New exact solitary wave solutions, bifurcation analysis and first order conserved quantities of resonance nonlinear Shrödinger's equation with Kerr law nonlinearity

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Abstract

The paper investigates exact solutions of the resonant nonlinear Shrödinger's equation (R-NLSE) with Kerr law nonlinearity by using the extended direct algebraic method. Graphs of some obtained solutions are presented with different values of parameters to describe their

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propagation. In order to understand the bifurcation structure of nonlinear and super-nonlinear travelling wave solutions of the considered equation, bifurcation analysis has been practiced. Moreover, a set of non-trivial and first-order conserved quantities are computed by multiplier approach.

Keywords: Shrödinger's equation; Bifurcation theory; Conservation laws.

1 Introduction

The nonlinear Shrödinger's equation (NLSE) plays an important role in different branches of science. Nonlinear optics, plasma physics, quantum mechanics and fluid dynamics are one of the fields where NLSE is appeared. The R-NLSE is utilized in the study of dynamics of solitons and Madelung fluids in many nonlinear systems ([3] and references therein):

$$i\Phi_t + \alpha\Phi_{xx} + \beta\Omega(|\Phi|^2)\Phi + \gamma\left(\frac{|\Phi|_{xx}}{|\Phi|}\right)\Phi = 0, \quad i = \sqrt{-1}. \tag{1}$$

In Eq. (1), non-dimensional distance along the fiber and temporal variable is described by independent variables x and t respectively. The complex valued dependent variable $\Phi(t,x)$ represents wave profile while constants α, β and γ are the coefficients of group-velocity dispersion, non-Kerr nonlinearity and resonant nonlinearity respectively.

In Eq. (1), real valued and n-times differentiable function Ω is defined as:

$$\Omega(|\Phi|^2)\Phi : \mathbb{C} \to \mathbb{C}.$$

where $\mathbb{C}$ represents the complex plane as a two dimensional linear space of $\mathbb{R}^2$. Here we are assuming $\Omega(y) = y$ which arises in nonlinear fiber optics or water waves when refractive index is proportional to intensity [6].

The R-NLSE equation has been discussed by many means in literature. In recent past, researchers computed the exact solutions of Eq. (1) by using different approaches. In [8], authors compute the exact solutions of Eq. (1) by using Jacobi elliptic approach and simplest equation technique. Baleanu et. al. [3] investigated the soliton solutions of the considered equation by using Ricatti-Bernoulli sub-ODE method. One of the objectives of this research is to find a new set of exact solutions. For this, the extended direct algebraic method is used to find the exact solutions of Eq. (1). In recent years, study of differential equations by means of bifurcation analysis is a hot topic of research. Dubinov et. al. in [10] characterized a new class of nonlinear and super-nonlinear waves. In [20], a class of solutions is obtained for Klein-Gordon-Zakharov equations by using bifurcation theory. More recently, Sharma-Tasso-Olver (STO) equation is dealt by means of bifurcation theory [1] and a complete classification of waves is presented. By the best of our knowledge, any study related to behaviour of nonlinear and super-nonlinear travelling waves for R-NLSE equation is not done before. Thus a depth analysis of Eq. (1) along these lines is interesting and is presented here.

Conservation laws have a vital role to play in solving differential equations and in many applications. The idea of conservation laws came from the conception of physical laws like mass, energy, and momentum. The use of Noether's theorem [5] proved by German mathematician Emmy Noether is a systematic method for finding conservation laws. Noether theorem states that each Euler - Lagrange equation's Noether symmetry corresponds to a difference equation's conservation law. Noether's theorem works only for variational differential equation, yet there are differential equations which have no Lagrangian equations which can be dealt with different approaches available in literature, some of them are [2, 11, 12] while computer package for construction of conserved quantities by using [2] is also developed [9] and utilized in this research. Here we search out the first order nontrivial conserved quantities of Eq. (1) by using multiplier approach [2]. Some latest works related to exact solutions and conservation laws are given in [4, 7, 13, 14, 16]

Paper is organized as: Section 2 is devoted for the exact solutions of Eq. (1) by using the extended direct algebraic method. Bifurcation analysis of Eq. (1) is presented in Section 3. Section 4 is devoted for conserved quantities, while conclusion is stated at the end.

2 Travelling wave solutions

2.1 Description of method

Suppose that a nonlinear partial differential equation:

$$Q(\Phi, \Phi_t, \Phi_x, \Phi_y, \Phi_{tt}, \Phi_{xx}, ...) = 0, \tag{2}$$

can be reduced into nonlinear ordinary differential equation:

$$G(u, u', u'', ...) = 0, \tag{3}$$

using the complex transformation

$$\Phi(t, x) = u(\xi)e^{iv(t,x)}, \tag{4}$$

where $\xi = K(x+st)$ and $v(t,x) = -kx + \omega t + \theta$ and here prime in Eq. (3) shows the derivative with respect to ξ.

Suppose that Eq. (3) has the solution of the form

$$u(\xi) = a_0 + \sum_{i=1}^{m} \Big[a_i W(\xi) \Big], \tag{5}$$

where

$$W'(\xi) = ln(\rho)\left(\mu + \nu W(\xi) + \zeta W^2(\xi)\right), \quad \rho \neq 0, 1, \tag{6}$$

where μ, ν and ζ are real constants.

