ON STABILITY AND HYPERSTABILITY OF DRYGAS FUNCTIONAL EQUATION IN QUASI-BANACH SPACE

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Abstract. In this paper, we investigate the Ulam-Hyers-Rassias stability for the Drygas functional equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y)$$

in the setting quasi-Banach spaces using the approach of Brzdek's fixed point theorem. Also, we gave a general result on the hyperstability of Drygas functional equation. The results obtained in this paper extend various previously known results to the setting of quasi-Banach space.

1. Introduction and Preliminaries

In the theory of Ulam's stability, one can find the efficient tools to evaluate the errors, that is to study the existence of an exact solution of the perturbed functional equation which is not far from given function. In 1940, the stability problem for the functional equations was first raised by S.M. Ulam [30]. Hyers [16] gave affirmative partial answer to the of Ulam in Banach space. After that Aoki [2] and Rassias [25] generalized Hyers theorem for additive and linear mapping by considering an unbounded Cauchy difference. In 1994, Găvruta [14] generalized Rassias' theorem and discuss the stability of linear functional equation.

A functional equation is hyperstable if a function satisfying this functional equation approximately is a true solution of it. In 1949, D.G. Bourgin [5] gave the first hyperstability result and concerned the ring homomorphisms. The hyperstability results of the several functional equation in the literature have been studied by many authors in recent years, (see, [3] , [10], [8], [9] [15], [18], [24] and references cited therein).

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$, the set of nonnegative integers by $\mathbb{N}_0 = \{0, 1, 2, 3....\}$ and the set of all natural numbers greater then or equal to the natural number m by $\mathbb{N}_m$. Let $\mathbb{R}$ be set

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of real numbers and $\mathbb{R}_+ = [0, \infty)$ the set of nonnegative real numbers. We write X^Y to mean the family of all functions mapping from a nonempty set X into a nonempty set Y, and we denote X^n the $n-$ary Cartesian power of X.

Definition 1. (see [4], [19]) A quasi-norm is a real-valued function on the linear space X satisfying the following:

(1) $\parallel x \parallel \ \geq \ 0$ for all $x \in X$ and $\parallel x \parallel = 0$ if and only if $x = 0$;
(2) $\parallel \lambda x \parallel = \mid \lambda \mid . \parallel x \parallel$ for all $\lambda \in \mathbb{R}$ and all $x \in X$;
(3) There is a constant $K \geq 1$ such that $\parallel x + y \parallel \leq K(\parallel x \parallel + \parallel y \parallel)$ for all $x, y \in \mathbb{X}$.

The pair $(X, \parallel . \parallel, K)$ is called a quasi-normed space if $\parallel . \parallel$ is a quasi-norm on X. A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\parallel . \parallel$ is called $p - norm$ $(0 < p \leq 1)$ if

$$\parallel x + y \parallel^p \leq \parallel x \parallel^p + \parallel y \parallel^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p-Banach space.

The difference between a norm and quasi-norm is that the modulus of concavity of a quasi-norm is greater then equal to 1, while that of a norm is equal to 1. The quasi-norm is not continuous in general, while a norm is always continuous. However, every $p-$norm is continuous quasi norm. By the Aoki-Rolewicz theorem [19] (see also [4]), each quasi-norm is equivalent to some p-norm. Firstly studied Stability in quasi-Banach spaces by Najati and Moghimi [20] and Najati and Eskandani [21]. After that many authors obtained very interesting results in the topic (see [1] , [12]).

Definition 2. (see [28]) Let X be a non empty set, Y be a normed space, $\varepsilon \in \mathbb{R}_+^{X^n}$ and $\mathcal{V}_1, \mathcal{V}_2$ be operators mapping from a non empty set $D \subset Y^X$ into Y^{X^n}. We say that the operators equation

$$\mathcal{V}_1\varphi(x_1, x_2, ...x_n) = \mathcal{V}_2\varphi(x_1, x_2, ...x_n) \tag{1.1}$$

for $x_1, x_2, ...x_n \in X$ is $\varepsilon-$hyperstable provided that every $\varphi_0 \in D$ which satisfies

$$\parallel \mathcal{V}_1\varphi(x_1, x_2, ...x_n) - \mathcal{V}_2\varphi(x_1, x_2, ...x_n) \parallel \leq \ \varepsilon(x_1, x_2, ...x_n)$$

fulfills the equation (1.1).

Using the concept of Brzdek [6], Dung et al. [12] proved the following result.

