Symmetries and special solutions of a parabolic chemotaxis system

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Abstract

In this paper we consider a class of chemotaxis models with two arbitrary constitutive functions $g(u)$ and $f(v)$. After having performed a complete symmetry group classification with respect to them the reduced systems are derived. By considering $g(u)$ of the logistic form wide classes of exact solutions are found.

1 Introduction

The phenomena concerned with the motion of bacteria and other unicellular organisms due to chemical stimuli (in a *chemical substrate*) is called *chemotaxis*. These phenomena are quite common in biology and depend on the essential characteristic of living organisms to sense signals in the environment and to adapt their motion consequently. These signals allow to approach the chemically favorable environments and to avoid the unfavorable ones.

In chemotaxis the chemical signals can come from external sources or they can be secreted by the organisms themselves. Chemotaxis can be either positive (chemoattraction) when cells move to the chemical or negative (chemorepulsion) when cells move away from the chemical. Keller and Segel in some remarkable papers [1, 2, 3], concerned with studies about chemotactical features of the slime mold amoebae *Dictyostelium discoideum* and *Escherechia coli*, considered a phenomenological model from which the existence and properties of migrating bacterial bands can be deduced. Following [3] the equations of a reduced model for a population of density u, with a chemoattractant of concentration v, could be written as

$$\begin{aligned} u_t &= \nabla \cdot (\mu(v)\nabla u - u\chi(v)\nabla v), \\ v_t &= D_v \nabla^2 v - k(v)u, \end{aligned} \tag{1}$$

where $k(v)$ is the rate of consumption of the substrate per cell, D_v is the diffusion constant of the substrate while χ is called the chemotactic sensitivity, it shows how strong the cells (species) react to the chemical. In general the sign of χ depends on the type of chemotaxis: in case of chemoattraction its sign is positive, whereas in case of chemorepulsion its sign is negative. The motility parameter μ takes the place of a diffusion coefficient and, in general, it may be considered as a function of v.

Hillen and Painter on 2001 [4] modified system (1) as follows

$$\begin{aligned} u_t &= \nabla \cdot (\nabla u - V(u,v)\nabla v), \\ v_t &= D_v \nabla^2 v + h(u,v). \end{aligned} \tag{2}$$

In the first equation they put $\mu = 1$ and rewritten $u\chi$ as a more general function V depending on u and v. While, in the second equation, the function describing production and degradation of the external stimulus has been generalized. In this way the population density directly modulates its own sensitivity response. In particular in [4] they substituted the *chemotactic cross-diffusion* $V(u,v)\nabla v$ with the following $u\beta(u)\chi(v)\nabla v$ to show global existence of classical solutions.

The same authors in [5] used a probabilistic approach to arrive at the following equations

$$\begin{aligned} u_t = D_u \nabla^2 u - \nabla \cdot (u\chi(u,v)\nabla v) + g(u,v) \\ v_t = D_v \nabla^2 v + h(u,v) \end{aligned} \tag{3}$$

that are of the same form originally proposed by Patlak [6], and by Keller and Segel [2], to whom have been added cell and chemical kinetics of the form $g(u,v)$ and $h(u,v)$ respectively.

In several biologically relevant processes, when the chemical substance is produced by the individuals of the population, the function h of (3) satisfies the inequality $h_u > 0$. So, as in the classical Keller-Segel system, it is possible to consider a degradation of v and simplify the term $h(u,v)$ by the linear expression $h(u,v) = h_0 u - h_1 v$ with $h_0 > 0$, $h_1 > 0$ that without loss of generality we could normalize putting both equal to 1.

In [7] Negreanu and Tello considered a volume filling model with fast diffusion process for the chemical substance with logistic growth term

$$\begin{aligned} u_t &= \nabla^2 u - c\nabla \cdot (u(N-u)\nabla v) + \lambda u(1-u), \\ \nabla^2 v &= v - u, \end{aligned} \tag{4}$$

where $c > 0$ is a constant referred tho the chemotactic sensitivity, while as in [5] a threshold value $N > 0$ is considered. The sign change of the chemotactic cross-diffusion term characterizes the possibility of the system to evolve from positive to negative taxis or viceversa.