The general solutions of Eq. (6) with respect to parameters μ, ν and ζ are given below [18]:

1): When $\nu^2 - 4\mu\zeta < 0$ and $\zeta \neq 0$,

$$W_1(\xi) = -\frac{\nu}{2\zeta} + \frac{\sqrt{-(\nu^2 - 4\mu\zeta)}}{2\zeta} \tan_\rho\left(\frac{\sqrt{-(\nu^2 - 4\mu\zeta)}}{2}\xi\right), \tag{7}$$

$$W_2(\xi) = -\frac{\nu}{2\zeta} - \frac{\sqrt{-(\nu^2 - 4\mu\zeta)}}{2\zeta} \cot_\rho\left(\frac{\sqrt{-(\nu^2 - 4\mu\zeta)}}{2}\xi\right), \tag{8}$$

$$W_3(\xi) = -\frac{\nu}{2\zeta} + \frac{\sqrt{-(\nu^2 - 4\mu\zeta)}}{2\zeta} \left(\tan_\rho\left(\sqrt{-(\nu^2 - 4\mu\zeta)}\xi\right) \pm \sqrt{mn}\sec_\rho\left(\sqrt{-(\nu^2 - 4\mu\zeta)}\xi\right)\right), \tag{9}$$

$$W_4(\xi) = -\frac{\nu}{2\zeta} + \frac{\sqrt{-(\nu^2 - 4\mu\zeta)}}{2\zeta} \left(\cot_\rho\left(\sqrt{-(\nu^2 - 4\mu\zeta)}\xi\right) \pm \sqrt{mn}\csc_\rho\left(\sqrt{-(\nu^2 - 4\mu\zeta)}\xi\right)\right), \tag{10}$$

$$W_5(\xi) = -\frac{\nu}{2\zeta} + \frac{\sqrt{-(\nu^2 - 4\mu\zeta)}}{4\zeta} \left(\tan_\rho\left(\frac{\sqrt{-(\nu^2 - 4\mu\zeta)}}{4}\xi\right) - \cot_\rho\left(\frac{\sqrt{-(\nu^2 - 4\mu\zeta)}}{4}\xi\right)\right). \tag{11}$$

2): When $\nu^2 - 4\mu\zeta > 0$ and $\zeta \neq 0$,

$$W_6(\xi) = -\frac{\nu}{2\zeta} - \frac{\sqrt{\nu^2 - 4\mu\zeta}}{2\zeta} \tanh_\rho\left(\frac{\sqrt{\nu^2 - 4\mu\zeta}}{2}\xi\right), \tag{12}$$

$$W_7(\xi) = -\frac{\nu}{2\zeta} - \frac{\sqrt{\nu^2 - 4\mu\zeta}}{2\zeta} \coth_\rho\left(\frac{\sqrt{\nu^2 - 4\mu\zeta}}{2}\xi\right), \tag{13}$$

$$W_8(\xi) = -\frac{\nu}{2\zeta} + \frac{\sqrt{\nu^2 - 4\mu\zeta}}{2\zeta}\left(-\tanh_\rho\left(\sqrt{\nu^2 - 4\mu\zeta}\xi\right) \pm i\sqrt{mn}\,\mathrm{sech}_\rho\left(\sqrt{\nu^2 - 4\mu\zeta}\xi\right)\right), \tag{14}$$

$$W_9(\xi) = -\frac{\nu}{2\zeta} + \frac{\sqrt{\nu^2 - 4\mu\zeta}}{2\zeta}\left(-\coth_\rho\left(\sqrt{\nu^2 - 4\mu\zeta}\xi\right) \pm \sqrt{mn}\,\mathrm{csch}_\rho\left(\sqrt{\nu^2 - 4\mu\zeta}\xi\right)\right), \tag{15}$$

$$W_{10}(\xi) = -\frac{\nu}{2\zeta} - \frac{\sqrt{\nu^2 - 4\mu\zeta}}{4\zeta}\left(\tanh_\rho\left(\frac{\sqrt{\nu^2 - 4\mu\zeta}}{4}\xi\right) + \coth_\rho\left(\frac{\sqrt{\nu^2 - 4\mu\zeta}}{4}\xi\right)\right). \tag{16}$$

3): When $\mu\zeta > 0$ and $\nu = 0$,

$$W_{11}(\xi) = \sqrt{\frac{\mu}{\zeta}}\tan_\rho\left(\sqrt{\mu\zeta}\xi\right), \tag{17}$$

$$W_{12}(\xi) = -\sqrt{\frac{\mu}{\zeta}}\cot_\rho\left(\sqrt{\mu\zeta}\xi\right), \tag{18}$$

$$W_{13}(\xi) = \sqrt{\frac{\mu}{\zeta}}\left(\tan_\rho\left(2\sqrt{\mu\zeta}\xi\right) \pm \sqrt{mn}\,\sec_\rho\left(2\sqrt{\mu\zeta}\xi\right)\right), \tag{19}$$

$$W_{14}(\xi) = \sqrt{\frac{\mu}{\zeta}}\left(-\cot_\rho\left(2\sqrt{\mu\zeta}\xi\right) \pm \sqrt{mn}\,\csc_\rho\left(2\sqrt{\mu\zeta}\xi\right)\right), \tag{20}$$

$$W_{15}(\xi) = \frac{1}{2}\sqrt{\frac{\mu}{\zeta}}\left(\tan_\rho\left(\frac{\sqrt{\mu\zeta}}{2}\xi\right) - \cot_\rho\left(\frac{\sqrt{\mu\zeta}}{2}\xi\right)\right). \tag{21}$$

