{hyp}}(t,s,\xi)$ is given by the following statement.

Lemma 1. The following estimate holds for the fundamental solution $E^V_{\text{hyp}} = E^V_{\text{hyp}}(t,s,\xi)$ with $(s,\xi),(t,\xi) \in Z_{\text{hyp}}$ and $t \geq s$:

$$(|E^V_{\text{hyp}}(t,s,\xi)|) \lesssim \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

For the proof of Lemma 1, we may follow the approach of Section 7.1 in[6]

2.2 | Considerations in the elliptic zone $Z_{\text{ell}}(\varepsilon, t_0)$

In the elliptic zone we introduce the micro-energy $V = \big(\langle\xi\rangle_{b(t)}\hat{v}, D_t\hat{v}\big)^{\mathrm{T}}$ for all $t \geq s$ and $(t,\xi),(s,\xi) \in Z_{\text{ell}}(\varepsilon, t_0)$. Then, the corresponding first-order system of the Cauchy problem (12) is stated as

$$D_tV = \begin{pmatrix} 0 & \langle\xi\rangle_{b(t)} \\ -\langle\xi\rangle_{b(t)} & 0 \end{pmatrix} V + \begin{pmatrix} \dfrac{D_t\langle\xi\rangle_{b(t)}}{\langle\xi\rangle_{b(t)}} & 0 \\ -\dfrac{b'(t)}{2\langle\xi\rangle_{b(t)}} & 0 \end{pmatrix} V.$$

Step 1. Diagonalization procedure: Our aim is to prove estimates and structural properties for the fundamental solution $E^V_{\text{ell}} = E^V_{\text{ell}}(t,s,\xi)$ corresponding to the micro-energy V. Performing the diagonalization procedure, we get after the second step of the diagonalization that the entries of the remainder matrix are uniformly integrable over the elliptic zone.

Lemma 2. The fundamental solution $E^V_{\text{ell}} = E^V_{\text{ell}}(t,s,\xi)$ can be estimated by

$$\big(|E^V_{\text{ell}}(t,s,\xi)|\big) \lesssim \frac{\langle\xi\rangle_{b(t)}}{\langle\xi\rangle_{b(s)}} \exp\Big(\int_s^t \langle\xi\rangle_{b(\tau)} d\tau\Big) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

with $(t,\xi),(s,\xi) \in Z_{\text{ell}}(\varepsilon, t_0) \cap \{t \geq t_0(\varepsilon)\}$ and $0 \leq s \leq t$.

For the proof on Lemma 2 we may follow Section 4.2.2 in[32].

Step 2. Transforming back to the original Cauchy problem: After obtaining estimates for $E^V_{\text{ell}} = E^V_{\text{ell}}(t,s,\xi)$ it is sufficient to apply the backward transformation to the original Cauchy problem. This means that we transform back $E^V_{\text{ell}} = E^V_{\text{ell}}(t,s,\xi)$ to estimate the fundamental solution $E_{\text{ell}} = E_{\text{ell}}(t,s,\xi)$ which is related to a first-order system for the micro-energy $\big(|\xi|^\sigma\hat{u}, D_t\hat{u}\big)^{\mathrm{T}}$ and gives the representation

$$E_{\text{ell}}(t,s,\xi) = T(t,\xi)E^V_{\text{ell}}(t,s,\xi)T^{-1}(s,\xi), \tag{16}$$

where the matrix $T(t,\xi)$ and its inverse matrix $T^{-1}(t,\xi)$ are given in the following way:

$$\begin{pmatrix} |\xi|^\sigma\hat{u} \\ D_t\hat{u} \end{pmatrix} = \underbrace{\begin{pmatrix} \dfrac{|\xi|^\sigma}{\lambda(t)h(t,\xi)} & 0 \\ i\dfrac{b(t)}{2\lambda(t)h(t,\xi)} & \frac{1}{\lambda(t)} \end{pmatrix}}_{T(t,\xi)} \begin{pmatrix} h(t,\xi)\hat{v} \\ D_t\hat{v} \end{pmatrix}, \qquad T^{-1}(t,\xi) = \begin{pmatrix} \dfrac{\lambda(t)h(t,\xi)}{|\xi|^\sigma} & 0 \\ -i\dfrac{b(t)\lambda(t)}{2|\xi|^\sigma} & \lambda(t) \end{pmatrix},$$

where the auxiliary function $\lambda = \lambda(t)$ is given by

$$\lambda(t) := \exp\Big(\frac{1}{2}\int_0^t b(\tau)d\tau\Big). \tag{17}$$

Lemma 3. The following inequalities hold:

1. in the elliptic zone it holds $\langle\xi\rangle_{b(t)} - \frac{b(t)}{2} \le -\frac{|\xi|^{2\sigma}}{b(t)}$,
2. $\frac{\lambda(s)}{\lambda(t)}\exp\Big(\int_s^t \langle\xi\rangle_{b(\tau)}d\tau\Big) \le \exp\Big(-|\xi|^{2\sigma}\int_s^t \frac{1}{b(\tau)}d\tau\Big)$,

where $\lambda = \lambda(t)$ is defined in (17).

Proof. Using the elementary inequality

$$\sqrt{x+y} \le \sqrt{x} + \frac{y}{2\sqrt{x}}$$

for any $x \ge 0$ and $y \ge -x$, the first statement is valid by plugging $x = \frac{b^2(t)}{4}$ and $y = -|\xi|^{2\sigma}$. The second statement follows directly from the first one together with the definition of $\lambda = \lambda(t)$. ☐

Step 3. A refined estimate for the fundamental solution in the elliptic zone: From Lemma 2 we get for $(t,\xi),(s,\xi) \in Z_{\text{ell}}(\varepsilon,t_0)$ the estimate

$$\big(|E^V_{\text{ell}}(t,s,\xi)|\big) \lesssim \frac{b(t)}{b(s)}\exp\Big(\int_s^t \langle\xi\rangle_{b(\tau)}d\tau\Big)\begin{pmatrix}1&1\\1&1\end{pmatrix}.$$

This yields in combination with (16) the estimate

$$\begin{aligned}\big(|E_{\text{ell}}(t,s,\xi)|\big) &\lesssim \begin{pmatrix}|\xi|^\sigma & 0\\ b(t) & b(t)\end{pmatrix}\exp\Big(\int_s^t\Big(\langle\xi\rangle_{b(\tau)}-\frac{b(\tau)}{2}\Big)d\tau\Big)\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}\frac{1}{|\xi|^\sigma} & 0\\ \frac{1}{|\xi|^\sigma} & \frac{1}{b(s)}\end{pmatrix}\\ &\lesssim \exp\Big(-|\xi|^{2\sigma}\int_s^t\frac{1}{b(\tau)}d\tau\Big)\begin{pmatrix}1 & \frac{|\xi|^\sigma}{b(s)}\\ \frac{b(t)}{|\xi|^\sigma} & \frac{b(t)}{b(s)}\end{pmatrix},\end{aligned}$$

where we used Lemma 3.

Lemma 4. The fundamental solution $E_{\text{ell}} = E_{\text{ell}}(t,s,\xi)$ satisfies the following estimate:

$$\big(|E_{\text{ell}}(t,s,\xi)|\big) \lesssim \exp\Big(-|\xi|^{2\sigma}\int_s^t\frac{1}{b(\tau)}d\tau\Big)\begin{pmatrix}1 & \frac{|\xi|^\sigma}{b(s)}\\ \frac{|\xi|^\sigma}{b(t)} & \frac{|\xi|^{2\sigma}}{b(s)b(t)}\end{pmatrix} + \frac{\lambda^2(s)}{\lambda^2(t)}\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

for all $t \ge s$ and $(t,\xi),(s,\xi) \in Z_{\text{ell}}(\varepsilon,t_0)$.

For the proof of Lemma 4 we can follow the idea of the proof of Lemma 4.19 in [32].

2.3 | Considerations in the reduced zone $Z_{\text{red}}(\varepsilon)$ and pseudo-differential zone $Z_{\text{pd}}(N,\varepsilon)$

In the reduced zone $Z_{\text{red}}(\varepsilon)$ we introduce the micro-energy $V = \big(\varepsilon\frac{b(t)}{2}\hat{v}, D_t\hat{v}\big)^{\text{T}}$. Then, by (12) the function V satisfies the following system:

$$D_tV = \underbrace{\begin{pmatrix}\frac{D_tb(t)}{b(t)} & \varepsilon\frac{b(t)}{2}\\ \frac{|\xi|^{2\sigma}-\frac{1}{4}b^2(t)-\frac{1}{2}b'(t)}{\varepsilon\frac{b(t)}{2}} & 0\end{pmatrix}}_{A^V(t,\xi)}V. \tag{18}$$

We want to estimate the fundamental solution $E^V_{\text{red}} = E^V_{\text{red}}(t,s,\xi)$ to (18), that is, the solution to

$$\begin{cases} D_t E^V_{\text{red}}(t,s,\xi) = A^V(t,\xi) E^V_{\text{red}}(t,s,\xi), \\ E^V_{\text{red}}(s,s,\xi) = I. \end{cases}$$

The norm of the coefficient matrix of (18) can be estimated by $\varepsilon b(t)$ for sufficiently large t.

Lemma 5. The fundamental solution $E^V_{\text{red}} = E^V_{\text{red}}(t,s,\xi)$ to (18) satisfies the following estimate:

$$\big(|E^V_{\text{red}}(t,s,\xi)|\big) \lesssim \exp\left(\varepsilon \int_s^t b(\tau)d\tau\right) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

for all $t \geq s \geq t_0$ with sufficiently large $t_0 = t_0(\varepsilon)$ and $(t,\xi),(s,\xi) \in Z_{\text{red}}(\varepsilon)$.

For the proof of Lemma 5, we see Section 2.3 in[34].

On the other hand, in $Z_{\text{pd}}(N,\varepsilon)$ we introduce the micro-energy $V = \big(\langle\xi\rangle_{b(t)}\hat{v}, D_t\hat{v}\big)^{\text{T}}$. Then, by (12) it holds

$$D_t V = \begin{pmatrix} 0 & \langle\xi\rangle_{b(t)} \\ \langle\xi\rangle_{b(t)} & 0 \end{pmatrix} V + \begin{pmatrix} \frac{D_t\langle\xi\rangle_{b(t)}}{\langle\xi\rangle_{b(t)}} & 0 \\ -\frac{b'(t)}{2\langle\xi\rangle_{b(t)}} & 0 \end{pmatrix} V. \tag{19}$$

Lemma 6. The fundamental solution $E^V_{\text{pd}} = E^V_{\text{pd}}(t,s,\xi)$ to (19) satisfies the following estimate:

$$\big(|E^V_{\text{pd}}(t,s,\xi)|\big) \lesssim \left(\frac{1+t}{1+s}\right)^C \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

for all $t \geq s \geq t_0$ with sufficiently large $t_0 = t_0(\varepsilon)$ and $(t,\xi),(s,\xi) \in Z_{\text{pd}}(N,\varepsilon)$.

For the proof of Lemma 6 we see Section 2.4 in[34].

3 | ENERGY ESTIMATES OF HIGHER ORDER

The main goal of this section is to prove on the one hand higher order energy estimates to solutions to (10). On the other hand, we want to derive higher order energy estimates for solutions to the corresponding family of (10) for parameter-dependent Cauchy problems with suitable initial data $\big(0, g(s,x)\big)$. The representation of the fundamental solutions obtained so far allows us to conclude estimates for solutions and their derivatives to these Cauchy problems.

3.1 | A family of parameter-dependent linear Cauchy problems

Let us consider the following family of parameter-dependent Cauchy problem:

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + b(t)u_t = 0, & (t,x) \in [s,\infty) \times \mathbb{R}^n, \quad s \geq 0, \\ u(s,x) = f(s,x), \quad u_t(s,x) = g(s,x), & x \in \mathbb{R}^n. \end{cases} \tag{20}$$

We apply the partial Fourier transformation to (20) with respect to the spatial variables. Denoting by $\hat{u} = \hat{u}(t,\xi)$ the partial Fourier transformation $\mathcal{F}_{x\to\xi}\big(u(t,x)\big)(t,\xi)$ we obtain

$$\begin{cases} \hat{u}_{tt} + |\xi|^{2\sigma}\hat{u} + b(t)\hat{u}_t = 0, & (t,\xi) \in [s,\infty) \times \mathbb{R}^n, \quad s \geq 0, \\ \hat{u}(s,\xi) = \hat{f}(s,\xi), \quad \hat{u}_t(s,\xi) = \hat{g}(s,\xi), & \xi \in \mathbb{R}^n. \end{cases} \tag{21}$$

Now we make the change of variables

$$y(t,\xi) = \frac{\lambda(t)}{\lambda(s)}\hat{u}(t,\xi) \qquad \text{with} \qquad \lambda(t) := \exp\left(\frac{1}{2}\int_0^t b(\tau)d\tau\right).$$

Then, we obtain the Cauchy problem

$$\begin{cases} y_{tt} + m(t,\xi)y = 0, & (t,\xi) \in [s,\infty) \times \mathbb{R}^n, \quad s \geq 0, \\ y(s,\xi) = \widehat{f}(s,\xi, & \xi \in \mathbb{R}^n, \\ y_t(s,\xi) = \dfrac{b(s)}{2}\widehat{f}(s,\xi) + \widehat{g}(s,\xi), & \xi \in \mathbb{R}^n, \end{cases} \tag{22}$$

where $m(t,\xi)$ is defined as in (13). Furthermore, we introduce the function $\langle \xi \rangle_{b(t)}$ as in (14). In the same manner as in Section 2, we divide the extended phase space $[s,\infty) \times \mathbb{R}^n$ into the following zones:

$$\begin{aligned} Z_{\text{hyp}}(N) &= \left\{ (t,\xi) \in [s,\infty) \times \mathbb{R}^n \,:\, \langle \xi \rangle_{b(t)} \geq N\frac{b(t)}{2} \right\} \cap \Pi_{\text{hyp}}, \\ Z_{\text{pd}}(N,\varepsilon) &= \left\{ (t,\xi) \in [s,\infty) \times \mathbb{R}^n \,:\, \varepsilon\frac{b(t)}{2} \leq \langle \xi \rangle_{b(t)} \leq N\frac{b(t)}{2} \right\} \cap \Pi_{\text{hyp}}, \\ Z_{\text{red}}(\varepsilon) &= \left\{ (t,\xi) \in [s,\infty) \times \mathbb{R}^n \,:\, \langle \xi \rangle_{b(t)} \leq \varepsilon\frac{b(t)}{2} \right\}, \\ Z_{\text{ell}}(\varepsilon,t_0) &= \left\{ (t,\xi) \in [s,\infty) \times \mathbb{R}^n \,:\, \langle \xi \rangle_{b(t)} \geq \varepsilon\frac{b(t)}{2} \right\} \cap \Pi_{\text{ell}} \cap \{t \geq t_0\}. \end{aligned}$$

Let $s \geq 0$ and $\xi \neq 0$. Let us distinguish between two cases with $b_\infty := \lim\limits_{t\to\infty} b(t)$:

- $b(t)$ is decreasing with $b_\infty \in [0,\infty)$ and $(s,\xi) \in Z_{\text{ell}}(\varepsilon,t_0)$,
- $b(t)$ is increasing with $b_\infty \in (0,\infty]$ and $(s,\xi) \in Z_{\text{hyp}}(N)$.