Negreanu and Tello in that paper proved the existence of global bounded classical solutions and analyzed the stability of the constant steady state $u = 1$, $v = 1$.

By assuming

$$V(u,v)\nabla v := cu\nabla v$$

we get the following simplified Keller-Segel system

$$\begin{aligned} u_t &= u_{xx} + cuv_{xx} - cu_x v_x, \\ 0 &= v_{xx} - \beta v + u, \end{aligned} \tag{5}$$

whose prperties have been studied in several papers see e.g.[8, 9, 10, 11, 12].

In particular in [9] and [10] Lie symmetries have been found and applied for construction of exact solutions.

Here, taking into account [5, 9, 10], we restrict ourselves to the following set of the constitutive relations

$$D_u = D_v = 1,\ V(u,v) = c(N-u),\ g(u,v) = g(u),\ h(u;v) = -f(v) + u,$$

where $f = f(v)$ and $g = g(u)$ are non-negative smooth functions. So we are able to rewrite (3) as

$$\begin{aligned} u_t &= \nabla^2 u - c(\nabla \cdot ((N-u)\nabla v)) + g(u) \\ v_t &= \nabla^2 v - f(v) + u, \end{aligned} \tag{6}$$

that in one space dimension reads:

$$\begin{aligned} &u_t = u_{xx} - c(N-u)v_{xx} + cu_x v_x + g(u) \\ &v_t = v_{xx} - f(v) + u. \end{aligned} \tag{7}$$

When growth effects in chemotaxis systems are considered to study the large time behavior, the growth term g in the first equation is defined by a logistic function and, after normalization, g has the following expression

$$g(u) = u(1-u). \tag{8}$$

In this paper we look for symmetries of system (7) whose knowledge allows to reduce the system of PDEs into a system of ordinary differential equations (ODEs) and to find exact solutions.

Lie group methods offer good tools as conservation laws, reductions, exact solutions, essentially they give a systematic way to analyze differential equations.

The structure of the work is as follows. In Section 2 we look for the Lie symmetry classification with respect to functions $f(u)$, $g(u)$ of system (7). We also derive,in Section 3 the reductions obtained from some Lie subalgebras admitted. In Section 4, by assuming the function $g(u)$ of the logistic type [16, 17, 18] together with some suitable specializations of $f(v)$, we find some exact solutions of system (7). For more general polynomial forms of $f(u)$ and $g(u)$, in Section 5, solutions of the kink type and soliton type are carried out. Conclusions are given in Section 6.

2 Lie symmetries

The search for solutions via classical Lie symmetries leads to the search for those that are invariant with respect to the same one parameter Lie groups of invariant transformations admitted by system (7). By following a well established procedure (for details the interested reader can see well known monographs e. g. [13, 14, 15]) the first step consists in determining the infinitesimal components $\xi(t,x,u,v)$, $\tau(t,x,u,v)$, $\eta(t,x,u,v)$ and $\phi(t,x,u,v)$. of transformation generator

$$\mathbf{v} = \xi\partial_x + \tau\partial_t + \eta\partial_u + \phi\partial_v. \tag{9}$$

Following the usual techniques, from the request invariance of system (7), by applying Lie infinitesimal criterion, we get a system of equations (*determining system*) to determine the infinitesimals components of $\mathbf{v}$.

