**A study on timelike sweeping surfaces and singularities in Minkowski 3–Space $\mathbb{E}_1^3$**

Rashad A. Abdel-Baky
Department of Mathematics, Faculty of Science,
University of Assiut, Assiut 71516,
Egypt rbaky@live.com

Fatemah Mofarreh
Mathematical Science Department Faculty of Science Princess Nourah bint Abdulrahman University. Riyadh 11546 Saudi Arabia.
fyalmofarrah@pnu.edu.sa

Nadia Alluhaibi
Science and Arts College, Rabigh Campus, King Abdulaziz University, Jeddah, Saudi Arabia
nallehaibi@kau.edu.sa

**Abstract**. The main aim of this paper is the use of the rotation minimizing frame for the singularity type classification of timelike sweeping surface in Minkowski 3–Space $\mathbb{E}_1^3$. Then, we give conditions for a timelike sweeping surface to be developable ruled surfaces. To apply and illustrate the main results, some examples are given.
**Key Words**. Rotation minimizing frame, Local singularities and convexity.
**AMS Subject Classification** (2010): Primary: 53C50. Secondary: 53A04, 53C40.

# 1 Introduction

The singularity theory of curves and surfaces is an active area of research in different branches of mathematics and physics. In the view of differential geometry, curves and surfaces are represented by functions with one variable and two variables, respectively. In recent years, singularity theory for the curves and surfaces became important tools for various interesting fields such as medical imaging and computer vision. (see e.g. [1-4]).

As we know, sweeping surface is the surface generated by the motion of a plane curve (the profile curve or generatrix) whilst the plane is moved through space in such away that the movement of the plane is always in the direction of the normal to the plane. Sweeping is a very important, powerful, and widespread method in geometric modelling. The basic idea is to choose some geometrical object (generator), which is then swept along a spine curve (trajectory) in the space. The result of such evolution, consisting of motion through space and intrinsic shape deformation, is a sweep object. The sweep object type is defined by the choice of the generator and the trajectory. Then, sweeping a curve along the other curve generates a sweeping surface. There are several different names of sweeping surface that we are familiar with, such as tubular surface, pipe surface, string, and canal surface [5-9].

One of the most appropriate methods to study curves and surfaces from the view differential geometry, Serret–Frenet frame, but not unique, there is another frame fields as the rotation minimizing frame (RMF) or Bishop frame [10]. Some applications of the Bishop frame can be found in [11-14]. Corresponding to Bishop frame in Euclidean space, there is a Minkowski version's frame which is named a Minkowski Bishop frame as applied to Minkowski geometry. By the investigation of the space curve, it is easier to use the Minkowski Bishop frame among the curve as an essential tool more than the Serret–Frenet frame type frame in Lorentzian space. Several papers focus on Minkowski Bishop frame, for example [15-17].

In this paper, we present the notion of timelike sweeping surfaces with rotation minimizing frames in Minkowski 3–Space $\mathbb{E}^3_1$. By applying singularity theory we classify the generic properties, and present new invariant connected to the singularity of this timelike sweeping surface. It is leads to the main generic singularity of this sweeping surface are the well known cuspidal edge and swallowtail, and the kind of them are characterized by this new invariant, respectively. Finally, to illustrate the principle results some examples are given and investigated in details.

## 2 Preliminaries

At this section, some notations on Minkowski 3-space are introduced. More concepts and properties are in [18, 19].

Suppose $\mathbb{R}^3 = \{(a_1, a_2, a_3) \mid, a_i \in \mathbb{R} \text{ (i=1, 2, 3)}\}$ be a 3-dimensional Cartesian space. For all $\mathbf{a} = (a_1, a_2, a_3)$, and $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$, the pseudo scalar product of $\mathbf{a}$, and $\mathbf{b}$ is denoted by

$$< \mathbf{a}, \mathbf{b} > = -a_1b_1 + a_2b_2 + a_3b_3. \tag{2.1}$$

$(\mathbb{R}^3, <, >)$ is called Minkowski 3-space. In fact, we use $\mathbb{E}^3_1$ rather than $(\mathbb{R}^3, <, >)$. We say that the non-zero vector $\mathbf{a} \in \mathbb{E}^3_1$ is spacelike, lightlike or timelike in case $< \mathbf{a}, \mathbf{a} > > 0$, $< \mathbf{a}, \mathbf{a} > = 0$ or $< \mathbf{a}, \mathbf{a} > < 0$ in the same order. The norm of the vector $\mathbf{a} \in \mathbb{E}^3_1$ is denoted to be $\|\mathbf{a}\| = \sqrt{|< \mathbf{a}, \mathbf{a} >|}$. For any two vectors $\mathbf{a}$, $\mathbf{c} \in \mathbb{E}^3_1$, we determine the vector $\mathbf{a} \times \mathbf{c}$ as

$$\mathbf{a} \times \mathbf{c} = \begin{vmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \left(-(a_2c_3 - a_3c_2), (a_3c_1 - a_1c_3), (a_1c_2 - a_2c_1)\right), \tag{2.2}$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is the canonical basis of $\mathbb{E}^3_1$. We can easily check that

$$\det(\mathbf{a}, \mathbf{c}, \mathbf{b}) = < \mathbf{a} \times \mathbf{c}, \mathbf{b} >, \tag{2.3}$$

so that $\mathbf{a} \times \mathbf{c}$ is pseudo orthogonal to any $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{E}_1^3$. The Lorentzian unit sphere with center in the origin of $\mathbb{E}_1^3$ is given as

$$\mathbb{S}_1^2 = \{\mathbf{x} \in \mathbb{E}_1^3 \mid < \mathbf{x}, \mathbf{x} > = 1\}. \tag{2.4}$$

Let $\beta = \beta(s)$ is the unit speed timelike curve; by $\kappa(s)$ and $\tau(s)$ we define the natural curvature and torsion of $\beta(s)$, in the same order. Let $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ be the Serret–Frenet frame related to $\beta(s)$. For all points on $\beta(s)$, the corresponding Serret-Frenet frame is:

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} = \omega \times \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{2.5}$$

where $\omega(s) = \tau\mathbf{T} + \kappa\mathbf{B}$ is Darboux vector of the Serret–Frenet formula. At this paper, dash indicats to the derivation with respect to the arc-length parameter $s$. It seems clear that

$$\mathbf{T} \times \mathbf{N} = \mathbf{B},\ \mathbf{T} \times \mathbf{B} = -\mathbf{N},\ \mathbf{N} \times \mathbf{B} = -\mathbf{T}. \tag{2.6}$$

**Definition 2.1**. A pseudo orthogonal moving frame $\{\xi_1, \xi_3, \xi_3\}$, along a non null space curve $\alpha(s)$, is rotation minimizing frame (RMF) respecting to $\xi_1$ in case the derivatives of $\xi_2$ and $\xi_3$ are both parallel to $\xi_1$, or its angular velocity $\omega$ satisfies $< \omega, \xi_1 >= 0$. Similarly, characterization stays hold in case $\xi_2$ or $\xi_3$ is selected as a reference direction.