Let us introduce the function

$$h = h(t,\xi) = \chi\left(\frac{\langle \xi \rangle_{b(t)}}{\varepsilon\frac{b(t)}{2}}\right)\varepsilon\frac{b(t)}{2} + \left(1 - \chi\left(\frac{\langle \xi \rangle_{b(t)}}{\varepsilon\frac{b(t)}{2}}\right)\right)\langle \xi \rangle_{b(t)}, \tag{23}$$

for our model (22). Here $\chi \in C^\infty\big([0,\infty)\big)$ is a localizing function with $\chi(\zeta) = 1$ for $0 \leq \zeta \leq \frac{1}{2}$ and $\chi(\zeta) = 0$ for $\zeta \geq 1$. We define $Y(t,\xi) = \big(h(t,\xi)y(t,\xi), D_t y(t,\xi)\big)^{\mathrm{T}}$. Then, from (22) we have

$$D_t Y(t,\xi) = \underbrace{\begin{pmatrix} \dfrac{D_t h(t,\xi)}{h(t,\xi)} & h(t,\xi) \\ \dfrac{m(t,\xi)}{h(t,\xi)} & 0 \end{pmatrix}}_{A^Y(t,\xi)} Y(t,\xi). \tag{24}$$

We denote by $E^Y = E^Y(t,t_1,\xi)$ the fundamental solution to (24) for any $t \geq t_1 \geq s$, i.e., the solution to

$$\begin{cases} D_t E^Y(t,t_1,\xi) = A^Y(t,\xi)E^Y(t,t_1,\xi), \\ E^Y(t_1,t_1,\xi) = I. \end{cases}$$

3.1.1 | Representation of the solutions

Let us turn now to the Cauchy problem (20). We introduce $\widehat{K}_1 = \widehat{K}_1(t,s,\xi)$ as the solution to (21) with initial conditions $\widehat{f}(s,\xi) = 0$ and $\widehat{g}(s,\xi) = 1$. Following the approach of[6] in Section 7.4, we have

$$\widehat{K}_1(t,s,\xi) = \frac{\lambda(s)}{\lambda(t)}\frac{E^Y_{12}(t,s,\xi)}{h(t,\xi)}, \tag{25}$$

$$D_t\widehat{K}_1(t,s,\xi) = \frac{\lambda(s)}{\lambda(t)}\Big(E^Y_{22}(t,s,\xi) - \frac{b(t)}{2h(t,\xi)}E^Y_{12}(t,s,\xi)\Big). \tag{26}$$

In the same way we consider $\widehat{K}_0 = \widehat{K}_0(t, 0, \xi)$ as the solution to (21) with $s = 0$ and initial conditions $\widehat{f}(0, \xi) = 1$ and $\widehat{g}(0, \xi) = 0$. Then, it holds

$$\begin{aligned} \widehat{K}_0(t, 0, \xi) &= \frac{1}{\lambda(t)} \frac{h(0, \xi)}{h(t, \xi)} E^Y_{11}(t, 0, \xi), \\ D_t \widehat{K}_0(t, 0, \xi) &= \frac{h(0, \xi)}{\lambda(t)} \Big(E^Y_{21}(t, 0, \xi) - \frac{b(t)}{2h(t, \xi)} E^Y_{11}(t, 0, \xi) \Big), \end{aligned}$$

where $h = h(t, \xi)$ is defined in (23) and $E^Y := E^Y(t, s, \xi)$ is the fundamental solution to the system (24). These above relations allow us to transfer properties of $E^Y = E^Y(t, s, \xi)$ to $\widehat{K}_1 = \widehat{K}_1(t, s, \xi)$ and $E^Y = E^Y(t, 0, \xi)$ to $\widehat{K}_0 = \widehat{K}_0(t, 0, \xi)$.

3.1.2 | Estimates for the multipliers and their time derivatives

In order to estimate the norm of the solution to our original Cauchy problem, we need to estimate our multipliers $|\widehat{K}_1(t, s, \xi)|$ and $|\widehat{K}_0(t, 0, \xi)|$ in each zone of the extended phase space. Similarly, we can derive estimates for $|\partial_t \widehat{K}_1(t, s, \xi)|$ and $|\partial_t \widehat{K}_0(t, 0, \xi)|$ as well.

Let us consider the following estimate for $|\widehat{K}_1(t, s, \xi)|$ in $Z_{\mathrm{red}}(\varepsilon)$:

$$|\widehat{K}^{\mathrm{red}}_1(t, s, \xi)| \lesssim \frac{1}{|\xi|^\sigma} \Big(\frac{\lambda(s)}{\lambda(t)} \Big)^{1-2\beta}, \tag{27}$$

where we choose $\varepsilon > 0$ such that $\beta := C\varepsilon < \frac{1}{2}$. Then, we can see easily that we can estimate $|\widehat{K}_1(t, s, \xi)|$ in $Z_{\mathrm{hyp}}(N)$ and $Z_{\mathrm{pd}}(N, \varepsilon)$ by (27). Thus, we can glue $Z_{\mathrm{red}}(\varepsilon)$ to the hyperbolic region and we define new region by

$$\Pi_{\mathrm{hyp}}(N, \varepsilon) = Z_{\mathrm{red}}(\varepsilon) \cup Z_{\mathrm{pd}}(N, \varepsilon) \cup Z_{\mathrm{hyp}}(N).$$

We denote by $t_{\mathrm{ell}} = t_{\mathrm{ell}}(|\xi|)$ the separating line between $Z_{\mathrm{ell}}(\varepsilon, t_0)$ and $\Pi_{\mathrm{hyp}}(N, \varepsilon)$. This curve is given by

$$\frac{b^2(t_{\mathrm{ell}})}{4} - |\xi|^{2\sigma} = \varepsilon^2 \frac{b^2(t_{\mathrm{ell}})}{4}, \quad \text{i.e.} \quad t_{\mathrm{ell}} = b^{-1}\Big(\frac{2|\xi|^\sigma}{\sqrt{1-\varepsilon^2}} \Big).$$

3.2 | Estimates for the multiplier $\widehat{K}_1$

Now we distinguish between two cases related to the setting of the zones in the extended phase space for a general $s \geq 0$.

Small frequencies $|\xi|^\sigma \leq \frac{b(s)}{2}\sqrt{1-\varepsilon^2}$. We have the following two cases:

- *Case 1*: $0 \leq s \leq t \leq t_{\mathrm{ell}}$. In this case (t, ξ) and (s, ξ) belong to $Z_{\mathrm{ell}}(\varepsilon, t_0)$. It holds $h(t, \xi) \sim b(t)$. Then, we have the following estimates from Lemma 4 for all $t \in [s, t_{\mathrm{ell}}]$:

$$\begin{aligned} |\widehat{K}_1(t, s, \xi)| &\lesssim \frac{1}{b(s)} \exp\big(- C|\xi|^{2\sigma} \mathscr{B}(s, t) \big), \\ |\partial_t \widehat{K}_1(t, s, \xi)| &\lesssim \frac{|\xi|^{2\sigma}}{b(s)b(t)} \exp\big(- C|\xi|^{2\sigma} \mathscr{B}(s, t) \big). \end{aligned}$$

- *Case 2*: $0 \leq s \leq t_{\mathrm{ell}} \leq t$. In this case we glue the estimates in $Z_{\mathrm{ell}}(\varepsilon, t_0)$ from Lemma 4 and in $\Pi_{\mathrm{hyp}}(N, \varepsilon)$ from (27). Hence, we arrive at the following estimates for all $(s, \xi) \in Z_{\mathrm{ell}}(\varepsilon, t_0)$ and $t \geq t_{\mathrm{ell}}$:

$$\begin{aligned} |\widehat{K}_1(t, s, \xi)| &\lesssim \frac{1}{b(s)} \exp\big(- C|\xi|^{2\sigma} \mathscr{B}(s, t) \big), \\ |\partial_t \widehat{K}_1(t, s, \xi)| &\lesssim \frac{|\xi|^{2\sigma}}{b(s)b(t)} \exp\big(- C|\xi|^{2\sigma} \mathscr{B}(s, t) \big). \end{aligned}$$

Large frequencies $|\xi|^\sigma \geq \frac{b(s)}{2}\sqrt{1-\varepsilon^2}$. We have the following two cases:

- *Case 1*: $0 \leq s \leq t \leq t_{\mathrm{ell}}$. If $b = b(t)$ is increasing, then (t, ξ) and (s, ξ) belong to $\Pi_{\mathrm{hyp}}(N, \varepsilon)$. Taking $h(t, \xi) \sim |\xi|^\sigma$ we have

$$|\widehat{K}_1(t, s, \xi)| \lesssim \frac{1}{|\xi|^\sigma} \Big(\frac{\lambda(s)}{\lambda(t)} \Big)^{1-2\beta}, \tag{28}$$

$$|\partial_t \widehat{K}_1(t, s, \xi)| \lesssim \Big(\frac{\lambda(s)}{\lambda(t)} \Big)^{1-2\beta}, \tag{29}$$

where these estimates are derived by using the representations (25) and (26) and the estimate in $\Pi_{\text{hyp}}(N,\varepsilon)$. Indeed, the representation (26) implies the estimate

$$\begin{aligned}|\partial_t \widehat{K}_1(t,s,\xi)| &\le \frac{\lambda(s)}{\lambda(t)}|E^Y_{22}(t,s,\xi)| + \frac{b(t)}{2h(t,\xi)}|E^Y_{12}(t,s,\xi)|\\ &\lesssim \frac{\lambda(s)}{\lambda(t)}\Big(\frac{\lambda(t)}{\lambda(s)}\Big)^{2\beta} \lesssim \Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\beta},\end{aligned}$$

where we used the estimates of $E^Y_{12}(t,s,\xi)$ and $E^Y_{22}(t,s,\xi)$ from Lemma 6.

We remark that the estimates (28) and (29) remain true for large frequencies in the case that $b=b(t)$ is decreasing. If $b=b(t)$ is decreasing, then we have only $Z_{\text{hyp}}(N)$ for large frequencies.

- *Case 2*: $0\le s\le t_{\text{ell}}\le t$. We remark that this case comes into play only if $b=b(t)$ is increasing and there is no separating line if $|\xi|\ge b_\infty\sqrt{1-\varepsilon^2}$. Then, we have the following estimates for all $t_{\text{ell}}\le t$:

$$\begin{aligned}|\widehat{K}_1(t,s,\xi)| &\lesssim \frac{1}{b(s)}\exp\big(-C'|\xi|^{2\sigma}\mathscr{B}(s,t)\big),\\ |\partial_t\widehat{K}_1(t,s,\xi)| &\lesssim \frac{|\xi|^{2\sigma}}{b(s)b(t)}\exp\big(-C'|\xi|^{2\sigma}\mathscr{B}(s,t)\big).\end{aligned}$$

3.3 | Final estimates

For any $t\ge s$ and $s\in[0,\infty)$ let us define

$$\Omega(s,t):=\left(\max\big\{b(s),b(t)\big\}\frac{\sqrt{1-\varepsilon^2}}{2}\right)^{\frac{1}{\sigma}}.$$

Remark 4. We distinguish between small and large frequencies as follows: Small frequencies satisfy the condition $|\xi|\le\Omega(s,t)$, while, large frequencies satisfy the condition $|\xi|\ge\Omega(s,t)$.

Summarizing we arrived at the following statements for the estimates of $|\widehat{K}_1(t,s,\xi)|$ and $|\partial_t\widehat{K}_1(t,s,\xi)|$ with $t\ge s\ge 0$.

Corollary 1. If $|\xi|\ge\Omega(s,t)$, then we have the following estimates:

$$|\widehat{K}_1(t,s,\xi)| \lesssim \frac{1}{|\xi|^\sigma}\Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\beta}, \tag{30}$$

$$|\partial_t\widehat{K}_1(t,s,\xi)| \lesssim \Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\beta}. \tag{31}$$

If $|\xi|\le\Omega(s,t)$, then we have the following estimates:

$$|\widehat{K}_1(t,s,\xi)| \lesssim \frac{1}{b(s)}\exp\big(-C'|\xi|^{2\sigma}\mathscr{B}(s,t)\big), \tag{32}$$

$$|\partial_t\widehat{K}_1(t,s,\xi)| \lesssim \frac{|\xi|^{2\sigma}}{b(s)b(t)}\exp\big(-C'|\xi|^{2\sigma}\mathscr{B}(s,t)\big). \tag{33}$$

We have similar results for the estimates of $|\widehat{K}_0(t,0,\xi)|$ and $|\partial_t\widehat{K}_0(t,0,\xi)|$.