Theorem 1. *[12] Let X be a non empty set, Y be a quasi-Banach space. $f_1, f_2......f_k : X \to X$ and $l_1, l_2....l_k : X \to \mathbb{R}_+$ be given mapping.*

Suppose that $\mathcal{T} : Y^X \to Y^X$ is an operator satisfies the inequality

$$\| \mathcal{T}\xi(x) - \mathcal{T}\mu(x) \| \leq \sum_{i=1}^{k} l_i(x) \| (\xi - \mu) f_i(x) \| \tag{1.2}$$

for all $\xi, \mu \in Y^X$, and for all $x \in X$ and assume that the function $\varepsilon : X \to \mathbb{R}_+$ and a mapping $\varphi : X \to Y$ satisfies conditions

$$\| \mathcal{T}\varphi(x) - \varphi(x) \|_Y \leq \varepsilon(x) \tag{1.3}$$

for every $x \in X$ and $\theta = log_{2K} 2$,

$$\varepsilon^*(x) = \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)^\theta (x) < \infty \tag{1.4}$$

where $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ be a linear operator defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} l_i(x)\delta(f_i(x)), \tag{1.5}$$

for $\delta \in \mathbb{R}_+^X$ and $x \in X$.
Then we have

(1) *For every $x \in X$, the limit*

$$\lim_{n\to\infty} \mathcal{T}^n \varphi(x) = \psi(x) \tag{1.6}$$

there exists a fixed point ψ of $\mathcal{T}$ with

$$\| \varphi(x) - \psi(x) \|^\theta \leq 4\varepsilon^*(x) \tag{1.7}$$

for all $x \in X$.

(2) *For every $x \in X$, if*

$$\varepsilon^*(x) \leq \left(M \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(x) \right)^\theta < \infty, \tag{1.8}$$

for some positive real number M, then the fixed point of $\mathcal{T}$ is unique.

Characterizing quasi-inner product spaces, Drygas [11] consider the following functional equation

$$f(x) + f(y) = f(x-y) + \{f(\frac{x+y}{2}) - f(\frac{x-y}{2})\}, \tag{1.9}$$

for all $x, y \in \mathbb{R}$, which can be reduced to the following equation (see, [26], Remark 9.2, pp. 131)

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y), \tag{1.10}$$

for all $x, y \in \mathbb{R}$. This equation is known in the literature as Drygas equation and is a generalization of the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \tag{1.11}$$

for all $x, y \in \mathbb{R}$.

The general solution of Drygas equation was given by Ebanks et al. [13]. It has the form $f(x) = A(x)+Q(y)$ for all $x \in \mathbb{R}$, where $A : \mathbb{R} \to \mathbb{R}$ is an additive function and $Q : \mathbb{R} \to \mathbb{R}$ is a quadratic function (see also [17]). A set-valued version of Drygas equation was considered by Smajdor [29]. Recently, the hyperstability of the Drygas functional equation has been studied by various authors see [23], [28] and [27]. In this paper, we discuss the generalized Hyers-Ulam-Rassias stability problem for the Drygas functional equation (1.10) in Banach spaces by using Theorem 1. Also, we obtain some hyperstability results for this equation.

2. Main Result

Throughout in this section X is a non empty set , we write $X_0 := X - \{0\}$, and we denoted by $Aut(X)$ for the family of all automorphisms of X. The identity function on X will be denoted by Id_X, and for each $u \in X^X$ we write $ux = u(x)$ for $x \in X$and we defined $-u$ by $-ux := -u(x), 2ux = ux + ux$ and $u' = u'x := (Id_X - u)x = x - ux$ for $x \in X$.

Theorem 2. *Let X be a quasi-normed space with norm $\| \cdot \|_X$ and Y be quasi-Banach space with the norm $\| \cdot \|$. Let $\varepsilon : X_0 \times X_0 \to [0, \infty]$ be a function and*

$$l(X) := \{u \in Aut(X) : -u, u', (Id_X - 2u) \in Aut(X), \alpha_u < 1\} \tag{2.1}$$

be an infinite set, where

$$\alpha_u := \lambda(u') + \lambda(u) + \lambda(-u) + \lambda(Id_X - 2u),$$

$$\lambda(u) := \inf\{t \in \mathbb{R}_+ : \varepsilon(ux, uy) \leq t\varepsilon(x, y) \forall x, y \in X_0\},$$

for $u \in Aut(X)$. Assume that $f : X \to Y$ be a mapping such that

$$||f(x+y) + f(x-y) - 2\ f(x) -\ f(y) - f(-y)|| \leq \varepsilon(x, y), \tag{2.2}$$

for all $x, y \in X_0$ such that $x + y \neq 0$ and $x - y \neq 0$. Then, for each nonempty subset $U \subset l(X)$ such that

$$u \circ v = v \circ u, (u, v \in U), \tag{2.3}$$

there exists a unique function $D : X \to Y$ satisfying (1.10) and

$$\| D(x) - f(x) \|^\theta \leq 4\varepsilon^*(x), \tag{2.4}$$

for all $x \in X_0$, where $\theta = \log_{2K} 2$ and $\varepsilon^*(x) := \inf \left\{ \frac{\varepsilon^\theta(u'x, ux)}{1-\alpha_u^\theta} : u \in U \right\}$.