After a first simplification of this one, we get that

$$\begin{aligned} \xi &= \frac{1}{2}\tau_t x + \omega(t), \quad \tau = \tau(t), \\ \eta &= \alpha(x,t)u + \beta(x,t,v), \quad \phi = \delta(x,t) \end{aligned}$$

where $\tau(t)$, $\omega(t)$, $\alpha(t,x)$, $\beta(x,t,v)$ and $\delta(x,t)$ must satisfy the following equations

$$\begin{aligned} &\beta_v\, c + \beta_{v\,v} = 0, \\ &\alpha_x\, c\, u + \beta_x\, c + 2\,\beta_{v\,x} = 0, \\ &\tau_t\, u + \alpha\, u - f\,\tau_t - \delta\, f_v - \delta_t + \delta_{x\,x} + \beta = 0, \\ &\frac{\tau_{t\,t}\, x}{2} + \omega_t + c\,\delta_x + 2\,\alpha_x = 0, \\ &\frac{\tau_{t\,t}\, x}{2} + \omega_t = 0, \\ &c(\alpha\, N + \beta) = 0, \\ &\alpha\, c\, u\, N - \alpha\, c\, f\, N + c\,\delta_{x\,x}\, N - \alpha\, g_u\, u - c\,\delta_{x\,x}\, u + \beta\, c\, u + \beta_v\, u - \alpha_{x\,x}\, u + \\ &\qquad \alpha_t\, u - g\,\tau_t - \beta\, g_u + \alpha\, g - \beta\, c\, f - \beta_v\, f - \beta_{x\,x} + \beta_t = 0. \end{aligned} \tag{10}$$

A further analysis allows to ascertain that

$$\xi = \frac{k_1\, x}{2} + k_3, \qquad \tau = k_1 t + k_2, \qquad \eta = k_1(N-u), \qquad \phi = \gamma(t),$$

with k_2, k_3 arbitrary constants, while the function $\gamma = \gamma(t)$ and the constant k_1 are related to the constitutive functions $f = f(v)$, and $g = g(u)$ by the following conditions

$$k_1\, N - f\, k_1 - \gamma_t - f_v\,\gamma = 0, \qquad k_1\,(g_u\, N - g_u\, u + 2\, g) = 0. \tag{11}$$

If f and g are arbitrary functions we obtain

$$\xi = k_3, \quad \tau = k_2, \quad \eta = 0, \quad \phi = 0, \tag{12}$$

and the only symmetries admitted by (7) are defined by the group of space and time translations,

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t,$$

and they constitute the principal Lie algebra $\mathcal{L}_{\mathcal{P}}$ of system (7). Moreover by discussing the classifying system (11) we get additional extensions of the principal Lie algebra and a classification with respect to the functions f and g.

The picture of the results is summarized in Table 1.

By considering the invariance generators listed in Table 2 and following the well known procedures we find the corresponding similarity variables and the structure of the invariant solutions so that we write the corresponding ODE reduced systems.

i	f	g	$\mathbf{v}_3^i$	$\mathbf{v}_4^i$
1	arbitrary	arbitrary		
2	α	arbitrary	∂_v	
3	$\alpha\, v + f_0,\ \alpha \neq 0$	arbitrary	$\exp(-\alpha\, t)\ \partial_v$	
4	α	$k\,(N-u)^2$	$x\partial_x + 2t\partial_t + 2(N-u)\partial_u + 2(N-\alpha)\, t\partial_v$	∂_v
5	$N + \alpha \exp(\beta\, v),$	$k(N-u)^2$	$x\partial_x + 2t\partial_t + 2(N-u)\partial_u - \dfrac{2}{\beta}\partial_v$	