Corollary 2. If $|\xi|\ge\Omega(0,t)$, then we have the following estimates:

$$\begin{aligned}|\widehat{K}_0(t,0,\xi)| &\lesssim \Big(\frac{1}{\lambda(t)}\Big)^{1-2\beta},\\ |\partial_t\widehat{K}_0(t,0,\xi)| &\lesssim |\xi|^\sigma\Big(\frac{1}{\lambda(t)}\Big)^{1-2\beta}.\end{aligned}$$

If $|\xi|\le\Omega(0,t)$, then we have the following estimates:

$$\begin{aligned}|\widehat{K}_0(t,0,\xi)| &\lesssim \exp\big(-C'|\xi|^{2\sigma}\mathscr{B}(0,t)\big),\\ |\partial_t\widehat{K}_0(t,0,\xi)| &\lesssim \frac{|\xi|^{2\sigma}}{b(t)}\exp\big(-C'|\xi|^{2\sigma}\mathscr{B}(0,t)\big).\end{aligned}$$

3.4 | Matsumura-type estimates with additional regularity of the data

In this section, let us consider the following two linear Cauchy problems:

$$\begin{cases} v_{tt} + (-\Delta)^\sigma v + b(t)v_t = 0, & (t,x) \in [s,\infty) \times \mathbb{R}^n, \quad s \geq 0, \\ v(s,x) = 0, \quad v_t(s,x) = g(s,x), & x \in \mathbb{R}^n, \end{cases} \tag{34}$$

and

$$\begin{cases} w_{tt} + (-\Delta)^\sigma w + b(t)w_t = 0, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ w(0,x) = f(x), \quad w_t(0,x) = 0, & x \in \mathbb{R}^n. \end{cases} \tag{35}$$

We denoted the solutions to (34) and (35) by $K_1 = K_1(t,s,x)$ and $K_0 = K_0(t,0,x)$ with initial data $g = \delta_0$ and $f = \delta_0$, respectively, where δ_0 is the Dirac distribution with respect to spatial variables in $x = 0$. Thus, we may conclude the following estimates for the solutions $v = v(t,x)$ and $w = w(t,x)$:

$$\|v(t,\cdot)\|_{L^2} = \|\hat{v}(t,\cdot)\|_{L^2} \leq \|\hat{K}_1(t,s,\xi)\hat{g}(s,\xi)\|_{L^2},$$
$$\|w(t,\cdot)\|_{L^2} = \|\hat{w}(t,\cdot)\|_{L^2} \leq \|\hat{K}_0(t,0,\xi)\hat{f}(\xi)\|_{L^2}.$$

In order to estimate the L^2 norm of $\partial_t^\ell \partial_x^\alpha K_1(t,s,x) *_{(x)} g(s,x)$ and $\partial_t^\ell \partial_x^\alpha K_0(t,0,x) *_{(x)} f(x)$ for $\ell = 0,1$ and for any $\alpha \geq 0$, we can follow the techniques used in[6] and[27]. More precisely, we assume additional L^m regularity for the data, with $m \in [1,2)$ to prove Matsumura-type estimates for solutions and their first partial derivatives to (34) and (35).

Lemma 7. We have the following estimates for large frequencies $|\xi| \geq \Omega(s,t)$:

$$\Big\| |\xi|^\alpha \partial_t^\ell \hat{K}_1(t,s,\cdot)\hat{g}(s,\cdot) \Big\|_{L^2\{|\xi|\geq\Omega(s,t)\}} \lesssim \Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\beta} \|g(s,\cdot)\|_{H^{\alpha+(\ell-1)\sigma}},$$

with $\alpha \geq 0$ and $\ell \in \{0,1\}$ provided that $\alpha + (\ell-1)\sigma \geq 0$, and $\beta = C\varepsilon < \frac{1}{2}$ was given in (27). Moreover, when $\alpha \in [0,\sigma)$, the following estimate holds:

$$\Big\| |\xi|^\alpha \hat{K}_1(t,s,\cdot)\hat{g}(s,\cdot) \Big\|_{L^2\{|\xi|\geq\Omega(s,t)\}} \lesssim \Big(\frac{1}{\max\{b(s),b(t)\}}\Big)^{\sigma-\alpha} \Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\beta} \|g(s,\cdot)\|_{L^2}.$$

Proof. At first, one derives the following estimate for $\alpha + (\ell-1)\sigma \geq 0$:

$$\Big\| |\xi|^\alpha \partial_t^\ell \hat{K}_1(t,s,\cdot)\hat{g}(s,\cdot) \Big\|_{L^2\{|\xi|\geq\Omega(s,t)\}} \leq \Big\| |\xi|^{(1-\ell)\sigma} \partial_t^\ell \hat{K}_1(t,s,\xi) \Big\|_{L^\infty\{|\xi|\geq\Omega(s,t)\}} \Big\| |\xi|^{\alpha+(\ell-1)\sigma} \hat{g}(s,\cdot) \Big\|_{L^2\{|\xi|\geq\Omega(s,t)\}}.$$

The second term on the right-hand side can be estimated by $\|g(s,\cdot)\|_{H^{\alpha+(\ell-1)\sigma}}$. Now let us control the L^∞ norm of $K_1(t,s,\xi)$ and its derivatives with respect to t. Indeed, by using the estimates from (30) and (31) we get

$$|\xi|^{(1-\ell)\sigma} |\partial_t^\ell \hat{K}_1(t,s,\xi)| \lesssim \Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\beta}.$$

In the case of $\alpha \in [0,\sigma)$, we want to utilize the relation $\frac{1}{|\xi|} \leq \frac{1}{\Omega(s,t)}$ to obtain

$$|\xi|^\alpha |\hat{K}_1(t,s,\xi)| \lesssim \Big(\frac{1}{\Omega(s,t)}\Big)^{\sigma-\alpha} \Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\beta}.$$

Finally, we arrive at

$$\Big\| |\xi|^\alpha \hat{K}_1(t,s,\cdot)\hat{g}(s,\cdot) \Big\|_{\{L^2|\xi|\geq\Omega(s,t)\}} \lesssim \Big(\frac{1}{\Omega(s,t)}\Big)^{\sigma-\alpha} \Big(\frac{\lambda(s)}{\lambda(t)}\Big)^{1-2\beta} \|g(s,\cdot)\|_{L^2}.$$

This completes the proof. □

Lemma 8. The following estimates hold for small frequencies $|\xi| \leq \Omega(s,t)$:

$$\Big\| |\xi|^\alpha \partial_t^\ell \hat{K}_1(t,s,\cdot)\hat{g}(s,\cdot) \Big\|_{L^2\{|\xi|\leq\Omega(s,t)\}} \lesssim \frac{1}{b(s)b^\ell(t)} \big(\mathscr{B}(s,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\alpha}{2\sigma}-\ell} \|g(s,\cdot)\|_{L^m}$$

with $\alpha \geq 0$, $t \geq s \geq 0$ and $\ell = 0,1$, where $m \in [1,2)$.

Proof. With the assumption $m \in [1,2)$, let us choose m_1 and m_2 such that $\frac{1}{m} + \frac{1}{m_1} = 1$ and $\frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{2}$. Therefore, it holds $\frac{1}{m_2} = \frac{1}{m} - \frac{1}{2}$. Then, applying Hölder's inequality with $\frac{2}{m_1} + \frac{2}{m_2} = 1$ we obtain the following estimate:

$$\Big\| |\xi|^\alpha \partial_t^\ell \hat{K}_1(t,s,\cdot)\hat{g}(s,\cdot) \Big\|_{L^2\{|\xi|\leq\Omega(s,t)\}} \leq \Big\| |\xi|^\alpha \partial_t^\ell \hat{K}_1(t,s,\cdot) \Big\|_{L^{m_2}\{|\xi|\leq\Omega(s,t)\}} \|\hat{g}(s,\cdot)\|_{L^{m_1}\{|\xi|\leq\Omega(s,t)\}}.$$

We can estimate $\|\hat{g}(s,\cdot)\|_{L^{m_1}}$ by $\|g(s,\cdot)\|_{L^m}$ due to the Hausdorff-Young inequality. For this reason, we have only to control the L^{m_2} norm of the multiplier. Thanks to (32) and (33) one achieves the following estimate:

$$\Big\||\xi|^\alpha \partial_t^\ell \hat{K}_1(t,s,\cdot)\Big\|_{L^{m_2}\{|\xi|\le\Omega(s,t)\}} \lesssim \frac{1}{b(s)b^\ell(t)}\Big(\int\limits_{\{|\xi|\le\Omega(s,t)\}} |\xi|^{m_2(\alpha+2\ell\sigma)}\exp\Big(-Cp|\xi|^{2\sigma}\mathcal{B}(s,t)\Big)d\xi\Big)^{\frac{1}{m_2}}.$$

The application of the change of variables

$$r = Cm_2|\xi|^{2\sigma}\mathcal{B}(s,t), \quad dr = 2Cm_2\sigma|\xi|^{2\sigma-1}d|\xi|\mathcal{B}(s,t).$$

leads to

$$\int\limits_{\{|\xi|\le\Omega(s,t)\}} |\xi|^{m_2(\alpha+2\ell\sigma)}\exp\Big(-Cm_2|\xi|^{2\sigma}\mathcal{B}(s,t)\Big)d\xi \lesssim \big(\mathcal{B}(s,t)\big)^{-\frac{m_2(\alpha+2\ell\sigma)+n}{2\sigma}}\int\limits_0^\infty r^{\frac{m_2(\alpha+2\ell\sigma)+n}{2\sigma}-1}e^{-r}dr.$$

The integral on the right-hand side is bounded and we get the function

$$\frac{1}{b(s)b^\ell(t)}\big(\mathcal{B}(s,t)\big)^{-\frac{n}{2m_2\sigma}-\frac{\alpha}{2\sigma}-\ell} = \frac{1}{b(s)b^\ell(t)}\big(\mathcal{B}(s,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\alpha}{2\sigma}-\ell}.$$

This completes the proof. □

The main result for the family of one-parameter dependent Cauchy problems (34), which follows from the statements of the Lemmas 7 and 8, reads as follows:

Proposition 1. The Sobolev solution $v = v(t,x)$ to (34) satisfies the following Matsumura-type estimates for $t \ge s \ge 0$, $m \in [1,2)$ and $\alpha \ge 0$:

$$\|v(t,\cdot)\|_{\dot{H}^\alpha} \lesssim \frac{1}{b(s)}\big(1+\mathcal{B}(s,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\alpha}{2\sigma}}\|g(s,\cdot)\|_{L^m\cap H^{[\alpha-\sigma]_+}},$$

$$\|v_t(t,\cdot)\|_{\dot{H}^\alpha} \lesssim \frac{1}{b(s)b(t)}\big(1+\mathcal{B}(s,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\alpha}{2\sigma}-1}\|g(s,\cdot)\|_{L^m\cap H^\alpha}.$$

We know that the solution $u = u(t,x)$ to the linear Cauchy problem (10) can be represented as

$$u(t,x) = K_0(t,0,x) *_{(x)} u_0(x) + K_1(t,0,x) *_{(x)} u_1(x),$$

so we obtain the following statement.

Corollary 3. The Sobolev solution to (10) satisfies the following estimates with $m \in [1,2)$ and $\alpha \ge 0$:

$$\|u(t,\cdot)\|_{\dot{H}^\alpha} \lesssim \big(1+\mathcal{B}(0,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\alpha}{2\sigma}}\big(\|u_0\|_{L^m\cap H^\alpha} + \|u_1\|_{L^m\cap H^{[\alpha-\sigma]_+}}\big),$$

$$\|u_t(t,\cdot)\|_{\dot{H}^\alpha} \lesssim \frac{1}{b(t)}\big(1+\mathcal{B}(0,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\alpha}{2\sigma}-1}\big(\|u_0\|_{L^m\cap H^{\alpha+\sigma}} + \|u_1\|_{L^m\cap H^\alpha}\big).$$

4 | GLOBAL (IN TIME) EXISTENCE OF SOLUTIONS

We denote by $K_0(t,0,x)$ and $K_1(t,0,x)$ the fundamental solutions to the corresponding linear equation of (4), namely,

$$u^{\text{lin}} := K_0(t,0,x) *_{(x)} u_0(x) + K_1(t,0,x) *_{(x)} u_1(x).$$

is the solution to the linear Cauchy problem (10). For $T > 0$, we define the operator N such that

$$N : u \in X(T) \to Nu = Nu(t,x) := u^{\text{lin}}(t,x) + u^{\text{non}}(t,x),$$

where $X(T)$ is an evolution space to be determined later and $u^{\text{non}}(t,x)$ is written by the following integral operator:

$$u^{\text{non}}(t,x) := \int\limits_0^t K_1(t,s,x) *_{(x)} |u(s,x)|^p\,ds$$

thanks to Duhamel's principle. Then, we will prove global (in time) Sobolev solution to the semi-linear Cauchy problem (4) as a fixed point of the operator N. To demonstrate this, we will show that the mapping N satisfies the following two estimates:

$$\|Nu\|_{X(T)} \lesssim \|(u_0,u_1)\|_{\mathcal{D}^\alpha_m} + \|u\|^p_{X(T)}, \tag{36}$$

$$\|Nu - Nv\|_{X(T)} \lesssim \|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big), \tag{37}$$

where the data space $\mathcal{D}^\alpha_m$ is fixed in the statement of Theorem 1. Providing that $\|(u_0,u_1)\|_{\mathcal{D}^\alpha_m} = \varepsilon$ is sufficiently small, then the estimates (36) and (37) result the existence of a unique local (in time) large data solution and a unique global (in time) small data solution in $X(T)$ by using Banach's fixed point theorem.

To show the proof of Theorem 1, the following ingredients are useful.

Proposition 2 (Fractional Gagliardo-Nirenberg inequality[13]). Let $q, q_0, q_1 \in (1,\infty)$ and $\kappa \in [0,r)$ with $r > 0$. Then, the following inequality holds for all $f \in L^{q_0}(\mathbb{R}^n) \cap \dot{H}^r_{q_1}(\mathbb{R}^n)$:

$$\|f\|_{\dot{H}^\kappa_q} \lesssim \|f\|^{1-\theta}_{L^{q_0}}\ \|f\|^{\theta}_{\dot{H}^r_{q_1}},$$

where $\theta = \theta_{\kappa,r}(q,q_0,q_1,n) = \big(\frac{1}{q_0} - \frac{1}{q} + \frac{\kappa}{n}\big)\big/\big(\frac{1}{q_0} - \frac{1}{q_1} + \frac{r}{n}\big)$ and $\theta \in [\kappa/r, 1]$.