According to the Definition 2.1, it is observed that the Serret–Frenet frame is RMF respecting to the principal normal $\mathbf{N}$, but not respecting to the tangent $\mathbf{T}$ and the binormal $\mathbf{B}$. Despite the fact that Serret–Frenet frame is not RMF respecting to $\mathbf{T}$, it is not difficult to derive such a RMF from it. New normal plane vectors $(\mathbf{N}_1,\mathbf{N}_2)$ are particular among the rotation of $(\mathbf{N},\mathbf{B})$ as following

$$\begin{pmatrix} \mathbf{T}_1 \\ \mathbf{N}_1 \\ \mathbf{N}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{2.7}$$

with a certain spacelike angle $\vartheta(s) \geq 0$. Here, we will call the set $\{\mathbf{T}_1, \mathbf{N}_1, \mathbf{N}_2\}$ as RMF or Bishop frame. The RMF vector satisfy the relations

$$\mathbf{T}_1 \times \mathbf{N}_1 = \mathbf{N}_2,\ \mathbf{T}_1 \times\ \mathbf{N}_2 = -\mathbf{N}_1,\ \mathbf{N}_1 \times \mathbf{N}_2 = -\mathbf{T}_1. \tag{2.8}$$

As a result, the alternative frame equations are

$$\begin{pmatrix} \mathbf{T}_1' \\ \mathbf{N}_1' \\ \mathbf{N}_2' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(s) & -\kappa_2(s) \\ \kappa_1(s) & 0 & 0 \\ -\kappa_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}_1 \\ \mathbf{N}_1 \\ \mathbf{N}_2 \end{pmatrix} = \widetilde{\omega} \times \begin{pmatrix} \mathbf{T}_1 \\ \mathbf{N}_1 \\ \mathbf{N}_2 \end{pmatrix}, \tag{2.9}$$

where $\widetilde{\omega}(s) = \kappa_2\mathbf{N}_1 + \kappa_1\mathbf{N}_2$ is RMF Darboux vector. Here, the Bishop curvatures are defined by $\kappa_1(s) = \kappa\cos\vartheta$, and $\kappa_2(s) = \kappa\sin\vartheta$. One can show that

$$\left.\begin{array}{l} \kappa(s) = \sqrt{\kappa_1^2+\kappa_2^2}, \text{ and } \vartheta = \tanh^{-1}\left(\frac{\kappa_2}{\kappa_1}\right);\ \kappa_1 \neq 0, \\ \vartheta(s) = -\int\limits_{s_0}^{s}\tau ds + \vartheta_0,\ \vartheta_0 = \vartheta(0). \end{array}\right\} \tag{2.10}$$

By the comparison of Eq. (2.4) and Eq. (2.8), it is observed that the relative velocity is

$$\widetilde{\omega}(s) - \omega(s) = \tau\mathbf{T}. \tag{2.11}$$

Consequently, the Serre-Frenet frame and the RMF are identical iff $\beta(s)$ is a planar, i.e. $\tau(s) = 0$. Now we define the spacelike Bishop spherical Darboux image $\mathbf{e} : I \to \mathbb{S}_1^2$, by

$$\mathbf{e}(s) = \frac{\widetilde{\omega}(s)}{\|\widetilde{\omega}(s)\|} = \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}}\left(\frac{\kappa_2}{\kappa_1}\mathbf{N}_1 + \mathbf{N}_2\right). \tag{2.12}$$

Therefore, we consider a new geometric invariant $\rho(s) = \kappa_1\kappa_2^{'} - \kappa_2\kappa_1^{'}$.

**Definition 2.2**. A sweeping surface along $\beta(s)$ is a surface defined by

$$M : \mathbf{X}(s,u) = \beta(s) + T(s)\mathbf{x}(u) = \alpha(s) + x_1(u)\mathbf{N}_1(s) + x_2(u)\mathbf{N}_2(s), \tag{2.13}$$

where $\beta(s)$ is called the (at least $C^1$-continuos). $\mathbf{x}(u)$ is planar profile (cross-section) curve defined as he parametric representation $\mathbf{x}(u) = (0, x_1(u),\ x_2(u))^t$, 't' as a symbol indicates to the transposition, in addition to another parameter $u \in I \subseteq \mathbb{R}$. The semi orthogonal matrix $T(s) = \{\mathbf{T}_1,\ \mathbf{N}_1,\ \mathbf{N}_2\}$ specifies the RMF along $\beta(s)$.

# 3 Timelike sweeping surface

In this section, we present timelike sweeping surface in Minkowski 3-space $\mathbb{E}_1^3$ . Consider the planar profile curve given by $\mathbf{x}(u) = (0, \cos u, \sin u)$. By using Eq. (2.11), it follows that

$$M : \mathbf{X}(s,u) = \beta(s) + \cos u\mathbf{N}_1 + \sin u\mathbf{N}_2. \tag{3.1}$$

By the formulae expressed in Eq. (2.8), we can calculate

$$\left.\begin{array}{l} \mathbf{X}_s(s,u) = (1+\kappa_1\cos u - \kappa_2\sin u)\,\mathbf{T}_1, \\ \mathbf{X}_u(s,u) = \cos u\mathbf{N}_1 + \sin u\mathbf{N}_2, \end{array}\right\} \tag{3.2}$$

where $\mathbf{X}_i = \frac{\partial\mathbf{X}}{\partial i}$. The unit normal vector of $M$ is

$$\mathbf{U}(s,u) := \frac{\mathbf{X}_s\times\mathbf{X}_u}{\|\mathbf{X}_s\times\mathbf{X}_u\|} = -\sin u\mathbf{N}_1 + \cos u\mathbf{N}_2. \tag{3.3}$$

Note that $\|\mathbf{N}(s,u)\|^2 = 1$ means that $M$ is a timelike surface. Our aim of this work is the following theorem:

**Theorem 3**.1. Suppose $\beta$: $I \to \mathbb{E}_1^3$ is the unit speed timelike curve with $\kappa_1 > 0$. Then, for any fixed $\mathbf{x} \in \mathbb{S}_1^2$, one has the following:
A- (1) $\mathbf{e}(s)$ is locally diffeomorphic to the line $\{\mathbf{0}\}\times\mathbb{R}$ at $s_0$ iff $\rho(s_0) \neq 0$;
(2) $\mathbf{e}(s)$ is locally diffeomorphic to the cusp $C \times \mathbb{R}$ at $s_0$ iff $\rho(s_0) = 0$, and $\rho^{'}(s_0) \neq 0$.
B- (1).$M$ is locally diffeomorphic to Cuspidal edge $CE$ at $(s_0, u_0)$ iff $\mathbf{x} = \pm\mathbf{e}(s_0)$, and $\rho(s_0) \neq 0$.
(2) $M$ is locally diffeomorphic to Swallowtail SW at $(s_0, u_0)$ iff $\mathbf{x} = \pm\mathbf{e}(s_0)$, $\rho(s_0) = 0$, and $\rho^{'}(s_0) = 0$.
The proof will appear later. Her,

$$
\begin{aligned}
C \times \mathbb{R} &= \left\{(x_1, x_2)|x_1^2 = x_2^3\right\} \times \mathbb{R}, \\
CE &= \left\{(x_1, x_2, x_3)|\mathrm{x}_1\text{=u, } \mathrm{x}_2\text{=}\mathrm{v}^2\text{, } \mathrm{x}_3\text{=}\mathrm{v}^3\right\}, \\
W &= \left\{(x_1, x_2, x_3)|x_1 = u,\ x_2 = 3v^2 + uv^2,\ x_3 = 4v^3 + 2uv\right\}.
\end{aligned}
$$

The pictures of $C \times \mathbb{R}$, $CE$, and $SW$ will be seen in Figs 1, 2, 3.