Proof. Fix $u \in U$. Replacing x by $u'x$ and y by ux in (2.2), we have

$$\|f(x)+f((Id_X-2u)x)-2f(u'x)-f(ux)-f(-ux)\|_Y \leq \varepsilon(u'x,ux) := \varepsilon_u(x), \tag{2.5}$$

for all $x \in X_0$. We define the operators $T_u : Y^{X_0} \to Y^{X_0}$ and $\Lambda_u : \mathbb{R}_+^{X_0} \to \mathbb{R}_+^{X_0}$ by

$$\begin{aligned} &T_u\xi(x) := 2\xi(u'x)+\xi(ux)+\xi(-ux)-\xi((Id_x-2u)x) \\ &\Lambda_u\delta(x) := K\left(2\delta(u'x)+\delta(ux)+\delta(-ux)+\delta((Id_X-2u)x)\right), \end{aligned} \tag{2.6}$$

for all $x \in X_0$, $\xi \in Y^{X_0}$ and $\delta \in \mathbb{R}_+^{X_0}$.
Then (2.5) becomes $\|f(x)-T_uf(x)\|_Y \leq \varepsilon_u(x)$, for all $x \in X_0$. The operator Λ_u has the form given by (1.5) with $s=4$ and $f_1(x)=u'x, f_2(x)=ux, f_3(x)=-ux, f_4(x)=(Id_X-2u)x,$ $l_1(x)=2K,$ $l_2(x)=l_3(x)=l_4(x)=K$ for all $x \in X_0$. Further, we have

$$\begin{aligned} &\|T_u\xi(x)-T_u\mu(x)\|_Y \\ &= \|2\xi(u'x)+\xi(ux)+\xi(-ux)-\xi((id_X-2u)x)-2\mu(u'x)-\mu(ux)-\mu(-ux)+ \\ &\quad \mu((id_X-2u)x)\| \\ &\leq K[2\|\xi(u'x)-\mu(u'x)\|_Y+\|\xi(ux)-\mu(ux)\|_Y+\|\xi(-ux)-\mu(-ux)\|_Y \\ &\quad +\|\xi((Id_X-2u)x)-\mu((Id_X-2u)x)\|_Y], \\ &= \sum_{r=0}^{4} l_i(x) \parallel \xi(f_i(x))-\mu(f_i(x)) \parallel, \end{aligned}$$

for all $x \in X_0$ and $\xi, \mu \in Y^{X_0}$. Using the definition of $\lambda(u)$, $\varepsilon(ux,uy) \leq \lambda(u)\varepsilon(u'x,ux)$ for all $x, y \in X_0$ we have to show that $\Lambda_u^r\varepsilon_u(x) \leq K^r\alpha_u^r\varepsilon(u'x,ux)$ for all $x \in X_0$, where $\alpha_u = 2\lambda(u')+\lambda(u)+\lambda(-u)+\lambda(id_X-2u)$.

If $r=0$, then $\varepsilon_u(x)=\varepsilon(u'x,ux)$. If $r=1$, we have

$$\begin{aligned} \Lambda_u\varepsilon(x) &= K\left(2\varepsilon_u(u'x)+\varepsilon_u(ux)+\varepsilon_u(-ux)+\varepsilon_u((id_X-2u)x)\right) \\ &= 2K\varepsilon\left(u'(u'x),u(u'x)\right)+K\varepsilon\left(u'(ux),u(ux)\right)+K\varepsilon\left(u'(-ux,u(-ux)\right) \\ &\quad +K\varepsilon\left(u'((Id_X-2u)x),u((Id_X-2u)x)\right) \\ &= 2K\varepsilon\left(u'(u'x),u'(ux)\right)+K\varepsilon\left(u(u'x),u(ux)\right)+K\varepsilon\left(-u(u'x),-u(ux)\right) \\ &\quad +K\varepsilon\left((Id_X-2u)(u'(x),(Id_X-2u)(ux)\right) \\ &\leq 2K\lambda(u')\varepsilon(u'x,ux)+K\lambda(u)\varepsilon(u'x,ux)+K\lambda(-u)\varepsilon(u'x,ux) \\ &\quad +K\lambda(Id_X-2u)\varepsilon(u'x,ux) \\ &= K\left(2\lambda(u')+\lambda(u)+\lambda(-u)+\lambda(id_X-2u)\right)\varepsilon(u'x,ux) \\ &= K\alpha_u\varepsilon(u'x,ux) \end{aligned}$$

Now, further, if $r = 2$, we have

$$\begin{aligned}
\Lambda^2\varepsilon_u(x) &= \Lambda\left[\Lambda\varepsilon_u(x)\right] \\
&= K\left[2\Lambda_u\varepsilon_u(u'x) + \Lambda_u\varepsilon_u(ux) + \Lambda_u\varepsilon_u(-ux) + \Lambda_u\varepsilon_u((Id_x - 2u)x)\right] \\
&= 2K^2\alpha_u\varepsilon\left(u'(u'x), u(u'x)\right) + K^2\alpha_u\varepsilon\left(u'(ux), u(ux)\right) + K^2\alpha_u\varepsilon\left(u'(-ux, u(-ux)\right) \\
&\quad + K^2\varepsilon\left(u'((Id_X - 2u)x), u((Id_X - 2u)x)\right) \\
&= 2K^2\alpha_u\varepsilon\left(u'(u'x), u'(ux)\right) + K^2\alpha_u\varepsilon\left(u(u'x), u(ux)\right) + K^2\alpha_u\varepsilon\left(-u(u'x), -u(ux)\right) \\
&\quad + K^2\alpha_u\varepsilon\left((Id_X - 2u)(u'x), (Id_X - 2u)(ux)\right) \\
&\leq 2K^2\alpha_u\lambda(u')\varepsilon(u'x, ux) + K^2\alpha_u\lambda(u)\varepsilon(u'x, ux) + K^2\alpha_u\lambda(-u)\varepsilon(u'x, ux) \\
&\quad + K\lambda(Id_X - 2u)\varepsilon(u'x, ux) \\
&= K^2\alpha_u\left(2\lambda(u') + \lambda(u) + \lambda(-u) + \lambda(id_X - 2u)\right)\varepsilon(u'x, ux) \\
&= K^2\alpha_u^2\varepsilon(u'x, ux).
\end{aligned}$$