Lemma 9 (see[6]). From the conditions in Definition 1, we may conclude that $\mathscr{B}(0,t)$ is positive, strictly increasing and $\mathscr{B}(0,t) \to \infty$ as $t \to \infty$. In addition, the primitive $\mathscr{B}(s,t)$ satisfies the following properties:

$$\mathscr{B}(s,t) \approx \frac{t}{b(t)} - \frac{s}{b(s)} \quad \text{for all} \quad s \in [0,t],$$
$$\mathscr{B}(s,t) \approx \mathscr{B}(0,t) \quad \text{for all} \quad s \in \big[0,t/2\big],$$
$$\mathscr{B}(0,s) \approx \mathscr{B}(0,t) \quad \text{for all} \quad s \in \big[t/2,t\big].$$

Proof of Theorem 1. We define the evolution space of solutions $X(T)$ by

$$X(T) := \mathcal{C}([0,T], H^\alpha)$$

with its corresponding norm

$$\|u\|_{X(T)} := \sup_{0\le t\le T}\Big[\big(1+\mathscr{B}(0,t)\big)^{\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)}\|u(t,\cdot)\|_{L^2} + \big(1+\mathscr{B}(0,t)\big)^{\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{\alpha}{2\sigma}}\|\,|D|^\alpha u(t,\cdot)\|_{L^2}\Big].$$

The application of the fractional Gagliardo-Nirenberg inequality from Proposition 2 and the definition of the evolution space leads to

$$\|\,|u(\tau,\cdot)|^p\|_{L^2} = \|u(\tau,\cdot)\|^p_{L^{2p}} \lesssim \big(1+\mathscr{B}(0,\tau)\big)^{-\frac{n}{2\sigma m}p+\frac{n}{4\sigma}}\|u\|^p_{X(T)}, \tag{38}$$

$$\|\,|u(\tau,\cdot)|^p\|_{L^m\cap L^2} = \|u(\tau,\cdot)\|^p_{L^{mp}} + \|u(\tau,\cdot)\|^p_{L^{2p}} \lesssim \big(1+\mathscr{B}(0,\tau)\big)^{-\frac{n}{2\sigma m}(p-1)}\|u\|^p_{X(T)}, \tag{39}$$

provided that

$$p \in \Big[\frac{2}{m},\infty\Big) \quad \text{if} \quad n \le 2\alpha, \quad \text{or} \quad p \in \Big[\frac{2}{m}, \frac{n}{n-2\alpha}\Big] \quad \text{if} \quad 2\alpha < n \le \frac{4\alpha}{2-m}.$$

First let us prove the inequality (36). From the estimates for solutions to (10), which are shown in Corollary 3, one may derive

$$\|u^{\mathrm{lin}}(t,\cdot)\|_{L^2} \lesssim \big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)}\|(u_0,u_1)\|_{\mathcal{D}^\alpha_m},$$
$$\|\,|D|^\alpha u^{\mathrm{lin}}(t,\cdot)\|_{L^2} \lesssim \big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\alpha}{2\sigma}}\|(u_0,u_1)\|_{\mathcal{D}^\alpha_m}.$$

This means that the linear part fulfills

$$\big(1+\mathscr{B}(0,t)\big)^{\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)}\|u^{\mathrm{lin}}(t,\cdot)\|_{L^2} + \big(1+\mathscr{B}(0,t)\big)^{\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)+\frac{\alpha}{2\sigma}}\|\,|D|^\alpha u^{\mathrm{lin}}(t,\cdot)\|_{L^2} \lesssim \|(u_0,u_1)\|_{\mathcal{D}^\alpha_m}.$$

For this reason, we immediately claim $u^{\mathrm{lin}} \in X(T)$. Therefore, it remains to prove

$$\|u^{\mathrm{non}}\|_{X(T)} \lesssim \|u\|^p_{X(T)}.$$

We may proceed $|D|^\alpha u^{\rm non}(t,\cdot)$ in the L^2 norm by applying $(L^2\cap L^m)-L^2$ estimates in $[0,t/2]$ and L^2-L^2 estimates in $[t/2,t]$ from Proposition 1 as follows:

$$\|\,|D|^\alpha u^{\rm non}(t,\cdot)\|_{L^2}\lesssim\int_0^{\frac{t}{2}} b(s)^{-1}\big(1+\mathscr{B}(s,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\alpha}{2\sigma}}\|\,|u(s,\cdot)|^p\|_{L^m\cap L^2}\,ds+\int_{\frac{t}{2}}^{t} b(s)^{-1}\big(1+\mathscr{B}(s,t)\big)^{-\frac{\alpha}{2\sigma}}\|\,|u(s,\cdot)|^p\|_{L^2}\,ds.$$

On the one hand, we have the following estimates for the first integral:

$$\begin{aligned}&\int_0^{\frac{t}{2}} b(s)^{-1}\big(1+\mathscr{B}(s,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\alpha}{2\sigma}}\|\,|u(s,\cdot)|^p\|_{L^m\cap L^2}\,ds\\
&\qquad\lesssim\|u\|^p_{X(T)}\int_0^{\frac{t}{2}} b(s)^{-1}\big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\alpha}{2\sigma}}\big(1+\mathscr{B}(0,s)\big)^{-\frac{n}{2\sigma m}p+\frac{n}{2\sigma m}}\,ds\\
&\qquad\lesssim\|u\|^p_{X(T)}\big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\alpha}{2\sigma}}\int_0^{\frac{t}{2}} b(s)^{-1}\big(1+\mathscr{B}(0,s)\big)^{-\frac{n}{2\sigma m}p+\frac{n}{2\sigma m}}\,ds\\
&\qquad\lesssim\|u\|^p_{X(T)}\big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\alpha}{2\sigma}},\end{aligned}$$

where we used inequality (39), Lemma 9 and $\|\cdot\|_{X(\tau)}\lesssim\|\cdot\|_{X(T)}$ for any $0\le\tau\le T$. Since $p>1+\frac{2m\sigma}{n}$ it follows immediately $-\frac{n}{2\sigma m}p+\frac{n}{2\sigma m}<-1$. On the other hand, for the second integral using (38) and Lemma 9 we have

$$\begin{aligned}\int_{\frac{t}{2}}^{t} b(s)^{-1}\big(1+\mathscr{B}(s,t)\big)^{-\frac{\alpha}{2\sigma}}\|\,|u(s,\cdot)|^p\|_{L^2}ds&\lesssim\|u\|^p_{X(T)}\int_{\frac{t}{2}}^{t} b(s)^{-1}\big(1+\mathscr{B}(s,t)\big)^{-\frac{\alpha}{2\sigma}}\big(1+\mathscr{B}(0,s)\big)^{-\frac{n}{2\sigma m}p+\frac{n}{4\sigma}}\,ds\\
&\lesssim\|u\|^p_{X(T)}\big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{2\sigma m}p+\frac{n}{4\sigma}}\int_{\frac{t}{2}}^{t} b(s)^{-1}\big(1+\mathscr{B}(s,t)\big)^{-\frac{\alpha}{2\sigma}}ds\\
&\lesssim\|u\|^p_{X(T)}\big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{2\sigma m}p+\frac{n}{4\sigma}}\big(1+\mathscr{B}(0,t)\big)^{1-\frac{\alpha}{2\sigma}}\\
&\lesssim\|u\|^p_{X(T)}\big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\alpha}{2\sigma}}\end{aligned}$$

for $p>1+\frac{2m\sigma}{n}$. Summarizing, we arrive at the estimate

$$\|\,|D|^\alpha u^{\rm non}(t,\cdot)\|_{L^2}\lesssim\big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\alpha}{2\sigma}}\|u\|^p_{X(T)}.$$

In the same way one can derive

$$\|u^{\rm non}(t,\cdot)\|_{L^2}\lesssim\big(1+\mathscr{B}(0,t)\big)^{\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)}\|u\|^p_{X(T)}.$$

From the definition of the norm $X(T)$ we obtain immediately inequality (36).

Next let us prove inequality (37). We have that

$$\|Nu-Nv\|_{X(T)}=\left\|\int_0^t K_1(t,s,x)\ast_{(x)}\big(|u(s,x)|^p-|v(s,x)|^p\big)ds\right\|_{X(T)}.$$

Thanks to the estimates for the solutions from Proposition 1 we can estimate

$$\begin{aligned}&\Big\|\,|D|^\alpha K_1(t,s,x)\ast_{(x)}\big(|u(s,x)|^p-|v(s,x)|^p\big)\Big\|_{L^2}\\
&\qquad\lesssim\begin{cases}b(s)^{-1}\big(1+\mathscr{B}(s,t)\big)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{\alpha}{2\sigma}}\Big\||u(s,x)|^p-|v(s,x)|^p\Big\|_{L^m\cap L^2} & \text{if } s\in\big[0,t/2\big],\\ b(s)^{-1}\big(1+\mathscr{B}(s,t)\big)^{-\frac{\alpha}{2\sigma}}\Big\||u(s,x)|^p-|v(s,x)|^p\Big\|_{L^2} & \text{if } s\in\big[t/2,t\big].\end{cases}\end{aligned}\tag{40}$$

So, by the fact that

$$\Big||u(s,x)|^p - |v(s,x)|^p\Big| \lesssim |u(s,x) - v(s,x)|\big(|u(s,x)|^{p-1} + |v(s,x)|^{p-1}\big)$$

and Hölder's inequality we obtain

$$\Big\||u(s,\cdot)|^p - |v(s,\cdot)|^p\Big\|_{L^m} \lesssim \|u(s,\cdot) - v(s,\cdot)\|_{L^{mp}}\big(\|u(s,\cdot)\|_{L^{mp}}^{p-1} + \|v(s,\cdot)\|_{L^{mp}}^{p-1}\big),$$
$$\Big\||u(s,\cdot)|^p - |v(s,\cdot)|^p\Big\|_{L^2} \lesssim \|u(s,\cdot) - v(s,\cdot)\|_{L^{2p}}\big(\|u(s,\cdot)\|_{L^{2p}}^{p-1} + \|v(s,\cdot)\|_{L^{2p}}^{p-1}\big).$$

In a similar way to the proof of (36) we use again the fractional Gagliardo-Nirenberg inequality from Proposition 2 to the following terms:

$$\|u(s,\cdot) - v(s,\cdot)\|_{L^h}, \quad \|u(s,\cdot)\|_{L^h} \quad \text{and} \quad \|v(s,\cdot)\|_{L^h}$$

with $h = mp$ and $h = 2p$. After deriving and plugging these estimates in (40), we follow the same ideas as we did in estimation for $\|\,|D|^\alpha u^{\mathrm{non}}(t,\cdot)\|_{L^2}$ to conclude the inequality (37). Hence, from the definition of $X(T)$ we may conclude the proof of the inequality (37). In this way the proof of Theorem 1 is completed. □

5 | BLOW-UP RESULT

Before giving the proof of Theorem 2, we would like to recall the following useful lemma which will be utilized in the sequel.

Lemma 10 (see[21,24]). Let us consider the initial value problem for the ordinary differential equation

$$\begin{cases} -g'(t) + b(t)g(t) = 1, & t > 0, \\ g(0) = \mathbb{B}_0, \end{cases} \tag{41}$$

where the constant $\mathbb{B}_0$ is defined in Theorem 2. Then, the solutions to (41) enjoy the following properties:

i) There exist positive constants T_0, $\mathbb{B}_1$ and $\mathbb{B}_2$ such that it holds for any $t \geq T_0$

$$\mathbb{B}_1 \leq b(t)g(t) \leq \mathbb{B}_2.$$

ii) There exists a positive constant T_0 such that it holds for any $t \geq T_0$

$$|g'(t)| \leq \frac{1 + \mathbb{B}_\infty}{1 - \mathbb{B}_\infty}.$$

Now, we are ready to present the proof of Theorem 2.

5.1 | The case that σ is an integer number

Proof. At first, we introduce the test functions $\eta = \eta(t)$ and $\varphi = \varphi(x)$ having the following properties (see, for example,[8,29]):

1. $\eta \in C_0^\infty\big([0,\infty)\big)$ and $\eta(t) = \begin{cases} 1 & \text{if } \ 0 \leq t \leq 1/2, \\ \text{decreasing} & \text{if } \ 1/2 \leq t \leq 1, \\ 0 & \text{if } \ t \geq 1, \end{cases}$

2. $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\varphi(x) = \begin{cases} 1 & \text{if } \ |x| \leq 1/2, \\ \text{decreasing} & \text{if } \ 1/2 \leq |x| \leq 1, \\ 0 & \text{if } \ |x| \geq 1, \end{cases}$

3. $$\eta^{-\frac{p'}{p}}(t)\big(|\eta'(t)|^{p'} + |\eta''(t)|^{p'}\big) \leq C \quad \text{for any } t \in [1/2,1], \tag{42}$$
$$\text{and} \quad \varphi^{-\frac{p'}{p}}(x)|\Delta^\sigma \varphi(x)|^{p'} \leq C \quad \text{for any } x \in \mathbb{R}^n \text{ such that } |x| \in [1/2,1], \tag{43}$$

where p' is the conjugate of p and C is a suitable positive constant. In addition, we suppose that $\varphi = \varphi(|x|)$ is a radial function satisfying $\varphi(|x|) \leq \varphi(|y|)$ for any $|x| \geq |y|$.