Table 1: Symmetry classification for system (7) with $c \neq 0$. Here k, α, β, f_0 are constitutive constants. Of course in this table appear only the extensions of $\mathcal{L}_{\mathcal{P}}$

i	$\mathbf{v}^i$	z	u	v
1	$\lambda\mathbf{v}_1 + \mu\mathbf{v}_2$	$\mu x - \lambda t$	$H(z)$	$F(z)$
2	$\lambda\mathbf{v}_1 + \mu\mathbf{v}_2 + \mathbf{v}_3^2$	$\mu x - \lambda t$	$H(z)$	$\frac{x}{\lambda} + F(z)$
3	$\lambda\mathbf{v}_1 + \mu\mathbf{v}_2 + \mathbf{v}_3^3$	$\mu x - \lambda t$	$H(z)$	$F(z) - \frac{e^{-\alpha\, t}}{\alpha\, \mu}$
4	$\mathbf{v}_3^4$	$\frac{x}{\sqrt{t}}$	$\frac{H(z)}{t} + N$	$F(z) + (N-\alpha)\, t$
5	$\mathbf{v}_3^5$	$\frac{x}{\sqrt{t}}$	$\frac{H(z)}{t} + N$	$F(z) - \frac{\ln\ t}{\beta}$

Table 2: Similarity variables and similarity solutions

i	ODE'$_i$
1	$H'\,\lambda - c\,\mu^2\,F''\,N + \mu^2\,H'' + c\,\mu^2\,F'\,H' + c\,\mu^2\,F''\,H + g = 0$ $f - F'\,\lambda - H - \mu^2\,F'' = 0.$
2	$H'\,\lambda + \frac{c\,\mu\,H'}{\lambda} - c\,\mu^2\,F''\,N + \mu^2\,H'' + c\,\mu^2\,F'\,H' + c\,\mu^2\,F''\,H + g = 0,$ $-F'\,\lambda - H - \mu^2\,F'' + \alpha = 0.$
3	$H'\,\lambda - c\,\mu^2\,F''\,N + \mu^2\,H'' + c\,\mu^2\,F'\,H' + c\,\mu^2\,F''\,H + g = 0,$ $-F'\,\lambda - H - \mu^2\,F'' + \alpha\,F + f_0 = 0.$
4	$2\,H'' + 2\,c\,F'\,H' + z\,H' + 2\,k\,H^2 + 2\,c\,F''\,H + 2\,H = 0,$ $2\,H + 2\,F'' + z\,F' = 0.$
5	$2\,H'' + 2\,c\,F'\,H' + z\,H' + 2\,k\,H^2 + 2\,c\,F''\,H + 2\,H = 0,$ $2\,\beta\,H + 2\,\beta\,F'' + \beta\,z\,F' - 2\,\alpha\,\beta\,e^{\beta\,F} + 2 \; = 0.$

Table 3: System of ODEs

3 Reduced Systems

The form of the invariant solutions admitted by system (7) is derived from their invariance conditions

$$\mathbf{v}[u,v]^T = 0 \tag{13}$$

where T denotes the transposed of the vector row $[u,v]$. By specializing (13) to the case of the principal Lie algebra that is for $\mathbf{v} = \lambda\partial_x + \mu\partial_t$ we get the following PDE system

$$\lambda u_x + \mu u_t = 0 \tag{14}$$

$$\lambda v_x + \mu v_t = 0 \tag{15}$$

from where we derive the invariant solutions

$$\begin{aligned} u(t,x) &= H(z), \\ v(t,x) &= F(z), \end{aligned} \tag{16}$$

with the similarity variable $z = \mu x - \lambda t$. Substituting (16) into (7) we obtain the system ODE'$_1$ of the Table 3 for the search of solution of the traveling wave type

$$\begin{aligned}&H'\,\lambda - c\,\mu^2\,F''\,N + \mu^2\,H'' + c\,\mu^2\,F'\,H' + c\,\mu^2\,F''\,H + g = 0,\\ &f - F'\,\lambda - H - \mu^2\,F'' = 0,\end{aligned} \tag{17}$$

where, of course, f and g are arbitrary functions.

In the following sections we focuse our attention only on the search for some special cases. Solutions and analyses of additional cases will be considered in further researches.