Proceeding on the similar lines, we get $\Lambda_u^r\varepsilon_u(x) \leq K^r\alpha_u^r\varepsilon(u'x, ux)$ for all $x \in X_0$ and $r \in \mathbb{N}_0$. Hence

$$\varepsilon^*(x) = \sum_{r=0}^{\infty}\left(\Lambda_u^r\varepsilon_u\right)^\theta(x) \leq \varepsilon^\theta(u'x, ux)\sum_{r=0}^{\infty}K^{r\theta}\alpha_u^{r\theta} = \frac{\varepsilon^\theta(u'x, ux)}{1 - K^\theta\alpha_u^\theta} < \infty,$$

for all $x \in X_0$. Therefore by the Theorem 1 there exists a unique solution $D_u : X \to Y$ of the equation

$$D_u(x) = 2D_u(u'x) + D_u(ux) + D_u(-ux) - D_u((Id_x - 2u)x), \tag{2.7}$$

for all $x \in X_0$, which is a fixed point of $\mathcal{T}_u$ such that

$$\|D_u(x) - f(x)\|_Y^\theta \leq 4\varepsilon^*(x), \tag{2.8}$$

for all $x \in X_0$. Moreover, $D_u(x) = \lim_{r\to\infty}\mathcal{T}_u^r f(x)$ for all $x \in X_0$.

Now, to prove that D_u satisfies the functional equation (1.10) on x_0, we have to prove the following inequality

$$\begin{aligned}
&\|\ \mathcal{T}_u^r f(x+y) + \mathcal{T}_u^r f(x-y) - 2\mathcal{T}_u^r f(x) - \mathcal{T}_u^r f(y) - \mathcal{T}_u^r f(-y)\ \|_Y \\
\leq\ & K^{r\theta}\alpha_u^{r\theta}\varepsilon^\theta(x, y),
\end{aligned} \tag{2.9}$$

for all $x, y \in X_0$ such that $x + y \neq 0$, $x - y \neq 0$, and $r \in \mathbb{N}_0$. Indeed if $r = 0$ then (2.9) is simply (2.2). So we suppose that (2.9) holds for $r \in \mathbb{N}$ and $x, y \in X_0$. Then from (2.6) and the triangle inequality, we get

$$\|\ \mathcal{T}_u^{r+1}f(x+y) + \mathcal{T}_u^{r+1}f(x-y) - 2\mathcal{T}_u^{r+1}f(x) - \mathcal{T}_u^{r+1}f(y) - \mathcal{T}_u^{r+1}f(-y)\ \|_Y$$