Let R be a large parameter in $[0,\infty)$. Because of the assumption (**B4**), let us choose a large parameter $T\in[0,\infty)$ fulfilling the relation

$$R^{2\sigma}=\int_0^T\frac{1}{b(s)}ds. \tag{$*$}$$

We define the following test function:

$$\phi_{T,R}(t,x)=\eta_T(t)\varphi_R(x),$$

where

$$\eta_T(t):=\eta(T^{-1}t)\quad\text{and}\quad\varphi_R(x):=\varphi(R^{-1}x).$$

Assume that $g(t)$ is the solution to (41). After multiplying the first equation in (4) by $g(t)$ and performing a direct calculation, one achieves

$$\big[g(t)u(t,x)\big]_{tt}+(-\Delta)^\sigma\big[g(t)u(t,x)\big]-\big[\big(g'(t)-1\big)u(t,x)\big]_t=g(t)|u(t,x)|^p. \tag{44}$$

Now we define the functionals

$$I_R=\int_0^\infty\int_{\mathbb{R}^n}g(t)|u(t,x)|^p\phi_{T,R}(t,x)dxdt=\int_{Q_{T,R}}g(t)|u(t,x)|^p\phi_{T,R}(t,x)d(x,t),$$

where

$$Q_{T,R}:=[0,T]\times B_R\quad\text{with}\quad B_R:=\big\{x\in\mathbb{R}^n:|x|\le R\big\}.$$

Let us assume that $u=u(t,x)$ is a global (in time) Sobolev solution from $C\big([0,\infty),L^2\big)$ to (4). Multiplying the equation (44) by $\phi_{T,R}=\phi_{T,R}(t,x)$, we carry out integration by parts to obtain

$$\begin{aligned}I_R+\int_{B_R}\big(u_0(x)+g(0)u_1(x)\big)\varphi_R(x)dx&=\int_{Q_{T,R}}g(t)u(t,x)\eta_T''(t)\varphi_R(x)d(x,t)+\int_{Q_{T,R}}g(t)u(t,x)\eta_R(t)(-\Delta)^\sigma\varphi_R(x)d(x,t)\\&\quad+\int_{Q_{T,R}}\big(g'(t)-1\big)u(t,x)\eta_T'(t)\varphi_R(x)d(x,t)\\&=:I_{1,R}+I_{2,R}+I_{3,R}.\end{aligned} \tag{45}$$

Employing Hölder's inequality with $\frac{1}{p}+\frac{1}{p'}=1$ we can proceed as follows:

$$\begin{aligned}|I_{1,R}|&\le\int_{Q_{T,R}}g(t)|u(t,x)|\,\Big|\eta_T''(t)\Big|\varphi_R(x)d(x,t)\\&\lesssim\Big(\int_{Q_{T,R}}\Big|g(t)^{\frac{1}{p}}u(t,x)\phi_{T,R}^{\frac{1}{p}}(t,x)\Big|^pd(x,t)\Big)^{\frac{1}{p}}\Big(\int_{Q_{T,R}}\Big|g(t)^{\frac{1}{p'}}\phi_{T,R}^{-\frac{1}{p}}(t,x)\eta_T''(t)\varphi_R(x)\Big|^{p'}d(x,t)\Big)^{\frac{1}{p'}}\\&\lesssim I_R^{\frac{1}{p}}\Big(\int_{Q_{T,R}}g(t)\eta_T^{-\frac{p'}{p}}(t)|\eta_T''(t)|^{p'}\varphi_R(x)d(x,t)\Big)^{\frac{1}{p'}}.\end{aligned}$$

After performing the change of variables $\tilde t:=T^{-1}t$ and $\tilde x:=R^{-1}x$, we derive

$$|I_{1,R}|\lesssim I_R^{\frac{1}{p}}T^{-2}R^{\frac{n}{p'}}\Big(\int_{T/2}^Tg(t)dt\Big)^{\frac{1}{p'}},$$

where we have utilized the relation

$$\eta_T''(t)=T^{-2}\eta''(\tilde t)$$

and the assumption (42). To estimate the above integral, the application of Lemma 10 plays a key role. The behavior $g(t)\sim b(t)^{-1}$ from the assertion i) of Lemma 10 leads to

$$|I_{1,R}|\lesssim I_R^{\frac{1}{p}}T^{-2}R^{\frac{n}{p'}}\Big(\int_{T/2}^T\frac{1}{b(t)}dt\Big)^{\frac{1}{p'}}\lesssim I_R^{\frac{1}{p}}T^{-2}R^{\frac{n+2\sigma}{p'}}. \tag{46}$$

Analogously, we may arrive at the following estimate:

$$|I_{2,R}| \lesssim I_R^{\frac{1}{p}} R^{-2\sigma+\frac{n}{p'}} \left(\int_0^T \frac{1}{b(t)} dt \right)^{\frac{1}{p'}} \lesssim I_R^{\frac{1}{p}} R^{-2\sigma+\frac{n+2\sigma}{p'}} \tag{47}$$

by noticing that it holds

$$(-\Delta)^\sigma \varphi_R(x) = R^{-2\sigma}(-\Delta)^\sigma \varphi(\tilde{x})$$

since σ is an integer number. Here the assumption (43) is also used in the previous inequality. Next, in order to handle the estimation for $I_{3,R}$, first one recognizes from Lemma 10 that the term $|g'(t)-1|$ is bounded. Combining this and the assertion i) of Lemma 10, we deduce

$$|g'(t)-1| \sim b(t)g(t).$$

Thus, it follows immediately

$$|I_{3,R}| \lesssim \int_{Q_{T,R}} b(t)g(t)|u(t,x)|\,|\eta_T'(t)|\varphi_R(x)d(x,t).$$

In an analogous procedure as we have estimated $I_{1,R}$, we may conclude that

$$|I_{3,R}| \lesssim I_R^{\frac{1}{p}} T^{-1} R^{\frac{n}{p'}} \left(\int_{T/2}^T g(t)b(t)^{p'} dt \right)^{\frac{1}{p'}} \lesssim I_R^{\frac{1}{p}} T^{-1} R^{\frac{n}{p'}} \max_{t\in[T/2,T]} b(t) \left(\int_{T/2}^T g(t)dt \right)^{\frac{1}{p'}},$$

where we have used the relation

$$\eta_T'(t) = T^{-1}\eta'(\tilde{t})$$

and the assumption (42). For any $t \in [T/2, T]$, one rewrites

$$g(t) = g(0) + \int_0^t g'(s)ds \sim g(0) + Ct \sim t,$$

thanks to the assertion ii) of Lemma 10, whenever T is chosen to be sufficiently large. This means that $b(t) \sim t^{-1} \sim T^{-1}$ for any $t \in [T/2, T]$ with a sufficiently large number T. From this observation, we obtain

$$|I_{3,R}| \lesssim I_R^{\frac{1}{p}} T^{-2} R^{\frac{n}{p'}} \left(\int_{T/2}^T g(t)dt \right)^{\frac{1}{p'}} \lesssim I_R^{\frac{1}{p}} T^{-2} R^{\frac{n+2\sigma}{p'}}. \tag{48}$$

So, combining the estimates from (45) to (48) we have shown that

$$I_R + \int_{B_R} \big(u_0(x) + g(0)u_1(x)\big)\varphi_R(x)dx \lesssim I_R^{\frac{1}{p}} T^{-2} R^{\frac{n+2\sigma}{p'}} + I_R^{\frac{1}{p}} R^{-2\sigma+\frac{n+2\sigma}{p'}}. \tag{49}$$

For any $t \in [0, T]$, recalling $(*)$ one may verify

$$\frac{g(t)^2}{R^{2\sigma}} \le \frac{g(0)^2 + 2\int_0^t g(s)g'(s)ds}{1 + \int_0^t g(s)ds} \le \frac{\mathbb{B}_0^2 + 2\|g'\|_{L^\infty} \int_0^t g(s)ds}{1 + \int_0^t g(s)ds} \le \max\left\{ \mathbb{B}_0^2, \frac{2(1+\mathbb{B}_\infty)}{1-\mathbb{B}_\infty} \right\},$$

which is equivalent to

$$g(t) \lesssim R^\sigma. \tag{50}$$

Obviously, the previous estimate leads to

$$R^{2\sigma} = 1 + \int_0^T g(t)dt \lesssim T \max_{[0,T]} g(t) \lesssim T R^\sigma, \qquad \text{that is,} \qquad R^\sigma \lesssim T.$$

As a result, both the above verification and the estimate (49) give

$$I_R + \int_{B_R} \big(u_0(x) + g(0)u_1(x)\big)\varphi_R(x)dx \lesssim I_R^{\frac{1}{p}}\, R^{-2\sigma+\frac{n+2\sigma}{p'}}. \tag{51}$$

Due to the assumption (6), there exists a sufficiently large constant $R_0 > 0$ so that it holds

$$\int_{B_R} \big(u_0(x) + \mathbb{B}_0 u_1(x)\big)\varphi_R(x)dx > 0,$$

that is,

$$\int_{B_R} \big(u_0(x) + g(0)u_1(x)\big)\varphi_R(x)dx > 0, \tag{52}$$

for any $R > R_0$. From (51) and (52), one gets

$$I_R^{1-\frac{1}{p}} \lesssim R^{-2\sigma+\frac{n+2\sigma}{p'}}. \tag{53}$$

It is clear that the assumption (7) is equivalent to $-2\sigma + \frac{n+2\sigma}{p'} \le 0$. For this reason, we shall split our consideration into two subcases as follows:

- **Case 1:** If
$$p < 1 + \frac{2\sigma}{n}, \text{ i.e. } -2\sigma + \frac{n+2\sigma}{p'} < 0,$$
then passing $R \to \infty$ in (53) we obtain
$$\int_0^\infty \int_{\mathbb{R}^n} g(t)|u(t,x)|^p dxdt = 0.$$
This implies $u \equiv 0$, which is a contradiction to the assumption (6). Therefore, there is no global (in time) Sobolev solution to (4) in the subcritical case.

- **Case 2:** If
$$p = 1 + \frac{2\sigma}{n}, \text{ i.e. } -2\sigma + \frac{n+2\sigma}{p'} = 0,$$
then from (53) there exists a positive constant C_0 such that
$$I_R = \int_{Q_{T,R}} g(t)|u(t,x)|^p \phi_R(t,x)d(x,t) \le C_0$$
for a sufficiently large R. Thus, it follows
$$\int_{\overline{Q}_{T,R}} g(t)|u(t,x)|^p \phi_R(t,x)d(x,t) \to 0 \quad \text{as } R \to \infty, \tag{54}$$
where we introduce the notation
$$\overline{Q}_{T,R} := Q_{T,R} \setminus \big([0,T/2] \times B_{R/2}\big) \quad \text{with} \quad B_{R/2} := \big\{x \in \mathbb{R}^n : |x| \le R/2\big\}.$$
Since
$$\partial_t^2 \phi_R(t,x) = \partial_t \phi_R(t,x) = (-\Delta)^\sigma \phi_R(t,x) = 0 \text{ in } \big(\mathbb{R}_+ \times \mathbb{R}^n\big) \setminus \overline{Q}_{T,R},$$
we may repeat several steps of the proofs from (45) to (51) to conclude the following estimates:
$$\begin{aligned} I_R + \int_{B_R} \big(u_0(x) + g(0)u_1(x)\big)\varphi_R(x)dx &\lesssim \Big(\int_{\overline{Q}_{T,R}} g(t)|u(t,x)|^p \phi_R(t,x)d(x,t)\Big)^{\frac{1}{p}} R^{-2\sigma+\frac{n+2\sigma}{p'}} \\ &= \Big(\int_{\overline{Q}_{T,R}} g(t)|u(t,x)|^p \phi_R(t,x)d(x,t)\Big)^{\frac{1}{p}}, \end{aligned} \tag{55}$$

because $-2\sigma+\frac{n+2\sigma}{p'}=0$. By using (54), we let $R\to\infty$ in (55) to derive

$$\int_0^\infty\int_{\mathbb{R}^n} g(t)|u(t,x)|^p dxdt+\int_{\mathbb{R}^n}\big(u_0(x)+g(0)u_1(x)\big)dx=0.$$

This is again a contradiction to the assumption (6), that is, there is no global (in time) Sobolev solution to (4) in the critical case.

Hence, our proof of Theorem 2 in the case that σ is an integer number is completed. □

5.2 | The case that σ is a fractional number

At first, let us recall some auxiliary knowledge of the modified test function $\psi=\psi(x):=\langle x\rangle^{-r}$ for any $r>0$.

Lemma 11 (Lemma 3 in[9]). Let $\sigma\ge 1$ be a fractional number. We denote $s:=\sigma-[\sigma]$. Then, the following estimates hold for all $x\in\mathbb{R}^n$:

$$|(-\Delta)^\sigma\psi(x)|\lesssim\begin{cases}\langle x\rangle^{-r-2\sigma} & \text{if } 0<r+2[\sigma]<n,\\ \langle x\rangle^{-n-2s}\log(e+|x|) & \text{if } r+2[\sigma]=n,\\ \langle x\rangle^{-n-2s} & \text{if } r+2[\sigma]>n.\end{cases}$$

Lemma 12 (Lemma 3.3 in[10]). Let $\sigma\ge 1$ be a fractional number. For any $R>0$, let ψ_R be a function defined by

$$\psi_R(x)=\psi\big(R^{-1}x\big)\quad \text{for all } x\in\mathbb{R}^n.$$

Then, $(-\Delta)^\sigma(\psi_R)$ satisfies the following scaling properties for all $x\in\mathbb{R}^n$:

$$(-\Delta)^\sigma(\psi_R)(x)=R^{-2\sigma}\big((-\Delta)^\sigma\psi\big)\big(R^{-1}x\big).$$

Lemma 13 (Lemma 7 in[9]). Let $s\in\mathbb{R}$. We assume that $\mu_1=\mu_1(x)\in H^s$ and $\mu_2=\mu_2(x)\in H^{-s}$. Then, the following relation holds:

$$\int_{\mathbb{R}^n}\mu_1(x)\,\mu_2(x)dx=\int_{\mathbb{R}^n}\widehat{\mu}_1(\xi)\,\widehat{\mu}_2(\xi)d\xi.$$

Proof. At first, we denote the constant $\overline{\sigma}:=\sigma-[\sigma]$. Since σ is a fractional number, it follows that $\overline{\sigma}\in(0,1)$. We introduce, on the one hand, the function $\eta=\eta(t)$ satisfying the same properties as in Section 5.1. On the other hand, we define the function $\varphi=\varphi(|x|)=\langle x\rangle^{-n-2\overline{\sigma}}$.