4 Exact solutions with $g(u)$ of the logistic form.

4.1 Case 1

Here we consider the ODE system (17) derived by reducing system (7) and specializing the constitutive functions $g(u)$ and $f(v)$ as:

$$g(u) = u(1-u), \quad f(v) = v, \tag{18}$$

that is by assuming the growth term of the first equation to be of logistic type. On the basis of these specializations the system (17) reads

$$\begin{aligned}&H'\,\lambda - c\,\mu^2\,F''\,N + \mu^2\,H'' + c\,\mu^2\,F'\,H' + c\,\mu^2\,F''\,H + H(1-H) = 0,\\ &F - F'\,\lambda - H - \mu^2\,F'' = 0.\end{aligned} \tag{19}$$

It is a simple matter to verify that in the case $N < 1$ an exact solution of the system (19) has the form

$$H = 1 - b_0\frac{(N-1)}{2}e^{b_1 z}, \tag{20}$$

$$F = 1 + b_0 e^{b_1 z} \tag{21}$$

with b_0, $b_1 \neq 0$ arbitrary constants and

$$z = \pm x\sqrt{\frac{1-N}{4b_1^2 c}} + \frac{1-N-2c(N+1)}{4b_1 c}t. \tag{22}$$

Then for the system (7) we get

$$u(t,x) = 1 - b_0\frac{N-1}{2}e^{b_1\left(\pm x\sqrt{\frac{1-N}{4b_1^2 c}} + \frac{1-N-2c(N+1)}{4b_1 c}t\right)}, \tag{23}$$

$$v(t,x) = 1 + b_0 e^{b_1\left(\pm x\sqrt{\frac{1-N}{4b_1^2 c}} + \frac{1-N-2c(N+1)}{4b_1 c}t\right)}. \tag{24}$$

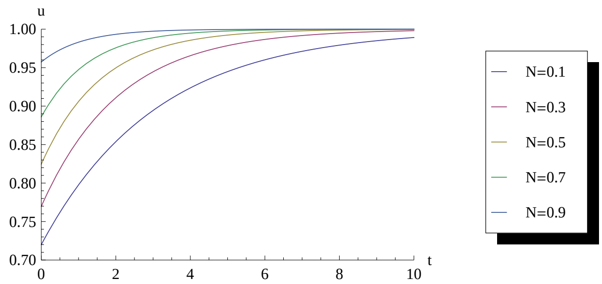

Figure 1: The solution $u(t,1)$ given by (23) with $b_0 = 1$, $b_1 = 1$, $c = 1$, $\lambda = 1$, $\mu = 1$.

In Fig.1 we show the density $u(t,1)$ given by (23) for different values of N.

Taking into account that with these forms of f and g we fall in the case 3 of Table 1, we can use the infinitesimal generator $\mathbf{v} = \lambda \mathbf{v_1} + \mu \mathbf{v_2} + \mathbf{v_3^3}$. Moreover we observe that the corresponding reduced system is the same of system (19) then, by using the same solution (20), (21), going back to the original variables for system (7) we get

$$
\begin{aligned}
u(t,x) &= 1 - b_0 \frac{N-1)}{2} e^{b_1\left(\pm x\sqrt{\frac{1-N}{4b_1^2 c}} + \frac{1-N-2c(N+1)}{4b_1 c}t\right)}, && (25)\\
v(t,x) &= 1 + b_0 e^{b_1\left(\pm x\sqrt{\frac{1-N}{4b_1^2 c}} + \frac{1-N-2c(N+1)}{4b_1 c}t\right)} - \frac{e^{-t}}{\sqrt{\frac{1-N}{4b_1^2 c}}}. && (26)
\end{aligned}
$$

4.2 Case 2.

Here, taking into account [5], [19], we consider $g(u)$ of the following generalized logistic form

$$
g(u) = g_0\, u(1-u^2), \tag{27}
$$

and

$$
f(v) = \alpha = const. \tag{28}
$$

We fall in the case 2 (Table 1), and we use the infinitesimal generator $\mathbf{v} = \lambda \mathbf{v_1} + \mu \mathbf{v_2} + \mathbf{v_2^3}$. From where we get the invariant solutions

$$
\begin{aligned}
u(x,t) &= H(z),\\
v(x,t) &= F(z) + \frac{x}{\lambda},
\end{aligned} \tag{29}
$$

with the similarity variable $z = \mu x - \lambda t$.