$$
\begin{aligned}
&= \| \, 2\mathcal{T}_u^r f(u'(x+y)) + \mathcal{T}_u^r f(u(x+y)) + \mathcal{T}_u^r f(-u(x+y)) - \mathcal{T}_u^r f((Id_X - 2u)(x+y)) \\
&\quad + 2\mathcal{T}_u^r f(u'(x-y)) + \mathcal{T}_u^r f(u(x-y)) + \mathcal{T}_u^r f(-u(x-y)) - \mathcal{T}_u^r f((Id_X - 2u)(x-y)) \\
&\quad - 4\mathcal{T}_u^r f(u'(x)) - 2\mathcal{T}_u^r f(u(x)) - 2\mathcal{T}_u^r f(-u(x)) + \mathcal{T}_u^r f((Id_X - 2u)(x)) \\
&\quad - 2\mathcal{T}_u^r f(u'(y)) - \mathcal{T}_u^r f(u(y)) - 2\mathcal{T}_u^r f(-u(y)) + \mathcal{T}_u^r f((Id_X - 2u)(y)) \\
&\quad + 2\mathcal{T}_u^r f(u'(-y)) + \mathcal{T}_u^r f(u(-y)) + 2\mathcal{T}_u^r f(-u(-y)) + \mathcal{T}_u^r f((Id_X - 2u)(-y)) \, \|_Y \\
&\leq 2K \, \| \, \mathcal{T}_u^r f(u'(x+y)) + \mathcal{T}_u^r f(u'(x-y)) - 2\mathcal{T}_u^r f(u'(x)) - \mathcal{T}_u^r f(u'(y)) - \mathcal{T}_u^r f(u'(-y)) \, \|_Y \\
&\quad + K \, \| \, \mathcal{T}_u^r f(u(x+y)) + \mathcal{T}_u^r f(u(x-y)) - 2\mathcal{T}_u^r f(u(x)) - \mathcal{T}_u^r f(u(y)) - \mathcal{T}_u^r f(u(-y) \, \|_Y \\
&\quad + K \, \| \, \mathcal{T}_u^r f(-u(x+y)) + \mathcal{T}_u^r f(-u(x-y)) - 2\mathcal{T}_u^r f(-u(x)) - \mathcal{T}_u^r f(-u(y)) - \mathcal{T}_u^r f(-u(-y) \, \|_Y \\
&\quad + K \, \| \, \mathcal{T}_u^r f((Id_X - 2u)(x+y)) + \mathcal{T}_u^r f((Id_X - 2u)(x-y)) - 2\mathcal{T}_u^r f((Id_X - 2u)(x)) \\
&\quad - \mathcal{T}_u^r f((Id_X - 2u)(y)) - \mathcal{T}_u^r f((Id_X - 2u)(-y) \, \|_Y \\
&\leq K^{r+1} \alpha_u^r \left[2\varepsilon(u'x, u'y) + \varepsilon(ux, uy) + \varepsilon(-ux, -uy) + \varepsilon((Id_X - 2u)x, (Id_X - 2u)y) \right] \\
&\leq K^{r+1} \alpha_u^r [2\lambda(u') + \lambda(u) + \lambda(-u) + \lambda(Id_x - 2u)] \varepsilon(x, y) \\
&\leq K^{r+1} \alpha_u^{r+1} \varepsilon(x, y).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\| \, \mathcal{T}_u^{r+1} f(x+y) + \mathcal{T}_u^{r+1} f(x-y) - 2\mathcal{T}_u^{r+1} f(x) - \mathcal{T}_u^{r+1} f(y) - \mathcal{T}_u^{r+1} f(-y) \, \| \\
&\leq \| \, \mathcal{T}_u^{r+1} f(x+y) + \mathcal{T}_u^{r+1} f(x-y) - 2\mathcal{T}_u^{r+1} f(x) - \mathcal{T}_u^{r+1} f(y) - \mathcal{T}_u^{r+1} f(-y) \, \|_Y^\theta \\
&\leq K^{(r+1)\theta} \alpha_u^{(r+1)\theta} \varepsilon^\theta(x, y).
\end{aligned}
$$

By induction, we have shown that (2.9) holds for all $x, y \in X_0$ such that $x + y \neq 0$, $x - y \neq 0$. Letting $r \to \infty$ in (2.9), we get

$$
D_u(x+y) + D_u(x-y) = 2D_u(x) + D_u(y) + D_u(-y), \tag{2.10}
$$

for all $x, y \in X$. Thus we have prove that for every for $u \in \mathcal{U}$ there exists a function $D_u : X_0 \to Y$ which is the solution of functional equation (1.10) on X_0 and satisfies

$$
\|f(x) - D_u(x)\|_Y^\theta \leq 4 \left(\frac{\varepsilon^\theta(u'x, ux)}{1 - K^\theta \alpha_u^\theta} \right) = 4\varepsilon^*(x),
$$

for all $x \in X_0$. Now we prove that each solution $D : X_0 \to Y$ of (1.10) satisfying the inequality

$$
\|f(x) - D(x)\|_Y \leq M\varepsilon^\theta(v'x, vx), \tag{2.11}
$$

for all $x \in X_0$ with some $M > 0$ and $v \in \mathcal{U}$, is equal to D_w for each $w \in \mathcal{U}$. So, fix $v, w \in \mathcal{U}, M > 0$ and $D : X_0 \to Y$ which is the solution of functional equation (1.10) on X_0 and satisfies (2.11). Note that, by (2.8) and (2.11), there is a $M_0 > 0$ such that

$$\begin{aligned}\| D(x) - D_w(x) \|_Y \leq &\ [K \| D(x) - f(x) \|_Y + K \| f(x) - D_w(x) \|_Y] \\ \leq &\ M_0 \left(\varepsilon^\theta(v'x, vx) + \varepsilon^\theta(w'x, wx)\right) . \sum_{r=0}^{\infty} K^{r\theta} \alpha_w^{r\theta},\end{aligned}$$

for all $x \in X_0$. On the other side, D and D_w are solutions of (2.7) because they satisfy (1.10). We show that, for each $j \in \mathbb{N}$ and there is $M_1 > 0$ such that

$$\| D(x) - D_w(x) \|_Y \leq \left(M_1 \left(\varepsilon^\theta(v'x, vx) + \varepsilon^\theta(w'x, wx)\right) . \sum_{r=j}^{\infty} K^{r\theta} \alpha_w^{r\theta} \right)^\theta \tag{2.12}$$