Let R and T be two large parameters in $[0,\infty)$ enjoying the relation $(*)$. We introduce the following test function:

$$\phi_{T,R}(t,x)=\eta_T(t)\varphi_R(x),$$

where $\eta_T(t):=\eta(T^{-1}t)$ and $\varphi_R(x):=\varphi(R^{-1}x)$. We define the functionals

$$I_R=\int_0^\infty\int_{\mathbb{R}^n} g(t)|u(t,x)|^p\phi_R(t,x)\,dxdt=\int_0^T\int_{\mathbb{R}^n} g(t)|u(t,x)|^p\phi_R(t,x)\,dxdt.$$

Let us assume that $u=u(t,x)$ is a global (in time) Sobolev solution from $C\big([0,\infty),L^2\big)$ to (4). After multiplying the equation (44) by $\phi_{T,R}=\phi_{T,R}(t,x)$, we perform partial integration to obtain

$$\begin{aligned} I_R+\int_{\mathbb{R}^n}\big(u_0(x)+g(0)u_1(x)\big)\varphi_R(x)dx &=\int_{T/2}^T\int_{\mathbb{R}^n} g(t)u(t,x)\eta_T''(t)\varphi_R(x)\,dxdt+\int_0^\infty\int_{\mathbb{R}^n} g(t)\eta_T(t)\varphi_R(x)\,(-\Delta)^\sigma u(t,x)\,dxdt\\ &\quad+\int_{T/2}^T\int_{\mathbb{R}^n}\big(g'(t)-1\big)\eta_T'(t)\varphi_R(x)\,u(t,x)\,dxdt\\ &=:I_{1,R}+I_{2,R}+I_{3,R}. \end{aligned}\tag{56}$$

Applying Hölder's inequality with $\frac{1}{p}+\frac{1}{p'}=1$ we may deal with $I_{1,R}$ as follows:

$$\begin{aligned}|I_{1,R}| &\leq \int_{T/2}^{T}\int_{\mathbb{R}^n} g(t)|u(t,x)|\left|\eta_T''(t)\right|\varphi_R(x)\,dxdt\\ &\lesssim \Big(\int_{T/2}^{T}\int_{\mathbb{R}^n} g(t)\Big|u(t,x)\phi_{T,R}^{\frac{1}{p}}(t,x)\Big|^p\,dxdt\Big)^{\frac{1}{p}}\Big(\int_{T/2}^{T}\int_{\mathbb{R}^n} g(t)\Big|\phi_{T,R}^{-\frac{1}{p}}(t,x)\eta_T''(t)\varphi_R(x)\Big|^{p'}\,dxdt\Big)^{\frac{1}{p'}}\\ &\lesssim I_R^{\frac{1}{p}}\Big(\int_{T/2}^{T}\int_{\mathbb{R}^n} g(t)\eta_T^{-\frac{p'}{p}}(t)\left|\eta_T''(t)\right|^{p'}\varphi_R(x)\,dxdt\Big)^{\frac{1}{p'}}.\end{aligned}$$

By using the change of variables $\tilde{t}:=T^{-1}t$ and $\tilde{x}:=R^{-1}x$, we compute directly to give

$$|I_{1,R}| \lesssim I_R^{\frac{1}{p}}\,T^{-2}\,R^{\frac{n}{p'}}\Big(\int_{T/2}^{T} g(t)dt\Big)^{\frac{1}{p'}}\Big(\int_{\mathbb{R}^n}\langle\tilde{x}\rangle^{-n-2\overline{\sigma}}\,d\tilde{x}\Big)^{\frac{1}{p'}}.$$

Here we used $\eta_T''(t)=T^{-2}\eta''(\tilde{t})$ and the assumption (42). After repeating the same argument as we did in Section 5.1, one finds

$$|I_{1,R}| \lesssim I_R^{\frac{1}{p}}\,T^{-2}\,R^{\frac{n+2\sigma}{p'}}\Big(\int_{\mathbb{R}^n}\langle\tilde{x}\rangle^{-n-2\overline{\sigma}}\,d\tilde{x}\Big)^{\frac{1}{p'}}. \tag{57}$$

In an analogous way, we may conclude the following estimate for $I_{3,R}$:

$$|I_{3,R}| \lesssim I_R^{\frac{1}{p}}\,T^{-2}\,R^{\frac{n+2\sigma}{p'}}\Big(\int_{\mathbb{R}^n}\langle\tilde{x}\rangle^{-n-2\overline{\sigma}}\,d\tilde{x}\Big)^{\frac{1}{p'}}. \tag{58}$$

Now let us focus our attention on estimating $I_{2,R}$. In the first step, since $\varphi_R\in H^{2\sigma}$ and $u\in\mathcal{C}\big([0,\infty),L^2\big)$, we apply Lemma 13 to derive the following identities:

$$\int_{\mathbb{R}^n}\varphi_R(x)\,(-\Delta)^\sigma u(t,x)\,dx=\int_{\mathbb{R}^n}|\xi|^{2\sigma}\widehat{\varphi}_R(\xi)\,\widehat{u}(t,\xi)\,d\xi=\int_{\mathbb{R}^n}u(t,x)\,(-\Delta)^\sigma\varphi_R(x)\,dx.$$

Therefore, we get

$$I_{2,R}=\int_0^\infty\int_{\mathbb{R}^n}g(t)\eta_T(t)\varphi_R(x)\,(-\Delta)^\sigma u(t,x)\,dxdt=\int_0^\infty\int_{\mathbb{R}^n}g(t)\eta_T(t)u(t,x)\,(-\Delta)^\sigma\varphi_R(x)\,dxdt.$$

The application of Hölder's inequality again as we estimated $I_{1,R}$ gives

$$|I_{2,R}|\leq I_R^{\frac{1}{p}}\Big(\int_0^T\int_{\mathbb{R}^n}g(t)\eta_T(t)\varphi_R^{-\frac{p'}{p}}(x)\,|(-\Delta)^\sigma\varphi_R(x)|^{p'}\,dxdt\Big)^{\frac{1}{p'}}.$$

In the second step, to control the above integral we shall apply results from Lemmas 11 and 12 as the key tools. In particular, carrying out the change of variables $\tilde{x}:=R^{-1}x$ we get the following relation from Lemma 12:

$$(-\Delta)^\sigma\varphi_R(x)=R^{-2\sigma}(-\Delta)^\sigma(\varphi)(\tilde{x}).$$

After using the change of variables $\tilde{t}:=T^{-1}t$ we achieve

$$|I_{2,R}|\lesssim I_R^{\frac{1}{p}}\,R^{-2\sigma+\frac{n+2\sigma}{p'}}\Big(\int_{\mathbb{R}^n}\varphi^{-\frac{p'}{p}}(\tilde{x})\,|(-\Delta)^\sigma(\varphi)(\tilde{x})|^{p'}\,d\tilde{x}\Big)^{\frac{1}{p'}}.$$

Next, the employment of Lemma 11 leads to the following estimate:

$$|I_{2,R}|\lesssim I_R^{\frac{1}{p}}\,R^{-2\sigma+\frac{n+2\sigma}{p'}}\Big(\int_{\mathbb{R}^n}\langle\tilde{x}\rangle^{-n-2\overline{\sigma}}\,d\tilde{x}\Big)^{\frac{1}{p'}}. \tag{59}$$

Let us now link the derived estimates from (56) to (59) and then repeat some steps in the proof of Section 5.1 to establish

$$I_R \lesssim I_R^{\frac{1}{p}} R^{-2\sigma+\frac{n+2\sigma}{p'}} \left(\int\limits_{\mathbb{R}^n} \langle \tilde{x} \rangle^{-n-2\overline{\sigma}} \, d\tilde{x} \right)^{\frac{1}{p'}} \lesssim I_R^{\frac{1}{p}} R^{-2\sigma+\frac{n+2\sigma}{p'}}.$$

This means

$$I_R^{1-\frac{1}{p}} \lesssim R^{-2\sigma+\frac{n+2\sigma}{p'}}. \tag{60}$$

We can see that the assumption (7) is equivalent to $-2\sigma + \frac{n+2\sigma}{p'} < 0$. For this reason, passing $R \to \infty$ in (60) we get

$$I_R = \int\limits_0^\infty \int\limits_{\mathbb{R}^n} g(t)|u(t,x)|^p \, dx dt = 0,$$

which follows $u \equiv 0$. This is a contradiction to the assumption (6), that is, there is no global (in time) Sobolev solution to (4) in the subcritical case. Summarizing, the proof of Theorem 2 in the case that σ is a fractional number is completed. □

6 | LIFESPAN ESTIMATES

6.1 | Upper bound of lifespan

To show Theorem 3, the following lemma is useful in our proof.

Lemma 14 (see[30] at page 202)**.** The following formula of derivative of composed function holds for any multi-index γ:

$$\partial_\xi^\gamma h\big(f(\xi)\big) = \sum_{k=1}^{|\gamma|} h^{(k)}\big(f(\xi)\big) \left(\sum_{\substack{\gamma_1+\cdots+\gamma_k \le \gamma \\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\, |\gamma_i|\ge 1}} \big(\partial_\xi^{\gamma_1} f(\xi)\big) \cdots \big(\partial_\xi^{\gamma_k} f(\xi)\big) \right),$$

where $h = h(z)$ and $h^{(k)}(z) = \dfrac{d^k h(z)}{d\, z^k}$.

Proof of Theorem 3. We assume that $u = u(t,x)$ is a local solution to (4) in $[0, T_\varepsilon) \times \mathbb{R}^n$ with $T_\varepsilon = \text{LifeSpan}(u)$. The proof of Theorem 3 can be divided into two cases as follows:

- **Case 1:** If

  $$p < 1 + \frac{2\sigma}{n} \quad \text{(subcritical case)},$$

  then we repeat some steps in the proof of Theorem 2 to obtain the following estimate:

  $$I_R + c\varepsilon \le C\, I_R^{\frac{1}{p}} R^{-2\sigma+\frac{n+2\sigma}{p'}}, \quad \text{i.e.} \quad c\varepsilon \le C\, I_R^{\frac{1}{p}} R^{-2\sigma+\frac{n+2\sigma}{p'}} - I_R, \tag{61}$$

  where c is a suitable constant being subject to

  $$\int\limits_{B_R} \big(u_0(x) + g(0)u_1(x)\big)\varphi_R(x)dx > c > 0$$

  for any $R > R_0$. Here $R_0 > 0$ stands for a sufficiently large constant. After applying the elementary inequality

  $$A\, y^\beta - y \le A^{\frac{1}{1-\beta}} \quad \text{for any } A > 0,\ y \ge 0 \text{ and } 0 < \beta < 1,$$

  to (61), one may directly calculate to arrive at

  $$\varepsilon \le C\, R^{-(2\sigma p'-n-2\sigma)} = C\, \mathscr{B}(0,T)^{-\frac{2\sigma p'-n-2\sigma}{2\sigma}},$$

  provided that $R^{2\sigma} = \mathscr{B}(0,T)$ holds, which is reasonable from $(*)$. Therefore, letting $T \to T_\varepsilon^-$ in the previous inequality we may conclude

  $$\mathscr{B}(0,T_\varepsilon) \le C\varepsilon^{-\frac{2\sigma(p-1)}{2\sigma-n(p-1)}},$$

  which is the first desired estimate in (8).

- **Case 2:** If

$$p = 1 + \frac{2\sigma}{n} \quad \text{(critical case)},$$

then we introduce another test function $\eta^* = \eta^*(s)$ given by

$$\eta^*(s) := \begin{cases} 0 & \text{if } s \in [0, 1/2), \\ \eta(s) & \text{if } s \in [1/2, \infty), \end{cases}$$

where the test function $\eta = \eta(s)$ is defined as in the proof of Theorem 2. Moreover, for a large parameter $R \in (0, \infty)$, we denote $\psi_R = \psi_R(t,x)$ and $\psi_R^* = \psi_R^*(t,x)$ as follows:

$$\psi_R(t,x) := \left(\eta\left(\frac{\mathcal{B}(0,t) + |x|^{2\sigma}}{R}\right)\right)^{n+2\sigma} \quad \text{and} \quad \psi_R^*(t,x) := \left(\eta^*\left(\frac{\mathcal{B}(0,t) + |x|^{2\sigma}}{R}\right)\right)^{n+2\sigma}.$$

Obviously, one recognizes that

$$\begin{aligned} &\operatorname{supp} \psi_R \subset \left\{(t,x) \ : \ (t,|x|) \in [0,T] \times \left[0, R^{\frac{1}{2\sigma}}\right]\right\}, \\ &\operatorname{supp} \psi_R^* \subset \left([0,T] \times \left[0, R^{\frac{1}{2\sigma}}\right]\right) \setminus \left\{(t,x) \ : \ \mathcal{B}(0,t) + |x|^{2\sigma} \leqslant \frac{R}{2}\right\}, \end{aligned}$$

where T is chosen to be a large number enjoying the relation

$$R = \int_0^T \frac{1}{b(s)} ds. \tag{$**$}$$

By multiplying the equation (44) by $\psi_R = \psi_R(t,x)$, we perform integration by parts to derive

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} g(t)|u(t,x)|^p \psi_R(t,x)\, dxdt + \varepsilon \int_{\mathbb{R}^n} \big(u_0(x) + g(0)u_1(x)\big)\psi_R(0,x)dx \\ &= \int_0^T \int_{\mathbb{R}^n} g(t)u(t,x)\partial_t^2 \psi_R(t,x)dxdt + \int_0^T \int_{\mathbb{R}^n} g(t)u(t,x)(-\Delta)^\sigma \psi_R(t,x)dxdt \\ &\quad + \int_0^T \int_{\mathbb{R}^n} \big(g'(t) - 1\big)u(t,x)\partial_t \psi_R(t,x)dxdt \\ &=: I_{1,R} + I_{2,R} + I_{3,R}. \end{aligned} \tag{62}$$

Because of the assumption (6), there exists a sufficiently large constant $R_0 > 0$ such that for any $R > R_0$ we have

$$\int_{\mathbb{R}^n} \big(u_0(x) + g(0)u_1(x)\big)\varphi_R(x)dx > c > 0 \quad \text{for a suitable constant } c. \tag{63}$$