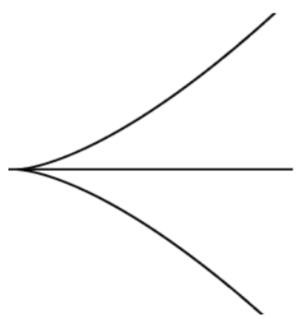

Figure 1: C×ℝ

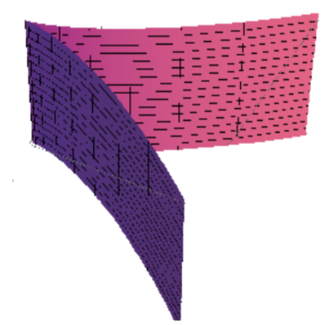

Figure 2: CE.

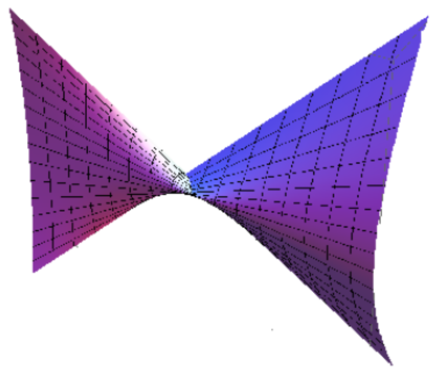

Figure 3: SW

## 3.1 Lorentzian height functions

Next, let us present two different families of Lorentzian height functions that will be used to study the singularties of $M$ as follows [1-3]: $H : I \times \mathbb{S}_1^2 \to \mathbb{R}$, by $H(s, \mathbf{x}) =< \beta(\mathbf{s}), \mathbf{x} >$. We call it Lorentzian height function. We use the notation $h_{\mathbf{x}}(s) = H(s, \mathbf{x})$ for any fixed $\mathbf{x} \in \mathbb{S}_1^2$. We also define $\widetilde{H} : I \times \mathbb{S}_1^2 \times \mathbb{R} \to \mathbb{R}$, by $\widetilde{H}(s, \mathbf{x}, w) =< \beta, \mathbf{x} > -w$ that is called the extended Lorentzian height function of $\beta(\mathbf{s})$. We denote that $\widetilde{h}_{\mathbf{x}}(s) = \widetilde{H}(s, \mathbf{x})$. From now on, we shall often not write the parameter $s$. Then, we give the next proposition:

**Proposition 3.1**. Suppose $\beta$: $I \to \mathbb{E}_1^3$ is the unit speed timelike curve with $\kappa_1 \neq 0$. Then the following holds:
(A).
(1)- $h_{\mathbf{x}}^{'}(s) = 0$ iff $\mathbf{x} = a_1\mathbf{N}_1 + a_2\mathbf{N}_2$, and $a_1^2 + a_2^2 = 1$.
(2)- $h_{\mathbf{x}}^{'}(s) = h_{x}^{''}(s) = 0$ iff $\mathbf{x} = \pm\mathbf{e}(s)$;
(3)- $h_{\mathbf{x}}^{'}(s) = h_{x}^{''}(s) = h_{\mathbf{x}}^{'''}(s) = 0$ iff $\mathbf{x} = \pm\mathbf{e}(s)$, and $\rho(s) = 0$.
(4)- $h_{\mathbf{x}}^{'}(s) = h_{x}^{''}(s) = h_{\mathbf{x}}^{'''}(s) = h_{\mathbf{x}}^{(4)}(s) = 0$ iff $\mathbf{x} = \pm\mathbf{e}(s)$, and $\rho(s) = \rho^{'}(s) = 0$.
(5)- $h_{\mathbf{x}}^{'}(s) = h_{x}^{''}(s) = h_{\mathbf{x}}^{'''}(s) = h_{\mathbf{x}}^{(4)}(s) = h_{\mathbf{x}}^{(5)}(s) = 0$ iff $\mathbf{x} = \pm\mathbf{e}(s)$, and $\rho(s) = \rho^{'}(s) = \rho^{''}(s) = 0$.
(B).
(1)- $\widetilde{h}_{\mathbf{x}}(s) = 0$ iff there exist $< \beta, \mathbf{x} >= w$;
(2)- $\widetilde{h}_{\mathbf{x}}(s) = \widetilde{h}_{x}^{'}(s) = 0$ iff there is $a_1$, $a_2 \in \mathbb{R}$ that is $\mathbf{x} = \cos u\mathbf{N}_1 + \sin u\mathbf{N}_2$, and $< \beta, \mathbf{x} >= w$.
(3)- $\widetilde{h}_{\mathbf{x}}(s) = \widetilde{h}_{\mathbf{x}}^{'}(s) = \widetilde{h}_{x}^{''}(s) = \widetilde{h}_{x}^{''}(s) = 0$ iff $\mathbf{x} = \pm\mathbf{e}(s)$, $< \beta, \mathbf{x} >= w$, and $\rho(s) = 0$;
(4)- $\widetilde{h}_{\mathbf{x}}(s) = \widetilde{h}_{\mathbf{x}}^{'}(s) = \widetilde{h}_{x}^{''}(s) = \widetilde{h}_{x}^{''}(s) = \widetilde{h}_{x}^{'''}(s) = 0$ iff $\mathbf{x} = \pm\mathbf{e}(s)$, $< \beta, \mathbf{x} >= w$, and $\rho(s) = \rho^{'}(s) = 0$.