For $j = 0$, it is exactly (2.12). So fix $\gamma \in \mathbb{N}_0$ and assume that (2.12) satisfies for $j = \gamma$. Then, in the view of definition of $\lambda(u)$,

$$\begin{aligned}&\| D(x) - D_w(x) \|_Y \\ &= \| 2D(w'x) + D(wx) + D(-wx) - D((Id_X - 2w)x) \\ &\quad - 2D_w(w'x) - D_w(wx) - D_w(-wx) + D_w((Id_X - 2w)x) \|_Y \\ &\leq 2K\|D(w'x) - D_w(w'x)\|_Y + K\|D(wx) - D_w(wx)\|_Y + K\|D(-wx) - D_w(-wx)\|_Y \\ &\quad + K \| D((Id_X - 2w)x) - D_w((Id_X - 2w)x) \|_Y \\ &\leq 2M_0K \left(\varepsilon^\theta(w'v'x, w'vx) + \varepsilon^\theta(w'w'x, w'wx)\right) . \sum_{r=\gamma}^{\infty} K^{r\theta} \alpha_w^{r\theta} \\ &\quad + M_0K \left(\varepsilon^\theta(wv'x, wvx) + \varepsilon^\theta(ww'x, wwx)\right) . \sum_{r=\gamma}^{\infty} K^{r\theta} \alpha_w^{r\theta} \\ &\quad + M_0K \left(\varepsilon^\theta(-wv'x, -wvx) + \varepsilon^\theta(-ww'x, -wwx)\right) . \sum_{r=\gamma}^{\infty} K^{r\theta} \alpha_w^{r\theta} \\ &\quad + M_0K \left(\varepsilon^\theta((Id_X - 2w)v'x, (Id_X - 2w)vx) + \varepsilon^\theta((Id_X - 2w)w'x, (Id_X - 2w)wx)\right) . \sum_{r=\gamma}^{\infty} K^{r\theta} \alpha_w^{r\theta} \\ &\leq M_0K \left(\varepsilon^\theta(v'x, vx) + \varepsilon^\theta(w'x, wx)\right) \left(2\lambda^\theta(w') + \lambda^\theta(w) + \lambda^\theta(-w) + \lambda^\theta(Id_x - 2w)\right) . \sum_{r=j}^{\infty} K^{r\theta} \alpha_w^{r\theta} \\ &\leq M_0K \left(\varepsilon^\theta(v'x, vx) + \varepsilon^\theta(w'x, wx)\right) \left(2\lambda(w') + \lambda(w) + \lambda(-w) + \lambda(Id_x - 2w)\right)^\theta . \sum_{r=j}^{\infty} K^{r\theta} \alpha_w^{r\theta} \\ &= M_0K(\varepsilon^\theta(v'x, vx) + \varepsilon^\theta(w'x, wx). \sum_{r=\gamma+1}^{\infty} K^{r\theta} \alpha_w^{r\theta}.\end{aligned}$$

So we have

$$\| D(x) - D_w(x) \| \leq \| D(x) - D_w(x) \|_Y^{\theta}$$
$$\leq \left(M_1 \left(\varepsilon^{\theta}(v'x, vx) + \varepsilon^{\theta}(w'x, wx) \right) . \sum_{r=\gamma+1}^{\infty} \alpha_w^{r\theta} \right)^{\theta} .$$

Hence we have (2.12). Now taking $j \to \infty$ in (2.12), we get

$$D(x) = D_w(x), \tag{2.13}$$

for all $x \in X_0$. Similarly, one can also prove that $D_u = D_w$ for each $u \in \mathcal{U}$, which yields

$$\|f(x) - D_w(x)\|_Y^{\theta} \leq 4 \frac{\varepsilon^{\theta}(u'x, ux)}{1 - K^{\theta} \alpha_u^{\theta}}$$

for all $x \in X_0$ and $u \in \mathcal{U}$. This implies (1.10) with $D = D_w$ and the uniqueness of D is given by (2.13). □

In the following theorem, we prove the hyperstability of the equation(1.11) in the Banach spaces.

Theorem 3. *Let X be a quasi-normed space and Y be a quasi-Banach space and ε be as in the above Theorem 2. Suppose that there exists a non empty set $\mathcal{U} \in l(X)$ such that $u \ o \ v = v \ o \ u$ for all $u, v \in \mathcal{U}$ and*

$$\begin{cases} \inf\limits_{u \in \mathcal{U}} \varepsilon^{\theta}(u'x, ux) & = 0 \\ \sup\limits_{u \in \mathcal{U}} \alpha_u < 1. \end{cases} \tag{2.14}$$

$x \in X_0$, then every $f : X \to Y$ satisfying (2.2) is a solution of (1.10) on X_0.

Proof. Suppose that $f : X \to Y$ be a mapping which is satisfying (2.2). Then , by the Theorem 2, there exists a mapping $D : X \to Y$, which satisfy (1.10) and $\| f(x) - D(x) \|_Y^{\theta} \leq \varepsilon^*(x)$ for all $x \in X_0$. Since, from (2.14), $\varepsilon^*(x) = 0$ for all $x \in X_0$. This implies that $f(x) = D(x)$ for all $x \in X_0$, where

$$f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y),$$

for all $x, y \in X$. Which is satisfies the functional equation (1.10) on X_0. □

From Theorem (2) and (3), we can obtain the following corollaries as natural results.