The reduced system is written in Table 3 (case 2) and, by setting $g_0 = \frac{3c}{N}$ it is possible to verify that a solution is

$$
\begin{aligned}
H(z) &= h_1 e^{\frac{-\lambda z}{2\mu^2}}, && (30)\\
F(z) &= \frac{2h_1^2\mu^2}{\lambda^2 N} e^{\frac{-\lambda z}{\mu^2}} + \frac{4\mu^2 h_1}{\lambda^2} e^{-\frac{\lambda z}{2\mu^2}} + \frac{(12c\mu^2 - 4cN^2\mu^2 - 2cN\mu - \lambda^2 N)z + 2f_0\lambda c N\mu^2}{2cN\lambda\mu^2} && (31)
\end{aligned}
$$

that for the system (7) becomes

$$u(t,x) = h_1 e^{\frac{-\lambda(\mu x - \lambda t)}{2\mu^2}} \tag{32}$$

$$v(t,x) = \frac{2h_1^2\mu^2}{\lambda^2 N} e^{\frac{-\lambda(\mu x-\lambda t)}{\mu^2}} + \frac{4\mu^2 h_1}{\lambda^2} e^{-\frac{\lambda(\mu x-\lambda t)}{2\mu^2}} + \frac{(12c\mu^2 - 4cN^2\mu^2 - 2cN\mu - \lambda^2 N)(\mu x - \lambda t) + 2f_0\lambda c N\mu^2}{2cN\lambda\mu^2} + \frac{x}{\lambda}. \tag{33}$$

5 Special travelling wave solutions

We consider solutions for system (17) of the following form

$$\begin{aligned} H(z) &= b_1 \tanh^s z + b_2, \\ F(z) &= b_3 \tanh^r z + b_4, \end{aligned} \tag{34}$$

where $b_i, (i = 1, ..4)$ are constants with $b_1 \neq 0$, $b_3 \neq 0$, that for system (7) read as

$$u(t,x) = b_1 \tanh^s(\mu x - \lambda t) + b_2, \tag{35}$$

$$v(t,x) = b_3 \tanh^r(\mu x - \lambda t) + b_4. \tag{36}$$

By choosing f and g in some more general polynomial forms it is a simple matter to verify that the system (17) admits solutions of the form (34).

In the following we consider special solutions of the kink type and soliton type by assuming, respectively, $r = s = 1$ and $r = s = 2$.