(5)- $\widetilde{h}_{\mathbf{x}}(s)=\widetilde{h}'_{\mathbf{x}}(s)=\widetilde{h}''_{x}(s)=\widetilde{h}''_{x}(s)=\widetilde{h}'''_{x}(s)=\widetilde{h}^{(4)}_{x}(s)=0$ iff $\mathbf{x}=\pm\mathbf{e}(s)$, $<\beta,\mathbf{x}>=w$ and $\rho(s)=\rho'(s)=\rho''(s)=0$. **Proof**. (A). (1) Since $h'_{\mathbf{x}}(s)=<\mathbf{T}_1,\mathbf{x}>$, and $\{\mathbf{T}_1,\ \mathbf{N}_1,\ \mathbf{N}_2\}$ is RMF along $\beta(s)$, then there exists $a_1,\ a_2\in\mathbb{R}$ such that $\mathbf{x}=a_1\mathbf{N}_1+a_2\mathbf{N}_2$. Moreover, in combination with $\mathbf{x}\in\mathbb{S}^2_1$, we get $a_1^2+a_2^2=1$, it follows that $h'_{\mathbf{x}}(s)=0$ iff $\mathbf{x}=a_1\mathbf{N}_1+a_2\mathbf{N}_2$, and $a_1^2+a_2^2=1$.

(2)- When $h'_{\mathbf{x}}(s)=0$, the assertion (2) follows from the fact that $h''_{\mathbf{x}}(s)=<\mathbf{T}'_1,\mathbf{x}>=<\kappa_1\mathbf{N}_1-\kappa_2\mathbf{N}_2,\mathbf{x}>=0$. Thus, we have that $a_1\kappa_1-a_2\kappa_2=0$. It follows from the fact $a_1^2+a_2^2=1$ that $a_1=\pm\kappa_2/\sqrt{\kappa_1^2+\kappa_2^2}$, and $a_2=\pm\kappa_1/\sqrt{\kappa_1^2+\kappa_2^2}$. Thereby, we have that

$$\mathbf{x}=\left(\mp\frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}}\left(\frac{\kappa_2}{\kappa_1}\mathbf{N}_1+\mathbf{N}_2\right)\right)(s)=\pm\mathbf{e}(s). \qquad (*)$$

Thus, we get that $h'_{\mathbf{x}}(s)=h''_{x}(s)=0$ iff $\mathbf{x}=\pm\mathbf{e}(s)$.

(3)- Under the condition that $h'_{\mathbf{x}}(s)=h''_{x}(s)=0$, $h'''_{\mathbf{x}}(s)=<\mathbf{T}''_1,\mathbf{x}>=<\left(\kappa_1^2+\kappa_2^2\right)\mathbf{T}_1+\kappa'_1\mathbf{N}_1-\kappa'_2\mathbf{N}_2,\mathbf{x}>=0$, and by Eq. (*), we have that

$$\pm\frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}}\left(\frac{\kappa_2\kappa'_1-\kappa_1\kappa'_2}{\kappa_1}\right)(s)=\pm\frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}}\left(\frac{\rho}{\kappa_1}\right)(s)=0.$$

Since $\kappa_1\neq0$, we get that $h'''_{\mathbf{x}}(s)=0$ iff $\mathbf{x}=\pm\mathbf{e}(s)$, and $\rho(s)=0$.

(4)- Since

$$\left.\begin{array}{l} h^{(4)}_{\mathbf{x}}(s)=<\mathbf{T}'''_1,\mathbf{x}>=<3(\kappa_1\kappa'_1+\kappa_2\kappa'_2)\mathbf{T}_1+\left(\kappa''_1+\kappa_1\left(\kappa_1^2+\kappa_2^2\right)\right)\mathbf{N}_1 \\ -\left(\kappa''_2+\kappa_2\left(\kappa_1^2+\kappa_2^2\right)\right)\mathbf{N}_2,\mathbf{x}>=0. \end{array}\right\}$$

Thus, making use of Eq. (*) in the above, we have that

$$\pm\frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}}\left(\frac{\left(\kappa_2\kappa'_1-\kappa_1\kappa'_2\right)'}{\kappa_1}\right)(s)=0.$$

This is equivalent to the condition $\rho(s)=\rho'(s)=0$.

(5)- Since $h^{(5)}_{\mathbf{x}}(s)=<\mathbf{T}^{(4)}_1,\mathbf{x}>0$, we have:

$$\left.\begin{array}{l} <\left(\left(\kappa_1^2+\kappa_2^2\right)^2+4\left(\kappa_2\kappa''_2+\kappa_1\kappa''_1\right)+3\left(\kappa'^2_1+\kappa'^2_2\right)\right)\mathbf{T}_1+ \\ \left(\kappa'''_1+5\kappa_1\left(\kappa'_1\kappa_1+\kappa'_2\kappa_2\right)+\kappa'_1\left(\kappa_1^2+\kappa_2^2\right)\right)\mathbf{N}_1- \\ \left(\kappa'''_2+5\kappa_2\left(\kappa'_2\kappa_2+\kappa'_1\kappa_1\right)+\kappa'_2\left(\kappa_1^2+\kappa_2^2\right)\right)\mathbf{N}_2,\mathbf{x}>=0. \end{array}\right\}$$

Similarly, by Eq. (*) in the above, we have that:

$$\pm\frac{1}{\sqrt{\kappa_1^2+\kappa_2^2}}\left(\frac{\kappa_2\kappa'''_1-\kappa_1\kappa'''_2+\left(\kappa_2\kappa'_1-\kappa_1\kappa'_2\right)\left(\kappa_1^2+\kappa_2^2\right)}{\kappa_1}\right)=0.$$

This is equivalent to the condition $\rho(s) = \rho^{'}(s) = \rho^{''}(s) = 0$. (B). Similar to the proof of (A), we have (B) $\blacksquare$.

**Proposition 3.2**. Suppose $\beta$: $I \to \mathbb{E}_1^3$ is the unit speed timelike curve with $\kappa_1 \neq 0$. Then , we have $\rho(s) = 0$ iff

$$\mathbf{e}(s) = \frac{\kappa_1}{\sqrt{\kappa_1^2 + \kappa_2^2}} \left( \frac{\kappa_2}{\kappa_1} \mathbf{N}_1 + \mathbf{N}_2 \right)$$

is a constant vector.
**Proof**. Let $\kappa_1 \neq 0$, using simple calculations, we have

$$\mathbf{e}^{'}(s) = \frac{\rho(s)}{\left(\sqrt{\kappa_1^2 + \kappa_2^2}\right)^3} \left( \kappa_1 \mathbf{N}_1 + \kappa_2 \mathbf{N}_2 \right).$$

Thus $\mathbf{e}^{'}(s) = \mathbf{0}$ iff $\rho(s) = \kappa_2 \kappa_1^{'} - \kappa_1 \kappa_2^{'} = 0 \blacksquare$.

**Proposition 3.3.** Let $\beta$: $I \to \mathbb{E}_1^3$ is the unit speed timelike curve with $\kappa_1 \neq 0$. Then we state the the following.

(a) $\beta$ is a slant helix iff $\kappa_2/\kappa_1$ is constant.