4): When $\mu\zeta < 0$ and $\nu = 0$,

$$W_{16}(\xi) = -\sqrt{-\frac{\mu}{\zeta}}\tanh_\rho\left(\sqrt{-\mu\zeta}\xi\right), \tag{22}$$

$$W_{17}(\xi) = -\sqrt{-\frac{\mu}{\zeta}}\coth_\rho\left(\sqrt{-\mu\zeta}\xi\right), \tag{23}$$

$$W_{18}(\xi) = \sqrt{-\frac{\mu}{\zeta}}\left(-\tanh_\rho\left(2\sqrt{-\mu\zeta}\xi\right) \pm i\sqrt{mn}\,\mathrm{sech}_\rho\left(2\sqrt{-\mu\zeta}\xi\right)\right), \tag{24}$$

$$W_{19}(\xi) = \sqrt{-\frac{\mu}{\zeta}}\left(-\coth_\rho\left(2\sqrt{-\mu\zeta}\xi\right) \pm \sqrt{mn}\,\mathrm{csch}_\rho\left(2\sqrt{-\mu\zeta}\xi\right)\right), \tag{25}$$

$$W_{20}(\xi) = -\frac{1}{2}\sqrt{-\frac{\mu}{\zeta}}\left(\tanh_\rho\left(\frac{\sqrt{-\mu\zeta}}{2}\xi\right) + \coth_\rho\left(\frac{\sqrt{-\mu\zeta}}{2}\xi\right)\right). \tag{26}$$

5): When $\nu = 0$ and $\mu = \zeta$,

$$W_{21}(\xi) = \tan_\rho(\mu\xi), \tag{27}$$

$$W_{22}(\xi) = -\cot_\rho(\mu\xi), \tag{28}$$

$$W_{23}(\xi) = \tan_\rho(2\mu\xi) \pm \sqrt{mn}\sec_\rho(2\mu\xi), \tag{29}$$

$$W_{24}(\xi) = -\cot_\rho(2\mu\xi) \pm \sqrt{mn}\csc_\rho(2\mu\xi), \tag{30}$$

$$W_{25}(\xi) = \frac{1}{2}\left(\tan_\rho\left(\frac{\mu}{2}\xi\right) - \cot_\rho\left(\frac{\mu}{2}\xi\right)\right). \tag{31}$$

6): When $\nu = 0$ and $\zeta = -\mu$,

$$W_{26}(\xi) = -\tanh_\rho(\mu\xi), \tag{32}$$

$$W_{27}(\xi) = -\coth_\rho(\mu\xi), \tag{33}$$

$$W_{28}(\xi) = -\tanh_\rho(2\mu\xi) \pm i\sqrt{mn}\,\mathrm{sech}_\rho(2\mu\xi), \tag{34}$$

$$W_{29}(\xi) = -\cot_\rho(2\mu\xi) \pm \sqrt{mn}\,\mathrm{csch}_\rho(2\mu\xi), \tag{35}$$

$$W_{30}(\xi) = -\frac{1}{2}\tanh_\rho\left(\frac{\mu}{2}\xi\right) + \cot_\rho\left(\frac{\mu}{2}\xi\right). \tag{36}$$

7): When $\nu^2 = 4\mu\zeta$,

$$W_{31}(\xi) = \frac{-2\mu(\nu\xi\ln\rho + 2)}{\nu^2\xi\ln\rho}. \tag{37}$$

8): When $\nu = p, \mu = pq, (q \neq 0)$ and $\zeta = 0$,

$$W_{32}(\xi) = \rho^{p\xi} - q. \tag{38}$$

9): When $\nu = \zeta = 0$,

$$W_{33}(\xi) = \mu\xi\ln\rho. \tag{39}$$

10): When $\nu = \mu = 0$,

$$W_{34}(\xi) = \frac{-1}{\zeta \xi \ln \rho}. \tag{40}$$

11): When $\mu = 0$ and $\nu \neq 0$,

$$W_{35}(\xi) - \frac{m\nu}{\zeta \left(\cosh_\rho (\nu\xi) - \sinh_\rho (\nu\xi) + m\right)}, \tag{41}$$

$$W_{36}(\xi) = -\frac{\nu \left(\sinh_\rho (\nu\xi) + \cosh_\rho (\nu\xi)\right)}{\zeta \left(\sinh_\rho (\nu\xi) + \cosh_\rho (\nu\xi) + n\right)}. \tag{42}$$

12): When $\nu = p, \zeta = pq,\ (q \neq 0$ and $\mu = 0)$,

$$W_{37}(\xi) = -\frac{m\rho^{p\xi}}{m - qn\rho^{p\xi}}. \tag{43}$$

$$\sinh_\rho(\xi) = \frac{m\rho^{\xi} - n\rho^{-\xi}}{2},\ \ \cosh_\rho(\xi) = \frac{m\rho^{\xi} + n\rho^{-\xi}}{2},$$

$$\tanh_\rho(\xi) = \frac{m\rho^{\xi} - n\rho^{-\xi}}{m\rho^{\xi} + n\rho^{-\xi}},\ \ \coth_\rho(\xi) = \frac{m\rho^{\xi} + n\rho^{-\xi}}{m\rho^{\xi} - n\rho^{-\xi}},$$

$$\mathrm{sech}_\rho(\xi) = \frac{2}{m\rho^{\xi} + n\rho^{-\xi}},\ \ \mathrm{csch}_\rho(\xi) = \frac{2}{m\rho^{\xi} - n\rho^{-\xi}},$$

$$\sin_\rho(\xi) = \frac{m\rho^{i\xi} - n\rho^{-i\xi}}{2i},\ \ \cos_\rho(\xi) = \frac{m\rho^{i\xi} + n\rho^{-i\xi}}{2},$$

$$\tan_\rho(\xi) = -i\frac{m\rho^{i\xi} - n\rho^{-i\xi}}{m\rho^{i\xi} + n\rho^{-i\xi}},\ \ \ \cot_\rho(\xi) = i\frac{m\rho^{i\xi} + n\rho^{-i\xi}}{m\rho^{i\xi} - n\rho^{-i\xi}},$$

$$\sec_\rho(\xi) = \frac{2}{m\rho^{\xi} + n\rho^{-\xi}},\ \ \ \csc_\rho(\xi) = \frac{2i}{m\rho^{\xi} - n\rho^{-\xi}},$$

where m and n are arbitrary constants greater than zero and called deformation parameters.