Corollary 1. *Let X be a quasi-normed space and Y be a quasi-Banach space and let $p, \ q \in \mathbb{R}, p < 0, \ q < 0$ and φ be a positive number. If $f : X \to Y$*

satisfies

$$\begin{aligned}&\| f(x+y)+f(x-y)-2f(x)-f(y)-f(-y) \|_Y \\ &\leq \varphi^\theta \left(\| x \|_X^p + \| X \|_X^q\right)^\theta\end{aligned} \tag{2.15}$$

for all $x, y \in X_0$, then f satisfies the functional equation (1.10) on X_0.

Proof. The proof follows from the above Theorem 3 by taking $\varepsilon^\theta(x,y) = \varphi^\theta \left(\| x \|_X^p + \| y \|_X^q\right)^\theta$ for all $x, y \in X_0$ with some real numbers $\varphi \geqslant 0$, $p < 0$ and $q < 0$. For each $m \in \mathbb{N}$, define u_m: $X_0 \to X_0$ by $u_m x := u_m(x) = -mx$ and u'_m:$X_0 \to X_0$ by $u'_m x := u'_m(x) = (1+m)x$. Then

$$\begin{aligned}\varepsilon^\theta(u_m x, u_k y) &= \varepsilon^\theta(-mx, -ky) \\ &= \left[\varphi\left(\| -mx \|_X^p + \| -ky \|_X^q\right)\right]^\theta \\ &= \left[\varphi m^p \| x \|_X^p + \varphi k^q \| y \|_X^q\right]^\theta \\ &= \left[(m^p + k^q)\varphi(\| x \|_X^p + \| y \|_X^q)\right]^\theta \\ &= (m^p + k^q)^\theta \varepsilon^\theta(x,y)\end{aligned}$$

for all $x, y \in X_0$ and $k, m \in \mathbb{N}$. Hence

$$\lim_{m\to\infty} \varepsilon^\theta(u'_m x, u_m y) \leq \lim_{m\to\infty} ((1+m)^p + m^q))^\theta \, \varepsilon^\theta(x,y) = 0$$

for all $x, y \in X_0$ and $k, m \in \mathbb{N}$. Then (2.14) is valid with $\lambda(u_m) = m^p + m^q$ for $m \in \mathbb{N}$, and there exists $n_0 \in \mathbb{N}$ such that $m \geq n_0$ and

$$\alpha_{u_m} = 2\left((1+m)^p + (1+m)^q\right) + 2(m^p + m^q) + (1+2m)^p + (1+2m)^q < 1.$$

Therefore we can say that (2.1) is satisfies with $\mathcal{U} := \{u_m \in Aut(X) : m \in \mathbb{N}_{n0}\}$. Hence, by the Theorem 3, every $f : X \to Y$ satisfying (2.15) is a solution of the functional equation (1.10) on X_0. □

Now, we extend the main result of Piszczek et al. [23] (Theorem 2) in the framework of quasi-Banach space.

Corollary 2. *Let X be a quasi-normed space and Y be a quasi-Banach space and let $p \in \mathbb{R}, p < 0$ and φ be a positive number. If $f : X \to Y$ satisfies*

$$\begin{aligned}&\| f(x+y)+f(x-y)-2f(x)-f(y)-(f-y) \|_Y \\ &\leq \ \varphi^\theta \left(\| x \|_X^p + \| y \|_X^p\right)^\theta\end{aligned} \tag{2.16}$$

for all $x, y \in X_0$, then f satisfies the functional equation (1.10) on X_0.

Proof. It is easily seen that the function ε given by

$$\varepsilon^\theta(x,y) = \left[\varphi\left(\| x \|_X^p + \| y \|_X^p\right)\right]^\theta$$

for all $x, y \in X_0$ satisfies (2.14), since

$$\varepsilon^\theta(mx, ky) = \left[\varphi \parallel mx \parallel_X^p + \varphi \parallel ky \parallel_X^p\right]^\theta \leq \left[\varphi(m^p + k^p)\left(\parallel x \parallel_Y^p + \parallel y \parallel_Y^p\right)\right]^\theta$$
$$= (m^p + k^p)^\theta \varepsilon^\theta(x, y)$$

foe all $x, y \in X_0, k, m \in \mathbb{N}$ and $km \neq 0$. The remaining part of the proof is similar to the Corollary 1. □

Remark 1. Piszczek et al. [23] obtained Corollary 2 in the setting of a Banach space.

If X is a normed space and $f : X \to Y$ satisfies (2.16) for $x, y \in X_0$, with $p < 0$, then by the Theorem 3 we know that f satisfies the Drygas functional equation on X_0. It is easy to see that if $f(0) = 0$, then f satisfies the Drygas functional equation on X. So we have the following corollary.

Corollary 3. *Let X be a quasi-normed space and Y be a quasi-Banach space and let $p \in \mathbb{R}, p < 0$ and φ be a positive number. If $f : X \to Y$ satisfies $f(0) = 0$ and inequality*

$$\begin{aligned} &\parallel f(x+y) + f(x-y-2f(x) - f(y) - f(-y) \parallel_Y \\ &\leq \ \varphi^\theta\left(\parallel x \parallel_X^p + \parallel y \parallel_X^p\right)^\theta \end{aligned} \tag{2.17}$$

for all $x, y \in X_0$, then f satisfies the functional equation (1.10) on X_0.