(b) $\mathbf{N}_2$ is a part of a circle on $\mathbb{S}_1^2$ whose center is the spacelike constant vector $\mathbf{e}_0$. **Proof**. (a) Suppose that $\rho(s) = \kappa_2 \kappa_1^{'} - \kappa_1 \kappa_2^{'} = 0$. Hence, we can write

$$\left( \frac{\kappa_2}{\kappa_1} \right)^{'} = \frac{\kappa_1 \kappa_2^{'} - \kappa_2 \kappa_1^{'}}{\kappa_1^2} = \frac{-\rho(s)}{\kappa_1^2} = 0.$$

This means that $\frac{\kappa_2}{\kappa_1}$=constant, that is, $\beta$ is a slant helix.

(b) Suppose that $\kappa_1 \neq 0$. Since

$$< \mathbf{e}, \mathbf{N}_2 >= \frac{\kappa_1}{\sqrt{\kappa_1^2 + \kappa_2^2}} < \left( \frac{\kappa_2}{\kappa_1} \mathbf{N}_1 + \mathbf{N}_2 \right), \mathbf{N}_2 >= \frac{1}{\sqrt{1 + \kappa_2^2/\kappa_1^2}} = const.$$

This means that $\mathbf{N}_2$ is a part of a circle on $\mathbb{S}_1^2$ whose center is the constant spacelike vector $\mathbf{e}_0(s)$ $\blacksquare$.

## 3.2 Unfolding of functions by one-variable

Now, some general results will be used on the singularity theory for families of function germs [1-3]. suppose $F$: $(\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ is the smooth function, and $f(s) = F_{x_0}(s, \mathbf{x}_0)$. Then $F$ is called an r-parameter unfolding of $f(s)$. We say that $f(s)$ has $\mathrm{A}_k$-singulaity at $s_0$ if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(s_0) \neq 0$. In addition, it is said that f has $\mathrm{A}_{\geqslant k}$-singulaity $(k \geqslant 1)$ at $\mathrm{s}_0$. Let the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at $s_0$ be $j^{(k-1)} \left( \frac{\partial F}{\partial x_i}(s, \mathbf{x}_0) \right)(s_0) = \Sigma_{j=0}^{k-1} L_{ji} (s - s_0)^j$ (without the constant term), for $i = 1, ..., r$. Therefore, $F(s)$ is named an $p$-versal unfolding in case the $k \times r$ matrix of coefficients $(L_{ji})$ of the rank $k$ ($k \leq r$). So, we write important set about the unfolding relative to the above notations. The discriminant set of $F$ is the set

$$\mathfrak{D}_F = \left\{ \mathbf{x}{\in}\mathbb{R}^r | \text{ there is } s \text{ with } F\left(s, \mathbf{x}\right) = \frac{\partial F}{\partial s}\left(s, \mathbf{x}\right) = 0 \text{ at } \left(s, \mathbf{x}\right) \right\}. \qquad (3.4)$$

The bifurcation set of $F$ is the set

$$\mathfrak{B}_F = \left\{ \mathbf{x}{\in}\mathbb{R}^r | \text{ there is } s \text{ with } \frac{\partial F}{\partial s}\left(s, \mathbf{x}\right) = \frac{\partial^2 F}{\partial s^2}\left(s, \mathbf{x}\right) = 0 \text{ at } \left(s, \mathbf{x}\right) \right\}. \qquad (3.5)$$

Then similar to [1-3], we state the following theorem:

**Theorem 3.2**. suppose $F$: $(\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ is an r-parameter unfolding of $f(s)$, that has the $A_k$ singularity at $s_0$.

Let $F$ is a $p$-versal unfolding.

(a) If $k = 1$, so $\mathfrak{D}_F$ is locally diffeomorphic to $\{\mathbf{0}\}{\times}\mathbb{R}^{r-1}$, and $\mathfrak{B}_F = \varnothing$;

(b) If $k = 2$, so $\mathfrak{D}_F$ is locally diffeomorphic to $\mathbf{C}{\times}\mathbb{R}^{r-2}$, and $\mathfrak{B}_F$ is locally diffeomorphic to $\{\mathbf{0}\}{\times}\mathbb{R}^{r-1}$;

(c) If $k = 3$, so $\mathfrak{D}_F$ is locally diffeomorphic to $\mathbf{SW}{\times}\mathbb{R}^{r-3}$, and $\mathfrak{B}_F$ is locally diffeomorphic to $\mathbf{C}{\times}\mathbb{R}^{r-2}$.

Hence, we give the following fundamental proposition:

**Proposition 3.4**. Suppose $\beta$: $I \to \mathbb{E}^3_1$ is the unit speed timelike curve $\kappa_1 \neq 0$. (1). If $h_{\mathbf{x}}(s) = H(s, \mathbf{x})$ has an $A_k$-singularity ($k = 2,\ 3$) at $s_0 \in \mathbb{R}$, therefore $H$ is a $p-$versal unfolding of $h_{\mathbf{x}_0}(s_0)$. (2). If $\widetilde{h}_{\mathbf{x}}(s) = \widetilde{H}(s, \mathbf{x}, w)$ has an $A_k$-singularity ($k = 2,\ 3$) at $s_0 \in \mathbb{R}$, then $\widetilde{H}$ is a $p-$versal unfolding of $\widetilde{h}_{\mathbf{x}_0}(s_0)$

**Proof**. (1) Because of $\mathbf{x} = (x_0{,}x_1, x_2) \in \mathbb{S}^2_1$, and $\beta(s){=}(\beta_0(s){,}\beta_1(s), \beta_2(s)) \in \mathbb{E}^3_1$ and Without loss of generality, Let $x_2 \neq 0$. So by $x_2 = \sqrt{1 + x_0^2 - x_1^2}$, we have

$$H(s, \mathbf{x}) = -x_0\beta_0(s) + x_1\beta_1(s) + \sqrt{1 + x_0^2 - x_1^2}\beta_2(s). \qquad (3.6)$$

Thus, we have that

$$\left.\begin{array}{l} \frac{\partial H}{\partial x_0} = -\beta_0(s) + \frac{x_0\beta_2(s)}{\sqrt{1+x_0^2-x_1^2}},\ \frac{\partial H}{\partial x_1} = \beta_1(s) - \frac{x_1\beta_2(s)}{\sqrt{1+x_1^2-x_2^2}}, \\ \frac{\partial^2 H}{\partial s \partial x_0} = -\beta_0^{'}(s) + \frac{x_0\beta_2^{'}(s)}{\sqrt{1+x_0^2-x_1^2}},\ \frac{\partial^2 H}{\partial s \partial x_1} = \beta_1^{'}(s) - \frac{x_1\beta_2^{'}(s)}{\sqrt{1+x_1^2-x_2^2}}. \end{array}\right\}$$