2.2 Application to Eq. (1)

The R-NLSE with Kerr law nonlinearity is considered for the calculation of the exact solutions via method purposed in [18]. For this let us substitute a complex envelope (4) in Eq. (1). After separating into real and imaginary parts, we get the following equations:

$$K^2(\gamma+\alpha)u^{''}-(k^2\alpha+\omega)u+\beta u^3=0, \quad s=2k\alpha. \tag{44}$$

After balancing the highest order derivative terms with the highest power of nonlinear terms in Eq. (44), one can take the solution of the form:

$$u=a_0+a_1W(\xi), \tag{45}$$

where $W(\xi)$ satisfies Eq. (6).

After substituting (45) in Eq. (44) and equating the coefficients of different powers of $W(\xi)$, leads to a system of following algebraic equations.

$$\begin{aligned}
&\Big(W(\xi)\Big)^0 &&: -k^2\alpha a_0-\omega a_0+\beta a_0^3+K^2\mu\nu\alpha a_1\log(\rho)^2+K^2\mu\nu\gamma a_1\log(\rho)^2=0,\\
&\Big(W(\xi)\Big)^1 &&: -k^2\alpha a_1-\omega a_1+3\beta a_0^2a_1+K^2\nu^2\alpha a_1\log(\rho)^2+2K^2\mu\zeta\alpha a_1\log(\rho)^2\\
& &&+K^2\nu^2\gamma a_1\log(\rho)^2+2K^2\mu\zeta\gamma a_1\log(\rho)^2=0,\\
&\Big(W(\xi)\Big)^2 &&: 3\beta a_0a_1^2+3K^2\nu\zeta\alpha a_1\log(\rho)^2+3K^2\nu\zeta\gamma a_1\log(\rho)^2=0,\\
&\Big(W(\xi)\Big)^3 &&: \beta a_1^3+2K^2\zeta^2\alpha a_1\log(\rho)^2+2K^2\zeta^2\gamma a_1\log(\rho)^2=0.
\end{aligned}$$

Solving above algebraic equations with the help of `Mathematica`, following set of solution is obtained:

$$a_0=\pm\frac{\nu\sqrt{\Theta}}{\beta\sqrt{\Pi}}, \quad a_1=\pm\frac{2\zeta\sqrt{\Theta}}{\beta\sqrt{\Pi}}, \quad K=\pm\frac{\sqrt{-2\Theta}}{\sqrt{\Pi(\alpha+\gamma)\log(\rho)^2}}, \tag{46}$$

where

$$\Theta=k^2\alpha+\omega, \quad \Pi=\nu^2-4\mu\zeta$$

Case 1. If $\Pi<0$ and $\zeta\neq0$, then
After substituting the values of a_0 and a_1 from (46) in Eq. (45) we get:

$$u_{1\pm}(\xi)=\mp\frac{\sqrt{-\Theta}}{\beta}\tan_\rho\left(\frac{\sqrt{-\Pi}}{2}\xi\right),$$

where $u_{1\pm}(\xi)$ with the complex transformation (4) yields the following solution of Eq. (1):

$$\Phi_{1\pm}(t,x) = \pm\frac{\sqrt{-\Theta}}{\beta}\tan_\rho\left(\frac{\sqrt{-\Pi}}{2}\xi\right)e^{i(-kx+\omega t+\theta)}.$$

Thus working on the same line following solutions are obtained.

$$\Phi_{2\pm}(t,x) = \pm\frac{\sqrt{-\Theta}}{\beta}\cot_\rho\left(\frac{\sqrt{-\Pi}}{2}\xi\right)e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{3\pm}(t,x) = \pm\frac{\sqrt{-\Theta}}{\beta}\left(\tan_\rho\left(\sqrt{-\Pi}\xi\right) \pm \sqrt{mn}\sec_\rho\left(\sqrt{-\Pi}\xi\right)\right)e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{4\pm}(t,x) = \pm\frac{\sqrt{-\Theta}}{\beta}\left(\cot_\rho\left(\sqrt{-\Pi}\xi\right) \pm \sqrt{mn}\csc_\rho\left(\sqrt{-\Pi}\xi\right)\right)e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{5\pm}(t,x) = \pm\frac{\sqrt{-\Theta}}{\beta}\left(\tan_\rho\left(\frac{\sqrt{-\Pi}}{4}\xi\right) - \cot_\rho\left(\frac{\sqrt{-\Pi}}{4}\xi\right)\right)e^{i(-kx+\omega t+\theta)}.$$

Case 2. If $\Pi > 0$ and $\zeta \neq 0$, then

$$\Phi_{6\pm}(t,x) = \mp\frac{\sqrt{\Theta}}{\beta}\tanh_\rho\left(\frac{\sqrt{\Pi}}{2}\xi\right)e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{7\pm}(t,x) = \mp\frac{\sqrt{\Theta}}{\beta}\coth_\rho\left(\frac{\sqrt{\Pi}}{2}\xi\right)e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{8\pm}(t,x) = \mp\frac{\sqrt{\Theta}}{\beta}\left(-\tanh_\rho\left(\sqrt{\Pi}\xi\right) \pm i\sqrt{mn}\,\mathrm{sech}_\rho\left(\sqrt{\Pi}\xi\right)\right)e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{9\pm}(t,x) = \mp\frac{\sqrt{\Theta}}{\beta}\left(-\coth_\rho\left(\sqrt{\Pi}\xi\right) \pm \sqrt{mn}\,\mathrm{csch}_\rho\left(\sqrt{\Pi}\xi\right)\right)e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{10\pm}(t,x) = \mp\frac{\sqrt{\Theta}}{\beta}\left(\tanh_\rho\left(\frac{\sqrt{\Pi}}{4}\xi\right) + \coth_\rho\left(\frac{\sqrt{\Pi}}{4}\xi\right)\right)e^{i(-kx+\omega t+\theta)}.$$