At the first stage, we will verify the following auxiliary estimates:

$$\left|\partial_t \psi_R(t,x)\right| \lesssim R^{-1} g(t)\big(\psi_R^*(t,x)\big)^{\frac{n+2\sigma-1}{n+2\sigma}}, \tag{64}$$

$$\left|\partial_t^2 \psi_R(t,x)\right| \lesssim R^{-1}\big(\psi_R^*(t,x)\big)^{\frac{n+2\sigma-2}{n+2\sigma}}, \tag{65}$$

$$\left|(-\Delta)^\sigma \eta_R(t,x)\right| \lesssim R^{-1}\big(\psi_R^*(t,x)\big)^{\frac{n}{n+2\sigma}}. \tag{66}$$

Namely, by straightforward calculations one finds

$$\partial_t \psi_R(t,x) = \frac{(n+2\sigma)}{Rb(t)}\left(\eta\left(\frac{\mathcal{B}(0,t) + |x|^{2\sigma}}{R}\right)\right)^{n+2\sigma-1} \eta'\left(\frac{\mathcal{B}(0,t) + |x|^{2\sigma}}{R}\right)$$

and

$$\begin{aligned}\partial_t^2\psi_R(t,x) &= \frac{(n+2\sigma)b'(t)}{Rb(t)^2}\left(\eta\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\right)^{n+2\sigma-1}\eta'\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\\ &\quad+\frac{(n+2\sigma)(n+2\sigma-1)}{R^2b(t)^2}\left(\eta\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\right)^{n+2\sigma-2}\left(\eta'\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\right)^2\\ &\quad+\frac{(n+2\sigma)}{R^2b(t)^2}\left(\eta\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\right)^{n+2\sigma-1}\eta''\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right).\end{aligned}$$

Since the properties

$$\eta'\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\not\equiv 0 \quad\text{and}\quad \eta''\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\not\equiv 0 \text{ for } \frac{R}{2}<\mathscr{B}(0,t)+|x|^{2\sigma}<R,$$

are valid, we may claim the estimates (64) and (65) by using the assertion i) of Lemma 10, the assumption (**B-L**) and noticing the relations $0<\psi_R^*(t,x)<1$ as well as $g(t)\lesssim\sqrt{R}$. Here we want to underline that the latter relation can be verified in the same procedure as we did in (50) by the help of $(**)$. What's more, to control $(-\Delta)^\sigma\psi_R(t,x)$, we shall apply Lemma 14 as a key tool. Particularly, we shall separate our computations into three steps as follows:

Step 1: The application of Lemma 14 with $h(z)=\dfrac{\mathscr{B}(0,t)+z^\sigma}{R}$ and $z=f(x)=|x|^2$ gives the following estimate for $|\gamma|\geq 1$:

$$\begin{aligned}\left|\partial_x^\gamma\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\right| &\leq \sum_{k=1}^{|\gamma|}\frac{|x|^{2\sigma-2k}}{R}\left(\sum_{\substack{|\gamma_1|+\cdots+|\gamma_k|=|\gamma|\\|\gamma_i|\geq 1}}\left|\partial_x^{\gamma_1}(|x|^2)\right|\cdots\left|\partial_x^{\gamma_k}(|x|^2)\right|\right)\\ &\leq \sum_{k=1}^{|\gamma|}\frac{|x|^{2\sigma-2k}}{R}\left(\sum_{\substack{|\gamma_1|+\cdots+|\gamma_k|=|\gamma|\\1\leq|\gamma_i|\leq 2}}\left|\partial_x^{\gamma_1}(|x|^2)\right|\cdots\left|\partial_x^{\gamma_k}(|x|^2)\right|\right)\\ &\lesssim \sum_{k=1}^{|\gamma|}\frac{|x|^{2\sigma-2k}}{R}\left(\sum_{\substack{|\gamma_1|+\cdots+|\gamma_k|=|\gamma|\\1\leq|\gamma_i|\leq 2}}|x|^{2-|\gamma_1|}\cdots|x|^{2-|\gamma_k|}\right)\\ &\lesssim \sum_{k=1}^{|\gamma|}\frac{|x|^{2\sigma-2k}}{R}|x|^{2k-|\gamma|}\lesssim\frac{|x|^{2\sigma-|\gamma|}}{R}.\end{aligned}$$

Step 2: Applying Lemma 14 with $h(z)=\eta(z)$ and $z=f(x)=\dfrac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}$ one gets for all $|\gamma|\geq 1$ the estimate

$$\begin{aligned}&\left|\partial_x^\gamma\eta\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\right|\\ &\leq\sum_{k=1}^{|\gamma|}\left|\eta^{(k)}\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\right|\left(\sum_{\substack{|\gamma_1|+\cdots+|\gamma_k|=|\gamma|\\1\leq|\gamma_i|\leq 2\sigma}}\left|\partial_x^{\gamma_1}\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\right|\cdots\left|\partial_x^{\gamma_k}\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\right|\right)\\ &\leq\sum_{k=1}^{|\gamma|}\left|\eta^{(k)}\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\right|\left(\sum_{\substack{|\gamma_1|+\cdots+|\gamma_k|=|\gamma|\\1\leq|\gamma_i|\leq 2\sigma}}\frac{|x|^{2\sigma-|\gamma_1|}}{R}\cdots\frac{|x|^{2\sigma-|\gamma_k|}}{R}\right)\\ &\lesssim\sum_{k=1}^{|\gamma|}\left(\frac{|x|^{2\sigma}}{R}\right)^k|x|^{-|\gamma|}\lesssim\frac{|x|^{2\sigma-|\gamma|}}{R}\qquad\left(\text{since } |x|^{2\sigma}\leq R\text{ in }\operatorname{supp}\psi_R^*\right).\end{aligned}$$

Step 3: After applying Lemma 14 with $h(z) = z^{n+2\sigma}$ and $z = f(x) = \eta\Big(\dfrac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\Big)$ we conclude

$$\begin{aligned}
\Big|(-\Delta)^\sigma \eta_R(t,x)\Big| &\lesssim \sum_{|\gamma|=2\sigma} \Big|\partial_x^\gamma \eta_R(t,x)\Big| \\
&\lesssim \sum_{k=1}^{2\sigma} \left(\eta\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\right)^{n+2\sigma-k} \left(\sum_{\substack{|\gamma_1|+\cdots+|\gamma_k|=2\sigma\\ |\gamma_i|\geq 1}} \left|\partial_x^{\gamma_1}\eta\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\right| \cdots \left|\partial_x^{\gamma_k}\eta\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\right| \right) \\
&\lesssim \sum_{k=1}^{2\sigma} \left(\eta^*\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\right)^{n+2\sigma-k} \sum_{\substack{|\gamma_1|+\cdots+|\gamma_k|=2\sigma\\ |\gamma_i|\geq 1}} \frac{|x|^{2\sigma-|\gamma_1|}}{R}\cdots\frac{|x|^{2\sigma-|\gamma_k|}}{R} \\
&\lesssim \sum_{k=1}^{2\sigma} \left(\eta^*\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\right)^{n+2\sigma-k} \frac{|x|^{2(k-1)\sigma}}{R^k} \\
&\lesssim R^{-1}\left(\eta^*\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\right)^{n} \qquad \Big(\text{since } \ |x|^{2\sigma}\leq R \text{ in } \operatorname{supp}\psi_R^*\Big),
\end{aligned}$$

which yields immediately the estimate (66).

Let us now turn to estimate $I_{1,R}$. By (65), the employment of Hölder's inequality leads to

$$\begin{aligned}
|I_{1,R}| &\leq R^{-1}\int_0^T\int_{\mathbb{R}^n} g(t)|u(t,x)|\big(\psi_R^*(t,x)\big)^{\frac{n+2\sigma-2}{n+2\sigma}}\,dxdt \\
&\leqslant R^{-1}\Big(\int_{\operatorname{supp}\psi_R^*} g(t)d(x,t)\Big)^{\frac{1}{p'}}\Big(\int_0^T\int_{\mathbb{R}^n} g(t)|u(t,x)|^p\big(\psi_R^*(t,x)\big)^{\frac{(n+2\sigma-2)p}{n+2\sigma}}\,dxdt\Big)^{\frac{1}{p}} \\
&\lesssim R^{(\frac{n}{2\sigma}+1)\frac{1}{p'}-1}\Big(\int_0^T\int_{\mathbb{R}^n} g(t)|u(t,x)|^p\big(\psi_R^*(t,x)\big)^{\frac{(n+2\sigma-2)p}{n+2\sigma}}\,dxdt\Big)^{\frac{1}{p}},
\end{aligned} \tag{67}$$

where we have utilized the estimate

$$\int_{\operatorname{supp}\psi_R^*} g(t)d(x,t) \lesssim \int_0^T\int_0^{R^{\frac{1}{2\sigma}}} |x|^{n-1}g(t)d|x|dt \lesssim R^{\frac{n}{2\sigma}}\mathscr{B}(T) \lesssim R^{\frac{n}{2\sigma}+1}$$

thanks to the relation (∗∗). Repeating the same procedure as the above together with the help of (66) and (64) one has

$$|I_{2,R}| \lesssim R^{(\frac{n}{2\sigma}+1)\frac{1}{p'}-1}\Big(\int_0^T\int_{\mathbb{R}^n} g(t)|u(t,x)|^p\big(\psi_R^*(t,x)\big)^{\frac{np}{n+2\sigma}}dxdt\Big)^{\frac{1}{p}} \tag{68}$$

and

$$|I_{3,R}| \lesssim R^{(\frac{n}{2\sigma}+1)\frac{1}{p'}-1}\Big(\int_0^T\int_{\mathbb{R}^n} g(t)|u(t,x)|^p\big(\psi_R^*(t,x)\big)^{\frac{(n+2\sigma-1)p}{n+2\sigma}}\,dxdt\Big)^{\frac{1}{p}}, \tag{69}$$

respectively. For this reason, in (62) we link the derived estimates (63) and (67)-(69) to establish

$$\begin{aligned}
c\varepsilon + \int_0^T\int_{\mathbb{R}^n} g(t)|u(t,x)|^p\psi_R(t,x)\,dxdt &\lesssim R^{(\frac{n}{2\sigma}+1)\frac{1}{p'}-1}\Big(\int_0^T\int_{\mathbb{R}^n} g(t)|u(t,x)|^p\big(\psi_R^*(t,x)\big)^{\frac{np}{n+2\sigma}}dxdt\Big)^{\frac{1}{p}} \\
&= \Big(\int_0^T\int_{\mathbb{R}^n} g(t)|u(t,x)|^p\psi_R^*(t,x)dxdt\Big)^{\frac{1}{p}}
\end{aligned} \tag{70}$$

by observing from the assumption $p = 1 + \frac{2\sigma}{n+2\sigma}$ that

$$\left(\frac{n}{2\sigma} + 1\right)\frac{1}{p'} - 1 = 0 \quad \text{and} \quad \frac{np}{n+2\sigma} = 1.$$

In the next stage, let us introduce the following auxiliary functions:

$$y(r) := \int_0^T \int_{\mathbb{R}^n} g(t)|u(t,x)|^p \psi_r^*(t,x)dxdt \quad \text{with } r \in (0,\infty)$$

and

$$Y(R) := \int_0^R y(r)r^{-1}dr.$$

By using the change of variable $s = \dfrac{\mathscr{B}(0,t) + |x|^{2\sigma}}{r}$, we achieve

$$\begin{aligned}
Y(R) &= \int_0^R \left(\int_0^T \int_{\mathbb{R}^n} g(t)|u(t,x)|^p \left(\eta^*\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{r}\right)\right)^{n+2\sigma} dxdt\right) r^{-1}dr \\
&= \int_0^T \int_{\mathbb{R}^n} g(t)|u(t,x)|^p \int_{\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}}^{\infty} \big(\eta^*(s)\big)^{n+2\sigma} s^{-1}dsdxdt \\
&\leqslant \int_0^T \int_{\mathbb{R}^n} g(t)|u(t,x)|^p \left(\int_{1/2}^1 \big(\eta^*(s)\big)^{n+2\sigma} s^{-1}ds\right)dxdt \qquad \big(\text{since } \operatorname{supp}\eta^* \subset [1/2,1]\big) \\
&= \int_0^T \int_{\mathbb{R}^n} g(t)|u(t,x)|^p \left(\int_{1/2}^1 \big(\eta(s)\big)^{n+2\sigma} s^{-1}ds\right)dxdt \qquad \big(\text{since } \eta^* \equiv \eta \text{ in } [1/2,1]\big) \\
&\le \int_0^T \int_{\mathbb{R}^n} g(t)|u(t,x)|^p \sup_{r\in(0,R)} \left(\eta\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{r}\right)\right)^{n+2\sigma} \left(\int_{1/2}^1 s^{-1}\,ds\right)dxdt \\
&\le \log 2 \int_0^T \int_{\mathbb{R}^n} g(t)|u(t,x)|^p \left(\eta\left(\frac{\mathscr{B}(0,t)+|x|^{2\sigma}}{R}\right)\right)^{n+2\sigma} dxdt \qquad \big(\text{since } \eta \text{ is decreasing}\big) \\
&= \log 2 \int_0^T \int_{\mathbb{R}^n} g(t)|u(t,x)|^p \psi_R(t,x)\,dxdt.
\end{aligned} \tag{71}$$

In addition, it holds

$$Y'(R)R = y(R) = \int_0^T \int_{\mathbb{R}^n} g(t)|u(t,x)|^p \psi_R^*(t,x)dxdt. \tag{72}$$