Therefore, the 2-jets of $\frac{\partial H}{\partial x_i}$ at $s_0$ (i=0, 1) are: Let $\mathbf{x}_0{=}(x_{00}{,}x_{10}, x_{20}) \in \mathbb{S}^2_1$, and assume $x_{20} \neq 0$, then

$$\left.\begin{array}{l} j^1\left(\frac{\partial H}{\partial x_0}(s, \mathbf{x}_0)\right) = \left(-\beta_0^{'}(s) + \frac{x_{00}\beta_2^{'}(s)}{x_{20}}\right)(s - s_0), \\ j^1\left(\frac{\partial H}{\partial x_1}(s, \mathbf{x}_0)\right) = \left(\beta_1^{'}(s) - \frac{x_{10}\beta_2^{'}(s)}{x_{20}}\right)(s - s_0), \end{array}\right\} \qquad (3.7)$$

and

$$\left.\begin{array}{r}j^2\left(\frac{\partial H}{\partial x_0}(s,\mathbf{x}_0)\right)=\left(-\beta_0^{'}(s)+\frac{x_{00}\beta_2(s)}{x_{20}}\right)(s-s_0)\\+\frac{1}{2}\left(-\beta_0^{''}+\frac{x_{00}\beta_2^{''}(s)}{x_{20}}\right)(s-s_0)^2,\\j^2\left(\frac{\partial H}{\partial x_1}(s,\mathbf{x}_0)\right)=\left(\beta_1^{'}(s)-\frac{x_{10}\beta_2^{'}(s)}{x_{20}}\right)(s-s_0)\\+\frac{1}{2}\left(\beta_1^{''}(s)-\frac{x_{10}\beta_2^{''}(s)}{x_{20}}\right)(s-s_0)^2\end{array}\right\}\tag{3.8}$$

(i) If $h_{\mathbf{x}_0}(s_0)$ has the $A_2$-singularity at $s_0$, then $h_{\mathbf{x}_0}^{'}(s_0)=0$. So the $(2-1)\times 2$ matrix of coefficients $(L_{ji})$ is:

$$A=\left(\begin{array}{cc}-\beta_0^{'}(s)+\frac{x_{00}\beta_2^{'}(s)}{x_{20}} & \beta_1^{'}(s)-\frac{x_{10}\beta_2^{'}(s)}{x_{20}}\end{array}\right).\tag{3.9}$$

Suppose that the rank of the matrix $A$ is zero, then we have:

$$\beta_0^{'}(s)=\frac{x_{00}\beta_2^{'}(s)}{x_{20}},\ \beta_1^{'}(s)=\frac{x_{10}\beta_2^{'}(s)}{x_{20}}.\tag{3.10}$$

Since $\left\|\beta^{'}(s_0)\right\|=\|\mathbf{T}_1(s_0)\|=1$, we have $\beta_2^{'}(s_0)\neq 0$, so that we have the contradiction as follows:

$$\begin{aligned}0&=<\left(\beta_0^{'}(s_0),\beta_1^{'}(s_0),\beta_2^{'}(s_0)\right),(x_{00},x_{10},x_{20})> &(3.11)\\&=-\beta_0^{'}(s_0)x_{00}+\beta_1^{'}(s_0)x_{10}+\beta_2^{'}(s_0)x_{20}\\&=-\frac{x_{00}^2\beta_2^{'}(s_0)}{x_{20}}+\frac{x_{10}^2\beta_2^{'}(s_0)}{x_{20}}+\beta_2^{'}(s_0)x_{20} &(3.12)\\&=\frac{\beta_2^{'}(s_0)}{x_{20}}\left(-x_{00}^2+x_{10}^2+x_{20}^2\right)\\&=\frac{\beta_2^{'}(s_0)}{x_{20}}\neq 0.\end{aligned}$$

Therefore $\mathrm{rank}(A)=1$, and $H$ is the (p) versal unfolding of $h_{\mathbf{x}_0}$ at $s_0$.
(ii) In case $h_{\mathbf{x}_0}(s_0)$ has the $A_3$-singularity at $s_0\in\mathbb{R}$, thus $h_{\mathbf{x}_0}^{'}(s_0)=h_{\mathbf{x}_0}^{''}(s_0)=0$, and by Proposition 3.1:

$$\mathbf{e}(s_0)=\frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}}\left(\frac{\kappa_2}{\kappa_1}\mathbf{N}_1+\mathbf{N}_2\right),\tag{3.13}$$

where $\rho^{'}(s_0)=0$, and $\rho^{''}(s_0)\neq 0$. So the $(3-1)\times 2$ matrix of the coefficients $(L_{ji})$ is

$$B=\left(\begin{array}{cc}L_{11} & L_{12}\\L_{21} & L_{22}\end{array}\right)=\left(\begin{array}{cc}-\beta_0^{'}(s)+\frac{x_{00}\beta_2(s)}{x_{20}} & \beta_1^{'}(s)-\frac{x_{10}\beta_2^{'}(s)}{x_{20}}\\-\beta_0^{''}+\frac{x_{00}\beta_2^{''}(s)}{x_{20}} & \beta_1^{''}(s)-\frac{x_{10}\beta_2^{''}(s)}{x_{20}}\end{array}\right).\tag{3.14}$$

For this purpose, we also require the $2 \times 2$ matrix $B$ to be non-singular, which always holds true. In fact, the determinate of this matrix at $s_0$ is

$$\begin{aligned} \det(B) &= \frac{1}{x_{20}} \begin{vmatrix} -\beta_0^{'} & \beta_1^{'} & \beta_2^{'} \\ -\beta_0^{''} & \beta_1^{''} & \beta_2^{''} \\ x_{00} & x_{10} & x_{20} \end{vmatrix} \quad (3.15) \\ &= \frac{1}{x_{20}} < \beta^{'} \times \beta^{''}, \mathbf{e_0} > \\ &= \mp \frac{\kappa_1}{x_{20}\sqrt{\kappa_1^2+\kappa_2^2}} < \beta^{'} \times \beta^{''}, \left(\frac{\kappa_2}{\kappa_1}\mathbf{N}_1 + \mathbf{N}_2\right) > \quad (3.16) \end{aligned}$$

Since $\beta^{'} = \mathbf{T}_1$, we have $\beta^{''} = \kappa_1\mathbf{N}_1 - \kappa_2\mathbf{N}_2$. Substituting these relations to the above equality, we have that

$$\det(B) = \mp \frac{\sqrt{\kappa_1^2+\kappa_2^2}}{x_{20}} \neq 0. \quad (3.17)$$

This means that $\mathrm{rank}(B) = 2$.
(2) Under similar notations as in (1), we have

$$\widetilde{H}(s, \mathbf{x}, x_2) = -x_0\beta_0(s) + x_1\beta_1(s) + \sqrt{1 + x_0^2 - x_1^2}\beta_2(s) - x_2. \quad (3.18)$$

We require the $2 \times 3$ matrix

$$G = \begin{pmatrix} -\beta_0^{'}(s) + \frac{x_{00}\beta_2(s)}{x_{20}} & \beta_1^{'}(s) - \frac{x_{10}\beta_2^{'}(s)}{x_{20}} & -1 \\ -\beta_0^{''} + \frac{x_{00}\beta_2^{''}(s)}{x_{20}} & \beta_1^{''}(s) - \frac{x_{10}\beta_2^{''}(s)}{x_{20}} & 0 \end{pmatrix},$$

to get the maximal rank. Using case (1) in Eq. (3.14), the second raw of G is not equal zero, so $\mathrm{rank}(G) = 2$ ■.