Case 3. If $\mu\zeta > 0$ and $\nu = 0$, then

$$\Phi_{11}(t,x) = \pm\frac{\sqrt{\Theta}}{\beta\sqrt{\Pi}}\left(\nu + 2\sqrt{\mu\zeta}\tan_\rho\left(\sqrt{\mu\zeta}\xi\right)\right)e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{12}(t,x) = \pm\frac{\sqrt{\Theta}}{\beta\sqrt{\Pi}}\left(\nu - 2\sqrt{\mu\zeta}\cot_\rho\left(\sqrt{\mu\zeta}\xi\right)\right)e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{13}(t,x) = \pm\frac{\sqrt{\Theta}}{\beta\sqrt{\Pi}}\left[\nu + 2\sqrt{\mu\zeta}\left\{\tan_\rho\left(2\sqrt{\mu\zeta}\xi\right) \pm \sqrt{mn}\sec_\rho\left(2\sqrt{\mu\zeta}\xi\right)\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{14}(t,x) = \pm\frac{\sqrt{\Theta}}{\beta\sqrt{\Pi}}\left[\nu + 2\sqrt{\mu\zeta}\left\{-\cot_\rho\left(2\sqrt{\mu\zeta}\xi\right) \pm \sqrt{mn}\csc_\rho\left(2\sqrt{\mu\zeta}\xi\right)\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{15}(t,x) = \pm\frac{\sqrt{\Theta}}{2\beta\sqrt{\Pi}}\left[\nu + 2\sqrt{\mu\zeta}\left\{\tan_\rho\left(\frac{\sqrt{\mu\zeta}}{2}\xi\right) - \cot_\rho\left(\frac{\sqrt{\mu\zeta}}{2}\xi\right)\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

Case 4. If $\mu\zeta < 0$ and $\nu = 0$, then

$$\Phi_{16}(t,x) = \pm\frac{\sqrt{\Theta}}{\beta\sqrt{\Pi}}\left\{\nu - 2\sqrt{-\mu\zeta}\tanh_\rho\left(\sqrt{-\mu\zeta}\xi\right)\right\}e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{17}(t,x) = \pm\frac{\sqrt{\Theta}}{\beta\sqrt{\Pi}}\left\{\nu - 2\sqrt{-\mu\zeta}\coth_\rho\left(\sqrt{-\mu\zeta}\xi\right)\right\}e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{18}(t,x) = \pm\frac{\sqrt{\Theta}}{\beta\sqrt{\Pi}}\left[\nu + 2\sqrt{\mu\zeta}\left\{-\tanh_\rho\left(2\sqrt{-\mu\zeta}\xi\right) \pm i\sqrt{mn}\,\mathrm{sech}_\rho\left(2\sqrt{-\mu\zeta}\xi\right)\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{19}(t,x) = \pm\frac{\sqrt{\Theta}}{\beta\sqrt{\Pi}}\left[\nu + 2\sqrt{\mu\zeta}\left\{-\coth_\rho\left(2\sqrt{-\mu\zeta}\xi\right) \pm \sqrt{mn}\,\mathrm{csch}_\rho\left(2\sqrt{-\mu\zeta}\xi\right)\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{20}(t,x) = \pm\frac{\sqrt{\Theta}}{2\beta\sqrt{\Pi}}\left[\nu - 2\sqrt{-\mu\zeta}\left\{\tanh_\rho\left(\frac{\sqrt{-\mu\zeta}}{2}\xi\right) + \coth_\rho\left(\frac{\sqrt{-\mu\zeta}}{2}\xi\right)\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

Case 5. If When $\nu = 0$ and $\mu = \zeta$, then

$$\Phi_{21}(t,x) = \pm\frac{\sqrt{\Theta}}{\beta\sqrt{\Pi}}\left[\nu + 2\zeta\tan_\rho\left(\mu\xi\right)\right]e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{22}(t,x) = \pm\frac{\sqrt{\Theta}}{\beta\sqrt{\Pi}}\left[\nu - 2\zeta\cot_\rho\left(\mu\xi\right)\right]e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{23}(t,x)=\pm\frac{\sqrt{\Theta}}{\beta\sqrt{\Pi}}\left[\nu+2\zeta\left\{\tan_\rho(2\mu\xi)\pm\sqrt{mn}\sec_\rho(2\mu\xi)\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{24}(t,x)=\pm\frac{\sqrt{\Theta}}{\beta\sqrt{\Pi}}\left[\nu+2\zeta\left\{-\cot_\rho(2\mu\xi)\pm\sqrt{mn}\csc_\rho(2\mu\xi)\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{25}(t,x)=\pm\frac{\sqrt{\Theta}}{2\beta\sqrt{\Pi}}\left[\nu+2\zeta\left\{\tan_\rho\left(\frac{\mu}{2}\xi\right)-\cot_\rho\left(\frac{\mu}{2}\xi\right)\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