After combining from (70) to (72), one has demonstrated the following estimate:

$$c\varepsilon + \frac{Y(R)}{\log 2} \le c\varepsilon + \int_0^T \int_{\mathbb{R}^n} g(t)|u(t,x)|^p \psi_R(t,x)\,dxdt \lesssim \big(Y'(R)R\big)^{\frac{1}{p}},$$

which is equivalent to

$$R^{-1}\left(c\varepsilon + \frac{Y(R)}{\log 2}\right)^p \le C\,Y'(R).$$

Thus, it follows that

$$R^{-1}\big(Y(R)\big)^p \le C(\log 2)^p Y'(R), \quad \text{i.e.} \quad R^{-1} \le C(\log 2)^p \frac{Y'(R)}{\big(Y(R)\big)^p} \tag{73}$$

and
$$c^p\varepsilon^p R^{-1} \le C\, Y'(R). \tag{74}$$
By putting $r := R$ and taking account of $R \geqslant R_0^2$, we integrate two sides of (73) over $[\sqrt{R}, R]$ to get
$$\frac{1}{2}\log R \le -\frac{C(\log 2)^p}{p-1}\big(Y(r)\big)^{-(p-1)}\Big|_{r=\sqrt{R}}^{r=R} \le \frac{C(\log 2)^p}{p-1}\big(Y(\sqrt{R})\big)^{-(p-1)}. \tag{75}$$
Again, denoting $r := R$ and taking integration of (74) over $[R_0, \sqrt{R}]$ one gains
$$c^p\varepsilon^p\Big(\log\sqrt{R} - \log R_0\Big) \le C\Big(Y(\sqrt{R}) - Y(R_0)\Big),$$
which results
$$\frac{1}{4}c^p\varepsilon^p \log R \le C\, Y(\sqrt{R}).$$
Finally, we plug the previous inequality in (75) to catch the estimate
$$\frac{1}{2}\log R \le \frac{C(\log 2)^p}{p-1}\left(\frac{1}{4C}c^p\varepsilon^p\log R\right)^{-(p-1)},$$
that is,
$$\log R \le \frac{C\log 2}{c^{p-1}}\left(\frac{2^{2p-1}}{p-1}\right)^{\frac{1}{p}}\varepsilon^{-(p-1)}.$$
For this reason, letting $R \to \mathscr{B}(0, T_\varepsilon)$, i.e. $T \to T_\varepsilon^-$, in the last estimate we may conclude that
$$\mathscr{B}(0,T_\varepsilon) \le \exp\left(\frac{C\log 2}{c^{p-1}}\left(\frac{2^{2p-1}}{p-1}\right)^{\frac{1}{p}}\varepsilon^{-(p-1)}\right)$$
what we wanted to prove.

Summarizing, the proof of Theorem 3 is complete. □

6.2 | Lower bound of lifespan

Proof of Theorem 4. First of all, let us define the evolution space of solutions $X(T)$ together with its corresponding norm as in the proof of Theorem 1 with $m = 1$. In the followings, we are going to use again some notations which are introduced before from Section 4. The main crux of our approach to indicate the desired lower bound estimates for the lifespan relies on a pair of inequalities as follows:
$$\|u\|_{X(T)} \leqslant \varepsilon C_0 + \begin{cases} C_1 \log\big(\mathrm{e} + \mathscr{B}(0,t)\big)\,\|u\|_{X(T)}^p & \text{if } p = 1 + \frac{2\sigma}{n}, \\ C_1\big(1+\mathscr{B}(0,t)\big)^{1-\frac{n}{2\sigma}(p-1)}\,\|u\|_{X(T)}^p & \text{if } p < 1 + \frac{2\sigma}{n}, \end{cases} \tag{76}$$
for all $t \in [0, T]$, where $C_0 = C_0(n, u_0, u_1)$ and C_1 is a positive constant independent of T. Clearly, it entails immediately the following estimate from the definition of the norm of $X(T)$ and Theorem 3:
$$\|u^{\mathrm{lin}}\|_{X(T)} \leqslant \varepsilon C_0(n, u_0, u_1).$$
For this reason, we need to establish
$$\|u^{\mathrm{non}}\|_{X(T)} \leqslant \begin{cases} C_1 \log\big(\mathrm{e} + \mathscr{B}(0,t)\big)\,\|u\|_{X(T)}^p & \text{if } p = 1 + \frac{2\sigma}{n}, \\ C_1\big(1+\mathscr{B}(0,t)\big)^{1-\frac{n}{2\sigma}(p-1)}\,\|u\|_{X(T)}^p & \text{if } p < 1 + \frac{2\sigma}{n}, \end{cases} \tag{77}$$
instead of (76). Indeed, repeating the same procedure and analogous arguments as we did in the proof of Theorem 1 one arrives at
$$\begin{aligned} \|\,|D|^{k\alpha}u^{\mathrm{non}}(t,\cdot)\|_{L^2} &\lesssim \big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{4\sigma}-\frac{k\alpha}{2\sigma}}\|u\|_{X(T)}^p \int_0^{\frac{t}{2}} b(s)^{-1}\big(1+\mathscr{B}(0,s)\big)^{-\frac{n}{2\sigma}(p-1)}\,ds \\ &\quad + \big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{2\sigma}p+\frac{n}{4\sigma}}\|u\|_{X(T)}^p \int_{\frac{t}{2}}^{t} b(s)^{-1}\big(1+\mathscr{B}(s,t)\big)^{-\frac{k\alpha}{2\sigma}}\,ds \\ &=: I_1 + I_2 \end{aligned}$$

with $k=0,1$. Concerning the second integral we use the change of variables $r=\mathscr{B}(s,t)$, i.e. $r=\mathscr{B}(0,t)-\mathscr{B}(0,s)$, to achieve

$$\int_{\frac{t}{2}}^{t} b(s)^{-1}\big(1+\mathscr{B}(s,t)\big)^{-\frac{k\alpha}{2\sigma}}ds=\int_{0}^{\mathscr{B}(t/2,t)}(1+r)^{-\frac{k\alpha}{2\sigma}}dr\lesssim\big(1+\mathscr{B}(t/2,t)\big)^{1-\frac{k\alpha}{2\sigma}}\lesssim\big(1+\mathscr{B}(0,t)\big)^{1-\frac{k\alpha}{2\sigma}}$$

since $\frac{k\alpha}{2\sigma}<1$, which implies

$$I_2\lesssim\big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{4\sigma}-\frac{k\alpha}{2\sigma}+1-\frac{n}{2\sigma}(p-1)}\|u\|^p_{X(T)}. \tag{78}$$

To control I_1, let us now separate our considerations into two cases as follows:

- **Case 1:** If
$$p=1+\frac{2\sigma}{n},\quad \text{i.e.}\quad \frac{n}{2\sigma}(p-1)=1,$$
then
$$\int_0^{\frac{t}{2}} b(s)^{-1}\big(1+\mathscr{B}(0,s)\big)^{-\frac{n}{2\sigma}(p-1)}ds\lesssim\log\big(\mathrm{e}+\mathscr{B}(0,t/2)\big)\lesssim\log\big(\mathrm{e}+\mathscr{B}(0,t)\big).$$
This yields that
$$I_1\lesssim\big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{4\sigma}-\frac{k\alpha}{2\sigma}}\log\big(\mathrm{e}+\mathscr{B}(0,t)\big)\|u\|^p_{X(T)}. \tag{79}$$
- **Case 2:** If
$$p<1+\frac{2\sigma}{n},\quad \text{i.e.}\quad \frac{n}{2\sigma}(p-1)<1,$$
then
$$\int_0^{\frac{t}{2}} b(s)^{-1}\big(1+\mathscr{B}(0,s)\big)^{-\frac{n}{2\sigma}(p-1)}ds\lesssim\big(1+\mathscr{B}(0,t/2)\big)^{1-\frac{n}{2\sigma}(p-1)}\lesssim\big(1+\mathscr{B}(0,t)\big)^{1-\frac{n}{2\sigma}(p-1)}.$$
This follows that
$$I_1\lesssim\big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{4\sigma}-\frac{k\alpha}{2\sigma}+1-\frac{n}{2\sigma}(p-1)}\|u\|^p_{X(T)}. \tag{80}$$

Summarizing, we link the obtained estimates from (78) to (80) to conclude

$$\|\,|D|^{k\alpha}u^{\mathrm{non}}(t,\cdot)\|_{L^2}\lesssim\begin{cases}\big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{4\sigma}-\frac{k\alpha}{2\sigma}}\log\big(\mathrm{e}+\mathscr{B}(0,t)\big)\|u\|^p_{X(T)} & \text{if } p=1+\frac{2\sigma}{n},\\ \big(1+\mathscr{B}(0,t)\big)^{-\frac{n}{4\sigma}-\frac{k\alpha}{2\sigma}+1-\frac{n}{2\sigma}(p-1)}\|u\|^p_{X(T)} & \text{if } p<1+\frac{2\sigma}{n},\end{cases}$$

for $k=0,1$. This completes the proof of (77).

What's more, motivated by the approach in [19] let us define

$$T^*:=\sup\big\{T\in(0,T_\varepsilon)\ \text{ such that }\ F(T):=\|u\|_{X(T)}\leqslant M\varepsilon\big\}$$

with $T_\varepsilon=\mathrm{LifeSpan}(u)$, where $M>0$ is a sufficiently large constant which will be determined later. From this definition and (76), it is clear to see that

$$F(T^*)=\|u\|_{X(T^*)}\le\begin{cases}\Big(C_0M^{-1}+C_1\log\big(\mathrm{e}+\mathscr{B}(0,T^*)\big)(M\varepsilon)^{p-1}\Big)M\varepsilon & \text{if } p=1+\frac{2\sigma}{n},\\ \Big(C_0M^{-1}+C_1\big(1+\mathscr{B}(0,T^*)\big)^{1-\frac{n}{2\sigma}(p-1)}(M\varepsilon)^{p-1}\Big)M\varepsilon & \text{if } p<1+\frac{2\sigma}{n}.\end{cases}$$

Now we take a large constant $M>0$ such that $C_0M^{-1}<\frac{1}{4}$. Assume that the following estimates hold:

$$\frac{1}{4}>\begin{cases}C_1\log\big(\mathrm{e}+\mathscr{B}(0,T^*)\big)(M\varepsilon)^{p-1} & \text{if } p=1+\frac{2\sigma}{n},\\ C_1\big(1+\mathscr{B}(0,T^*)\big)^{1-\frac{n}{2\sigma}(p-1)}(M\varepsilon)^{p-1} & \text{if } p<1+\frac{2\sigma}{n}.\end{cases} \tag{81}$$

Then, it follows immediately that

$$F(T^*)\le\frac{1}{2}M\varepsilon\le M\varepsilon.$$

Thanks to the continuity of the function $F(T)$, we claim that there exists $\overline{T} \in (T^*, T_\varepsilon)$ enjoying the relation $F(\overline{T}) \leq M\varepsilon$, which is a contradiction to the definition of T^*. This means that the assumption (81) is not true, i.e.

$$\frac{1}{4} \leq \begin{cases} C_1 \log\big(\mathrm{e} + \mathscr{B}(0,T^*)\big)(M\varepsilon)^{p-1} & \text{if } p = 1 + \frac{2\sigma}{n}, \\ C_1\big(1 + \mathscr{B}(0,T^*)\big)^{1-\frac{n}{2\sigma}(p-1)}(M\varepsilon)^{p-1} & \text{if } p < 1 + \frac{2\sigma}{n}. \end{cases}$$

More precisely, this observation gives

$$\mathscr{B}(0,T^*) \geq \begin{cases} \exp\big(c\varepsilon^{-(p-1)}\big) & \text{if } p = 1 + \frac{2\sigma}{n}, \\ c\varepsilon^{-\frac{2\sigma(p-1)}{2\sigma - n(p-1)}} & \text{if } p < 1 + \frac{2\sigma}{n}. \end{cases}$$

Therefore, the desired estimates for lifespan from the below (9) are established by the last inequality. □

7 | CONCLUDING REMARKS

Remark 5. It can be also expected to study the global (in time) existence of small data Sobolev solutions from suitable function spaces to weakly coupled systems of semi-linear σ-evolution models with time-dependent coefficients and different power non-linearities. For this reason, it is interesting to consider the following Cauchy problem for weakly coupled systems of semi-linear σ-evolution equations with time-dependent damping:

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + b_1(t)u_t = |\partial_t^j v|^p, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ v_{tt} + (-\Delta)^\sigma v + b_2(t)v_t = |\partial_t^k u|^q, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ (u,u_t,v,v_t)(0,x) = (u_0,u_1,v_0,v_1)(x), & x \in \mathbb{R}^n, \end{cases} \tag{82}$$

with any fractional number $\sigma \geq 1$, $j,k = 0,1$ and $p,q > 1$. Here the damping terms $b_1(t)u_t$ and $b_2(t)v_t$ are assumed to be effective (see Definition 1). Using the derived estimates to the single equation from Proposition 1 and Corollary 3 for the corresponding linear Cauchy problems of (82), we may prove global (in time) existence of small data Sobolev solutions to the weakly coupled systems of semi-linear σ-evolution model (82) with suitable function spaces. On the other hand, by using the modified test function method not only the blow-up of Sobolev solution, but also lifespans estimate can be studied to (82) when $\sigma \geq 1$ is any fractional number.

Remark 6. Motivated by the quite recent paper[11], where the authors investigated the equation

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + \dfrac{\mu}{1+t}u_t = |u|^p, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{83}$$

with $\mu > 0$, $\sigma > 1$ and $p > 1$, we can see that another interesting problem is to study the Cauchy problem (82) for weakly coupled system of semi-linear σ-evolution equations with with different scale-invariant time-dependent dissipation terms. Namely, the following damping terms $b_1(t)u_t$ and $b_2(t)v_t$ are of interest:

$$b_1(t) = \frac{\mu_1}{1+t} \quad \text{with} \quad \mu_1 > 0 \qquad \text{and} \qquad b_2(t) = \frac{\mu_2}{1+t} \quad \text{with} \quad \mu_2 > 0.$$

Depending on the size of the parameter μ_1 and μ_2, we expect to achieve blow-up results and lifespan estimates for solutions by an application of the same approaches used in this paper. Moreover, the critical curve of p,q for the global (in time) existence of small data solutions could be reasonable to conclude by the help of optimal $L^r - L^s$ decay estimates, with $1 \leq r \leq 2 \leq s \leq \infty$, for solutions to the corresponding linear Cauchy problem of (83).

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Conflict of interest

The authors declare no potential conflict of interests.

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