**Proof of Theorem 3.**1. (1) Using Proposition 3.1, the bifurcation set of $H(s, \mathbf{x})$ is

$$\mathfrak{B}_H = \left\{ \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}} \left(\frac{\kappa_2}{\kappa_1}\mathbf{N}_1 + \mathbf{N}_2\right) |s \in \mathbb{R}|s \in \mathbb{R} \right\}. \quad (3.19)$$

The assertion (1) of Theorem 3.1 follows from Proposition 3.1, Proposition 3.4, and Theorem 3.2. The discriminant set of $\widetilde{H}(s, \mathbf{x})$ is given as follows:

$$\mathfrak{D}_{\widetilde{H}} = \{\mathbf{x}_0 = \beta + \cos u\mathbf{N}_1 + \sin u\mathbf{N}_2 | s \in \mathbb{R}\}. \quad (3.20)$$

The assertion (1) of Theorem 3.1 follows from Proposition 3.1, and Proposition 3.4, and Theorem 3.2 ■.

**Example 3.1**. Given the timelike helix:

$$\beta(s) = (\sqrt{3}\sinh s, \sqrt{2}s, \sqrt{3}\cosh s), \quad -1 \leq s \leq 1,$$

Clearly

$$\left.\begin{array}{c}\mathbf{T}(s)=(\sqrt{3}\cosh s,\sqrt{2},\sqrt{3}\sinh s),\\ \mathbf{N}(s)=(\sinh s,0,\cosh s),\\ \mathbf{B}(s)=(-\sqrt{2}\cosh s,-\sqrt{3},-\sqrt{2}\sinh s),\\ \kappa(s)=\sqrt{3},\text{ and }\tau(s)=-\sqrt{2}.\end{array}\right\}$$

Taking $\theta_0 = 0$ we have $\theta(s) = \sqrt{2}s$. Using the Eq. (2.7), we obtain

$$\kappa_1(s)=\sqrt{3}\cos\sqrt{2}s,\text{ and }\kappa_2(s)=\sqrt{3}\sin\sqrt{2}s.$$

Hence, the geometric invariant is

$$\rho(s)=\sqrt{6}.$$

Therefore, the transformation matrix can be expressed as:

$$\begin{pmatrix}\mathbf{T}_1\\ \mathbf{N}_1\\ \mathbf{N}_2\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\sqrt{2}s & \sin\sqrt{2}s\\ 0 & -\sin\sqrt{2}s & \cos\sqrt{2}s\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix},$$

From this, we have

$$\begin{aligned}\mathbf{N}_1 &= \begin{pmatrix}N_{11}\\ N_{12}\\ N_{13}\end{pmatrix}=\begin{pmatrix}\sinh s\cos\sqrt{2}s-\sqrt{2}\cosh s\sin\sqrt{2}s\\ -\sqrt{3}\sin\sqrt{2}s\\ \cosh s\cos\sqrt{2}s-\sqrt{2}\sinh s\sin\sqrt{2}s\end{pmatrix},\\ \mathbf{N}_2 &= \begin{pmatrix}N_{21}\\ N_{22}\\ N_{23}\end{pmatrix}=\begin{pmatrix}\sinh s\sin\sqrt{2}s-\sqrt{2}\cosh s\cos\sqrt{2}s\\ -\sqrt{3}\cos\sqrt{2}s\\ \cosh s\cos\sqrt{2}s-\sqrt{2}\sinh s\cos\sqrt{2}s\end{pmatrix}.\end{aligned}$$

Hence, the spacelike sweeping surface is (Figure 4)

$$M:\mathbf{R}(s,u)=\left(\sqrt{3}\sinh s,\sqrt{2}s,\sqrt{3}\cosh s\right)+\cos u\begin{pmatrix}N_{11}\\ N_{12}\\ N_{13}\end{pmatrix}+\sin u\begin{pmatrix}N_{21}\\ N_{22}\\ N_{23}\end{pmatrix}.$$

The Bishop spherical Darboux image is (Figure 5)

$$\mathbf{e}(s)=\sin\sqrt{2}s\begin{pmatrix}N_{11}\\ N_{12}\\ N_{13}\end{pmatrix}+\cos\sqrt{2}s\begin{pmatrix}N_{21}\\ N_{22}\\ N_{23}\end{pmatrix}.$$

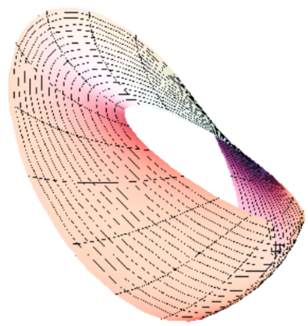

Figure 4: Timelike sweeping surface.

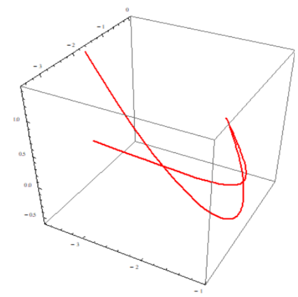

Figure 5: Bishop spherical Darboux image which has a cusp point.

### 3.3 Singularities of developable surfaces

Developable surfaces are considered as special cases of ruled surfaces. This kind of surfaces are used in different fields as the manufacture of automobile body parts, airplane wings, and ship hulls. Therefore, we analyze the case that the profile curve $\mathbf{x}$ degenerates into a line. Then, we have the following two timelike developable surfaces

$$R : \mathbf{Q}(s,u) = \beta(s) + u\mathbf{N}_2(s), \;\; u \in \mathbb{R}, \tag{3.21}$$

and

$$R^{\perp} : \mathbf{Q}^{\perp}(s,u) = \beta(s) + u\mathbf{N}_1(s), u \in \mathbb{R}. \tag{3.22}$$

Obviously, $\mathbf{R}(s,0) = \alpha(s)$ (resp. $\mathbf{R}^{\perp}(s,0) = \alpha(s)$), $0 \leq s \leq L$, that is, the surface $R$ (resp. $R^{\perp}$) interpolate the curve $\alpha(s)$. We can also calculate that

$$R : \mathbf{Q}_s \times \mathbf{Q}_u = -\left(1 - u\kappa_2\right)\mathbf{N}_1(s),$$

and

$$R^{\perp} : \mathbf{Q}_s^{\perp} \times \mathbf{Q}_u^{\perp} = \left(1 + u\kappa_1\right)\mathbf{N}_2(s).$$

Then we have $R$ (resp. $R^{\perp}$) is non-singular at $(s_0, u_0)$ iff $1 - u_0\kappa_2(s_0) \neq 0$ (resp. $(1 + u_0\kappa_1(s_0) \neq 0)$. Hence, by using $\kappa_2$ we classify the singularities as in the following.