Case 6. If $\nu=0$ and $\zeta=-\mu$, then

$$\Phi_{26}(t,x)=\pm\frac{\sqrt{\Theta}}{2\beta\sqrt{\Pi}}\left[\nu-2\zeta\tanh_\rho(\mu\xi)\right]e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{27}(t,x)=\pm\frac{\sqrt{\Theta}}{2\beta\sqrt{\Pi}}\left[\nu-2\zeta\coth_\rho(\mu\xi)\right],$$

$$\Phi_{28}(t,x)=\pm\frac{\sqrt{\Theta}}{2\beta\sqrt{\Pi}}\left[\nu+2\zeta\left\{-\tanh_\rho(2\mu\xi)\pm i\sqrt{mn}\,\mathrm{sech}_\rho(2\mu\xi)\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{29}(t,x)=\pm\frac{\sqrt{\Theta}}{2\beta\sqrt{\Pi}}\left[\nu+2\zeta\left\{-\coth_\rho(2\mu\xi)\pm\sqrt{mn}\,\mathrm{csch}_\rho(2\mu\xi)\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{30}(t,x)=\mp\frac{\sqrt{\Theta}}{4\beta\sqrt{\Pi}}\left[\nu+2\zeta\left\{\tanh_\rho\left(\frac{\mu}{2}\xi\right)+\coth_\rho\left(\frac{\mu}{2}\xi\right)\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

Case 7. If $\nu^2=4\mu\zeta$, then

$$\Phi_{31}(t,x)=\pm\frac{\sqrt{\Theta}}{2\beta\sqrt{\Pi}}\left[\nu+2\zeta\left\{\frac{-2\mu(\nu\xi\ln\rho+2)}{\nu^2\xi\ln\rho}\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

Case 8. If $\nu=p,\mu=pq,(q\neq 0)$ and $\zeta=0$, then

$$\Phi_{32}(t,x)=\pm\frac{\sqrt{\Theta}}{2\beta\sqrt{\Pi}}\left[\nu+2\zeta\left\{\rho^{p\xi}-q\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

Case 9. If $\nu=\zeta=0$, then

$$\Phi_{33}(t,x)=\pm\frac{\sqrt{\Theta}}{2\beta\sqrt{\Pi}}\left[\nu+2\zeta\mu\xi\ln\rho\right]e^{i(-kx+\omega t+\theta)}.$$

Case 10. If $\nu = \mu = 0$, then

$$\Phi_{34}(t,x) = \pm\frac{\sqrt{\Theta}}{2\beta\sqrt{\Pi}}\left[\nu + 2\zeta\left\{\frac{-1}{\zeta\xi\ln\rho}\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

Case 11. If $\mu = 0$ and $\nu \neq 0$ then

$$\Phi_{35}(t,x) = \pm\frac{\sqrt{\Theta}}{2\beta\sqrt{\Pi}}\left[\nu + 2\zeta\left\{-\frac{m\nu}{\zeta\left(\cosh_\rho(\nu\xi) - \sinh_\rho(\nu\xi) + m\right)}\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

$$\Phi_{36}(t,x) = \pm\frac{\sqrt{\Theta}}{2\beta\sqrt{\Pi}}\left[\nu + 2\zeta\left\{-\frac{\nu\left(\sinh_\rho(\nu\xi) + \cosh_\rho(\nu\xi)\right)}{\zeta\left(\sinh_\rho(\nu\xi) + \cosh_\rho(\nu\xi) + n\right)}\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

Case 12. If $\nu = p, \zeta = pq,\ (q \neq 0$ and $\mu = 0)$ then

$$\Phi_{37}(t,x) = \pm\frac{\sqrt{\Theta}}{2\beta\sqrt{\Pi}}\left[\nu + 2\zeta\left\{-\frac{m\rho^{p\xi}}{m - qn\rho^{p\xi}}\right\}\right]e^{i(-kx+\omega t+\theta)}.$$

3D-graphics, 2D-graphics and contour plots of different solutions $\mid \Phi_i \mid$ of Eq. (1) for $m = 2, n = 3, \rho = 3, \Pi = 1, \alpha = 0.5, \gamma = 0.5, \Theta = 1, \beta = 1, s = 0.2$ are presented to describe their behaviour in Fig (1-6).

3 Bifurcations behavior and phase portraits

By the theory of planar dynamical systems, equilibrium point (u_q, z_q) is called saddle point if $J < 0$, a center for $J > 0$ and $T_1 = 0$, a node if $J > 0$ and $T_1^2 - 4J > 0$ while a zero point when $J = 0$ and Poincaré index of (u_q, z_q) is zero. Where J and T_1 represent the Jacobian and trace of the coefficient matrix for the linearized system of (47).

For classification of different orbits in the phase portraits of dynamical system (47), following notations will be used:

(1): Supernonlinear periodic orbit is presented by SNPO(e,s),

(2): Nonlinear homoclinic orbit is presented by NHO(e,s),

(3): Nonlinear heteroclinic orbit is presented by NHTO(e,s),

(4): Nonlinear periodic orbit is presented by NPO(e,s),

where 'e' is the number of equilibrium points and 's' is the number of seperatix layers enveloped by the orbit. It is well known that each phase orbit is a closed non-self intersecting curve on the phase plane. Phase portrait of dynamical system is a set of such nested phase trajectories.

Eq. (44) can be written as a system of nonlinear dynamical equations:

$$\frac{\partial u}{\partial \xi} = z,$$

$$\frac{\partial z}{\partial \xi} = \frac{(k^2\alpha + \omega)u}{K^2(\gamma + \alpha)} + \frac{\beta u^3}{K^2(\gamma + \alpha)}. \tag{47}$$

The system (47) is a planar Hamiltonian system. Hamiltonian function can be obtained by integrating the system (47):

$$H(u, z) = \frac{z^2}{2} - \frac{(k^2\alpha + \omega)u^2}{2K^2(\gamma + \alpha)} - \frac{\beta u^4}{4K^2(\gamma + \alpha)} = h. \tag{48}$$

From (48), it can be verified that:

$$\frac{\partial u}{\partial \xi} = \frac{\partial H}{\partial z} \text{ and } \frac{\partial z}{\partial \xi} = -\frac{\partial H}{\partial u}. \tag{49}$$