Corollary 4. *Let X be a quasi-normed space and Y be a quasi-Banach space and let $p,\ q \in \mathbb{R},\ p+q < 0$ and φ be a positive number. If $f : X \to Y$ satisfies*

$$\begin{aligned} &\parallel f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y) \parallel_Y \\ &\leq \ \varphi^\theta\left(\parallel x \parallel_X^p \parallel y \parallel_X^q\right)^\theta \end{aligned} \tag{2.18}$$

for all $x, y \in X_0$, then f satisfies the functional equation (1.10) on X_0.

Proof. It is easily seen that the function ε given by

$$\varepsilon^\theta(x, y) = \left(\varphi\left(\parallel x \parallel_X^p \parallel y \parallel_X^q\right)\right)^\theta$$

for all $x, y \in X_0$ satisfies (2.12), since

$$\varepsilon^\theta(mx, ky) = \varphi^\theta\left(\parallel mx \parallel_X^p \parallel ky \parallel_X^q\right)^\theta \leq \varphi^\theta(m^p k^q)^\theta\left(\parallel x \parallel_X^p \parallel y \parallel_X^q\right)^\theta$$
$$= (m^p k^q)^\theta \varepsilon^\theta(x, y)$$

for all $x, y \in X_0$, $k, m \in \mathbb{N}$ and $km \neq 0$. The remainder of the proof is similar to the Corollary 1. □

By an analogous conclusion, the function ε given by

$$\varepsilon^\theta(x, y) = \varphi^\theta\left(\parallel x \parallel_X^p + \parallel y \parallel_X^q + \parallel x \parallel_X^p \parallel y \parallel_X^q\right)^\theta$$

for all $x, y \in X_0$ satisfies (2.12), since

$$\begin{aligned}\varepsilon^{\theta}(mx, ky) &= \varphi^{\theta}\left(\| mx \|_X^p + \| ky \|_X^q + \| mx \|_X^p \| ky \|_X^q\right)^{\theta} \\ &= \varphi^{\theta}\left(m^p \| x \|_X^p + k^q \| y \|_X^q + m^p k^q \| x \|_X^p \| y \|_X^q\right)^{\theta} \\ &\leq \left(m^p + k^q + m^p k^q\right)^{\theta} \varepsilon^{\theta}(x, y)\end{aligned}$$

for all $x, y \in X_0, k, m \in \mathbb{N}$ and $km \neq 0$. So we have the following corollary.

Corollary 5. *Let X be a complex quasi-normed space and Y be a quasi-Banach space and let $p,\ q \in \mathbb{R},\ p < 0,\ q < 0,\ p + q < 0$ and φ be a positive number. If $f : X \to Y$ satisfies*

$$\begin{aligned}&\| f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y) \|_Y \\ &\leq\ \varphi^{\theta}\left(\| x \|_X^p + \| y \|_X^q + \| x \|_X^p \| y \|_X^q\right)^{\theta}\end{aligned} \tag{2.19}$$

for all $x, y \in X_0$, then f satisfies the functional equation (1.10) on X_0.

The following result corresponds to the results on the nonhomogeneous Drygas functional equation (2.20).

Corollary 6. *Let X be a quasi-normed space and Y be a quasi-Banach space and ε as in Theorem 2 and $H : X^2 \to Y$. Suppose that $\| H(x, y) \|_Y^{\theta} \leq \varepsilon^{\theta}(x, y)$ for all $x, y \in X_0$, where $H(x_0, y_0) \neq 0$ for some $x_0, y_0 \in X_0$, and there exists a nonempty $\mathcal{U} \in l(X)$ such that (2.3) and (2.14) satisfies. Then the non homogeneous equation*

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) + f(-y) + H(x, y) \tag{2.20}$$

for all $x, y \in X_0$, has no solution in the class of functions $f : X \to Y$.

Proof. Let us assume the $f : X \to Y$ is a solution to (2.20). Then

$$\begin{aligned}&\| f(x + y) + f(x - y) - 2f(x) - 2f(y) - f(-y) \|_Y^{\theta} \\ &= \| 2f(x) + f(y) + f(-y) + H(x, y) - 2f(x) - f(y) - f(-y) \|_Y^{\theta} \\ &= \| H(x, y) \|_Y^{\theta} \\ &\leq \varepsilon^{\theta}(x, y),\end{aligned}$$

for all $x, y \in X_0$. Consequently, by Theorem 3, f is a solution of (1.10). Therefore, we have

$$H(x_0, y_0) = f(x_0 + y_0) + f(x_0 - y_0) - 2f(x_0) + f(y_0) - f(-y_0) = 0,$$

which is contradiction. Hence the result. □

Remark 2. If X is normed space and Y is Banach space and $K = 1$ in Theorem 2, we obtain the corresponding results of Sirouni et al. [28].

Question. Prove or disprove the conclusion of Theorem 2 in the case Y is a normed space.

Conflict of Interest. The authors declare that they have not any conflict of interest.

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