**Theorem 3.3**. Let $R$ be the timelike developable expressed by Eq. (3.21). Then we have the following
(1) $R$ is locally diffeomorphic to Cuspidal edge at $(s_0, u_0)$ iff $\kappa_2(s_0) = 0$, and $\kappa_2^{'}(s_0) \neq 0$;
(2) $R$ is locally diffeomorphic to Swallowtail at $(s_0, u_0)$ iff $\kappa_2(s_0) \neq 0$, and $\frac{\kappa_2^{'}(s_0)}{\kappa_2^{2}(s_0)} \neq 0$.

**Proof**. If there exists a parameter $s_0$ such that $\kappa_2(s_0) = 0$, and $u_0^{'} = \frac{\kappa_2^{'}(s_0)}{\kappa_2^{2}(s_0)} \neq 0$

$(\kappa_2'(s_0) \neq 0)$, then $R$ is locally diffeomorphic to Cuspidal edge at $(s_0, u_0)$. So, assertion (1) holds. Also, if there exists a parameter $s_0$ such that $u_0 = \frac{1}{\kappa_2(s_0)} \neq 0$, $u_0' = \frac{\kappa_2'(s_0)}{\kappa_2^2(s_0)} = 0$, and $\left(\frac{1}{\kappa_2(s_0)}\right)'' \neq 0$, then $R$ is locally diffeomorphic to Swallowtail at $(s_0, u_0)$, assertion (2) holds $\blacksquare$.

**Example 3.2**. By making using of Example 3.1, we have the following:
(1) If $s_0 = 0$, then $\kappa_2(s_0) = 0$, and $\kappa_2'(s_0) \neq 0$. The timelike developable surface

$$M : \mathbf{Q}(s,u) = \left(\frac{\sqrt{3}}{2}\sinh s, \frac{s}{2}, \frac{\sqrt{3}}{2}\cosh s\right) + u\begin{pmatrix} \frac{1}{2}\cosh\frac{s}{2}\cosh s - \sinh\frac{s}{2}\sinh s \\ -\frac{\sqrt{3}}{2}\cosh\frac{s}{2} \\ \frac{1}{2}\cosh\frac{s}{2}\sinh s - \sinh\frac{s}{2}\cosh s \end{pmatrix},$$

is locally diffeomorphic to the Cuspidal edge, $u \in \mathbb{R}$, see Figure 6.
(2) If $s_0 = 0$, then $\kappa_1(s_0) \neq 0$, and $\kappa_2'(s_0) = 0$. The timelike developable surface

$$M^{\perp} : \mathbf{Q}^{\perp}(s,u) = \left(\frac{\sqrt{3}}{2}\sinh s, \frac{s}{2}, \frac{\sqrt{3}}{2}\cosh s\right) + u\begin{pmatrix} \cosh\frac{s}{2}\sinh s - \frac{1}{2}\sinh\frac{s}{2}\cosh s \\ \frac{\sqrt{3}}{2}\sinh\frac{s}{2} \\ \cosh\frac{s}{2}\cosh s - \frac{1}{2}\sinh\frac{s}{2}\sinh s \end{pmatrix},$$

is locally diffeomorphic to Swallowtail, $u \in \mathbb{R}$, see Figure 7.

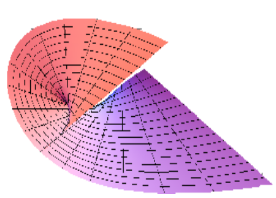

Figure 6: CE timelike developale surface.

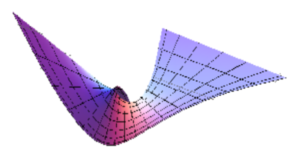

Figure 7: SW timelike developale surface.

# DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that can affect this work negatively.

# Data Availability

No data were used to support this study

# Acknowledgments

This research was funded by the Deanship of Scientific Research at Princess Nourah bint Abdulrahman University through the Fast-track Research Funding

Program.

## References


[1] JW Bruce, and PJ Giblin. Curves and Singularities. 2nd ed. Cambridge: Cambridge University Press, 1992.

[2] R. Cipolla, and PJ. Giblin. Visual Motion of Curves and Surfaces, Cambridge Univ. Press, 2000.

[3] K. Saji, M. Umehara, and K. Yamada. The geometry of fronts. Ann Math. (2009); 169: 491-529.

[4] K. Teramoto. Parallel and dual surfaces of cuspidal edges. Differential Geom Appl. (2016); 44:52-62.

[5] MP. do Carmo. Differential Geometry of Curves and Surface, Prentice-Hall, Englewood Cliffs, NJ, 1976MP.

[6] JS. Ro and DW. Yoon. Tubes of weingarten types in Euclidean 3-space," J. of the Chungcheong Math. Society, vol. 22, no. 3, (2009), 359–366,

[7] Z. Xu, and R. S. Feng. Analytic and algebraic properties of canal surfaces, J. of Computational and Applied Mathematics 195, (2006), 220-228.

[8] S. Izumiya, K. Saji, and N. Takeuchi. Circular surfaces, Advances in Geometry, 7 (2), (2007), 295–313.

[9] L. Cui, D. L Wang, and J.S. Dai. Kinematic geometry of circular surfaces with a fixed radius based on Euclidean invariants," ASME J. Mech. Vol. 131, October (2009).

[10] RL. Bishop. There is more than one way to frame a curve. Amer. Math. Monthly 82, (1975), 246–251.

[11] F. Klok. Two moving coordinate frames for sweeping along a 3D trajectory,. Comput. Aided Geom. Design 3, 217–229, 1986.

[12] W. Wang, and B. Joe. Robust computation of the rotation minimizing frame for sweep surface modelling, Computer-Aided Design, 29 :379–91, 1997.

[13] W. Wang, B. Jüttler, D. Zheng, andY. Liu. Computation of rotating minimizing frames, ACM Transactions on Graphics, 27: 1-18, 2008.

[14] RT. Farouki, C. Giannelli , and ML. Sampoli. Rotation-minimizing osculating frames, Comput Aided Geom Des 31(1): 27–4, 2011.

[15] B. Bükcu and M. K. Karacan. On the slant helices according to Bishop frame of the timelike curve in Lorentzian space, Tamkang J. Math. 39(3) (2008), 255–262.

[16] M. Grbovic and E. Nešovic. On the Bishop frames of pseudo null and null Cartan curves in Minkowski 3-space, J. Math. Anal. Appl. 461 (2018), 219–233.

[17] O. Keskin and Y. Yayli. An application of N-Bishop frame to spherical images for direction curves, Int. J. Geom. Methods Mod. Phys. 14(11) (2017), 1750162.

[18] B. O'Neil. Semi-Riemannian Geometry geometry, with applications to relativity, Academic Press, New York, 1983.

[19] J. Walfare. Curves and Surfaces in Minkowski Space, Ph.D. Thesis, K.U. Leuven, Faculty of Science, Leuven, 1995.