As system (47) is a planar Hamiltonian system and from (49), one can conclude that system (47) is conservative and thus phase orbits defined by the vector fields of (47) will posses all travelling wave solution of Eq. (44) (for detail see [15] and references therein).
Level curves $L_h(u, z)$ with respect to energy level h can be defined in following fashion:

$$L_h = \{(u, z) \in R \times R : H(u, z) = h\},$$

where $H(u, z)$ is defined in (48) and h is called the energy level. In the phase portraits against each energy level h one can have an orbit. In order to investigate the relations between energy level and closed orbits of system (47), let us define:

$$E_h(u) = h + \frac{(k^2\alpha + \omega)u^2}{2K^2(\gamma + \alpha)} + \frac{\beta u^4}{4K^2(\gamma + \alpha)}. \tag{50}$$

From (48), one can easily find the following relation:

$$z = \pm\sqrt{2h + \frac{(k^2\alpha + \omega)u^2}{K^2(\gamma + \alpha)} + \frac{\beta u^4}{2K^2(\gamma + \alpha)}}, \tag{51}$$

which means $\frac{z^2}{2} = E_h(u)$. Graphical illustration of (50) is given in Fig. 9(a-b).

Here all possible phase trajectories of dynamical system (47) are presented and classified. System (47) has three equilibrium points:

$$u_1 = (0, 0), \quad u_2 = (\sqrt{\frac{(k^2\alpha + \omega)}{\beta}}, 0), \quad u_3 = (-\sqrt{\frac{(k^2\alpha + \omega)}{\beta}}, 0).$$

The coefficient matrix of the linearized system (47) at an equilibrium point (u_q, z_q) is:

$$M = \begin{pmatrix} 0 & 1 \\ \frac{(k^2\alpha+\omega)}{K^2(\gamma+\alpha)} & 0 \end{pmatrix}, \tag{52}$$

while Jacobian for the system (47) is:

$$J = \begin{pmatrix} 0 & 1 \\ \frac{(k^2\alpha+\omega)}{K^2(\gamma+\alpha)} + \frac{3\beta u^2}{K^2(\gamma+\alpha)} & 0 \end{pmatrix}. \tag{53}$$

It yields the following cases:

3.0.1 $K^2(\gamma+\alpha) > 0, (-k^2\alpha-\omega) < 0, C > 0$ or $K^2(\gamma+\alpha) < 0, (-k^2\alpha-\omega) > 0, \beta < 0$

For this case, the system (47) has three equilibrium points u_1, u_2, and u_3. For this $J(u_1) < 0, J(u_2) > 0$, $J(u_3) > 0$ while $T_1(M(u_2)) = 0$ and $T_1(M(u_3)) = 0$. Above information helps to claim that u_1 is a saddle point and u_2, u_3 are center points (see Fig. 7(a)).
For this case, the phase portraits of nonlinear dynamical system (47) is presented in Fig. 10(a). This phase portrait contains a family of SNPO(3,1), where family of SNPO(3,1) carries all three equilibrium points of the considered dynamical system. It also carries two families of NPO(1,0), which accommodates u_2 and u_3. There is also a pair of NHO(1,0) at u_1 which carries u_2 and u_3.

3.0.2 $K^2(\gamma+\alpha) < 0, (-k^2\alpha-\omega) > 0, \beta > 0$.

For this case, the system (47) has one equilibrium points u_1, where $J(u_1) < 0$ thus u_1 is a saddle point. (see Fig. 7(b))

3.0.3 $K^2(\gamma+\alpha) > 0, (-k^2\alpha-\omega) > 0, C < 0$. or $K^2(\gamma+\alpha) < 0, (-k^2\alpha-\omega) < 0, \beta > 0$.

For this case, the system (47) has three equilibrium points u_1, u_2, and u_3. For this $J(u_1) > 0$ and $T_1(M(u_1)) = 0$, so u_1 is a center point, while $J(u_2) > 0$ and $J(u_3) > 0$ with Poincarè index is zero thus u_2, u_3 are cusp points, (see Fig. 8(a)). The phase portraits of the system of nonlinear ODEs (47) is given in Fig. 11(a). This phase portrait carries a family of NPO(1,0) which envelops u_1.

3.0.4 $K^2(\gamma+\alpha) > 0, (-k^2\alpha-\omega) > 0, \beta > 0$.

For this case, the system (47) has one equilibrium point u_1. For which $J(u_1) > 0$ and $T_1(M(u_1)) = 0$, thus u_1 is a center point. The phase portraits for this case is presented in Fig. 8(b), which shows that there is a family of NPO(1,0) which carries u_1.

4 Conserved quantities

In this section, nontrivial conserved quantities are computed by using the method given by Anco and Bluman [2]. They advocated a systematic approach to built non-trivial conservation laws.

4.1 Multiplier approach

A system containing two partial differential equations of second order with two dependent variables $\Psi = (u, v)$ and three independent variables $\chi = (t, x)$ is denoted by

$$\begin{aligned} R_1[\Psi] &= F_1\left(\chi, \Psi, \Psi_\chi, ..., \Psi_{\chi\chi}\right), \\ R_2[\Psi] &= F_2\left(\chi, \Psi, \Psi_\chi, ..., \Psi_{\chi\chi}\right), \end{aligned} \tag{54}$$

where Ψ_χ and $\Psi_{\chi\chi}$ stand for the first and second order partial derivatives of dependent variables with respect to independent variables, respectively. Let $U = (U^1, U^2)$ represents the set of arbitrary functions of independent variable χ, U_χ and $U_{\chi\chi}$ etc.

A set of multipliers (factors, characteristics) $\Lambda = (\Lambda_1, \Lambda_2)$ yields a divergence expression for the system given in Eq. (54) if the identity

$$\Lambda_1[U]R_1[\Psi] + \Lambda_2[U]R_2[\Psi] = D_\chi C^\chi[U] \tag{55}$$

holds for arbitrary function $U(\chi)$. In Eq. (55), T^χ are called the conserved densities (fluxes) while D_χ is the total derivative:

$$D_\chi = \frac{\partial}{\partial \chi} + \Psi_\chi \frac{\partial}{\partial \Psi_\chi} + \dots. \tag{56}$$

If $U = (U^1, U^2)$ is the solution of Eq. (54), form Eq. (55), one can derive the local conserved quantity by using the following equation

$$D_\chi T^\chi[\Psi] = 0. \tag{57}$$

System (54) contains the set of multipliers for the conserved quantities if and only if following identity holds:

$$\frac{\delta}{\delta \Psi}\Big(\Lambda_1[U]R_1[\Psi] + \Lambda_2[U]R_2[\Psi]\Big) = 0, \tag{58}$$

where $\frac{\delta}{\delta U}$ is said to be Euler operators and defined as:

$$\frac{\delta}{\delta \Psi} = \frac{\partial}{\partial \Psi} - D_\chi \frac{\partial}{\partial \Psi_\chi} + \dots. \tag{59}$$

Eq. (58) leads to a set of over-determined system of determining equations in terms of multipliers $\Lambda = (\Lambda_1, \Lambda_2)$. Solution of the obtained determining equations with some computation further gives the conserved quantities. In this section, first order nontrivial conserved quantities are computed by using the method given by Anco and Bluman [2]. They advocated a systematic approach to built non-trivial conservation laws. In this method, multipliers Λ of specific order for considered problem is required, which is further used to get their corresponding fluxes Υ. Each set of multipliers and fluxes produces a local conservation law $D\Upsilon = 0$ holding for all solution of the considered differential equation.

4.2 Conserved quantities

In this section, conserved quantities of Eq. (1) are computed [2, 9].
Eq. (44) with complex envelope:

$$\Phi(t,x) = u(t,x)e^{iv(t,x)} \tag{60}$$

converts into a complex partial differential equation, after splitting into real and imaginary parts it yields:

$$\beta u^3 - uv_t - \alpha u(v_x)^2 + (\alpha+\gamma)u_{xx} = 0, \quad u_t + 2\alpha u_x v_x + \alpha u v_{xx} = 0. \tag{61}$$

Substituting system (61) in Eq. (58) gives:

$$\frac{\delta}{\delta\Psi}\left[\Lambda_1(\beta u - uv_t - \alpha u(v_x)^2 + (\alpha+\gamma)u_{xx}) + \Lambda_2(u_t + 2\alpha u_x v_x + \alpha u v_{xx})\right] = 0. \tag{62}$$

Equating the coefficients of the derivatives of dependent variables with respect to independent variables in Eq. (62), we get linear homogeneous over-determined system of partial differential equations. After solving the obtained system of partial differential equations for

$$\Lambda_1 = \Lambda_1(t,x,u,v,u_t,v_t,u_x,v_x) \text{ and } \Lambda_2 = \Lambda_2(t,x,u,v,u_t,v_t,u_x,v_x)$$

with the help of Maple we get the following results:

$$\Lambda_1 = (c_1 t + c_3)u_x + c_2 u_t, \quad \Lambda_2 = \left(c_1 v_x t - \frac{c_1 x}{2\alpha} + c_2 v_t + c_3 v_x + c_4\right)u. \tag{63}$$

Next step is to find the fluxes by using the multiplier given in (63). For instance, the multipliers Λ_1 and Λ_2 for the constants c_i give the following conservation laws:

(i): $\Lambda_1^1 = tu_x, \quad \Lambda_2^1 = (v_x t - \frac{x}{2\alpha})u$

$$T_t^1 = \frac{u^2}{4\alpha}(2\alpha t v_x - x), \quad T_x^1 = \frac{1}{4}\left(\beta t u^4 - 2tv_t u^2 + 2\alpha t u^2 v_x^2 + 2\alpha t u_x^2 + 2\gamma t u_x^2 - 2v_x x u^2\right). \tag{64}$$

(ii): $\Lambda_1^2 = u_t, \quad \Lambda_2^2 = v_t u$

$$T_t^1 = \frac{1}{4}\left(\beta u^4 - 2\alpha u^2 v_x^2 - 2u_x^2\alpha - 2u_x^2\gamma\right), \quad T_x^1 = \alpha u^2 v_t v_x + \alpha u_t u_x + \gamma u_t u_x. \tag{65}$$

(iii): $\Lambda_1^3 = u_x, \quad \Lambda_2^3 = v_x u$

$$T_t^1 = \frac{u^2 v_x}{2}, \quad T_x^1 = \frac{1}{4}\left(\beta u^4 - 2v_t u^2 + 2\alpha u^2 v_x^2 + 2u_x^2\alpha + 2u_x^2\gamma\right). \tag{66}$$

(iv): $\Lambda_1^4 = 0, \quad \Lambda_2^4 = u$

$$T_t^1 = \frac{u^2}{2}, \qquad T_x^1 = v_x\alpha u^2. \tag{67}$$

4.3 Conclusion

To be brief, he extended direct algebraic method [18] is applied to find the exact solutions of the resonant non-linear Schrödinger equation with Kerr law nonlinearity. The proposed method gave a class of solutions which may be worthwhile for the explaining certain physical phenomena accurately. Moreover, physical composition of these solutions are described via their graphical presentation. Four portraits of dynamical system (47) are obtained and existence of the travelling wave solutions is discussed as well. Further, all possible cases of the system parameters are considered by using the phase portraits and the effect of different situations are shown in detail. Moreover, nontrivial, first order and new conserved quantities are given by using multiplier approach.

Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest concerning the publication of this manuscript.

Ethical standards

The authors state that this research complies with ethical standards. This research does not involve either human participants or animals.

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