5.1 Solutions of the kink type

For $r = s = 1$, specializing f and g in the form

$$g(u) = j_0 + j_1 u + j_2 u^2 + j_3 u^3 + j_4 u^4, \tag{37}$$

$$f(v) = k_3 v^3 + k_2 v^2 + k_1 v + k_0, \tag{38}$$

with the following constraints between the constitutive constants

$$
\begin{aligned}
j_0 &= \frac{(b_2-b_1)\,(b_1+b_2)\,(b_1^2\lambda+b_1^2b_3\,\mu^2c+2\,b_2\,b_1\,\mu^2-b_2^2b_3\,\mu^2c-2\,b_2\,c\mu^2N\,b_3)}{b_1^3},\\
j_1 &= 2\,\frac{3\,b_2^3b_3\,\mu^2c-3\,b_1^2c\mu^2b_3\,b_2-b_1^2c\mu^2N\,b_3+3\,b_2^2c\mu^2N\,b_3+b_1^3\mu^2-3\,b_2^2b_1\,\mu^2-b_2\,b_1^2\lambda}{b_1^3},\\
j_2 &= \frac{6\,b_2\,b_1\,\mu^2+4\,b_1^2b_3\,\mu^2c+b_1^2\lambda-12\,b_2^2b_3\,\mu^2c-6\,b_2\,c\mu^2N\,b_3}{b_1^3},\\
j_3 &= -2\,\frac{\mu^2\,(b_1-5\,cb_3\,b_2-cN\,b_3)}{b_1^3},\\
j_4 &= -3\,\frac{b_3\,\mu^2c}{b_1^3},\\
k_0 &= -\frac{-b_3^3\lambda-2\,b_4\,\mu^2b_3^2-b_3^2b_2+2\,b_4^3\mu^2+b_4^2b_3\,\lambda+b_4\,b_3\,b_1}{b_3^2},\\
k_1 &= \frac{-2\,\mu^2b_3^2+b_3\,b_1+6\,b_4^2\mu^2+2\,b_4\,b_3\,\lambda}{b_3^2},\\
k_2 &= -\frac{b_3\,\lambda+6\,b_4\,\mu^2}{b_3^2},\\
k_3 &= 2\,\frac{\mu^2}{b_3^2},
\end{aligned}
\tag{39}
$$

we obtain the solution

$$H(z) = b_1\tanh z + b_2, \tag{40}$$

$$F(z) = b_3\tanh z + b_4, \tag{41}$$

and for system (7) we have the kink solution

$$u(t,x) = b_1\tanh(\mu x-\lambda t) + b_2, \tag{42}$$

$$v(t,x) = b_3\tanh(\mu x-\lambda t) + b_4. \tag{43}$$

In Fig.2 we show the density u given by (42) for different values of x.

5.2 Solutions of the soliton type

For $r = s = 2$, specializing f and g in the form

$$g(u) = g_0+g_1u+g_2u^2+g_3u^3+g_4\left(\frac{u}{b_1}-\frac{b_2}{b_1}\right)^{\frac{3}{2}}+g_5\left(\frac{u}{b_1}-\frac{b_2}{b_1}\right)^{\frac{1}{2}}, \tag{44}$$

$$f(v) = c_0+c_1v+c_2v^2+c_3\left(\frac{v}{b_3}-\frac{b_4}{b_3}\right)^{\frac{3}{2}}+c_4\left(\frac{v}{b_3}-\frac{b_4}{b_3}\right)^{\frac{1}{2}}, \tag{45}$$

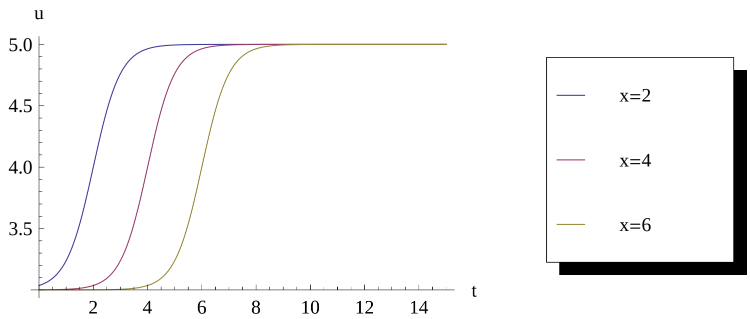

Figure 2: The kink solution u given by (42) with $b_1 = 1$, $b_2 = 2$, $\lambda = 1$, $\mu = 1$.

with

$$
\begin{aligned}
g_0 &= \frac{2\left(b_2+b_1\right)\mu^2\left(3b_2b_3cN + b_1b_3cN + 2{b_2}^2b_3c + 2b_1b_2b_3c - 3b_1b_2 - {b_1}^2\right)}{{b_1}^2},\\
g_1 &= -\frac{2\mu^2\left(6b_2b_3cN + 4b_1b_3cN + 9{b_2}^2b_3c + 12b_1b_2b_3c + 3{b_1}^2b_3c - 6b_1b_2 - 4{b_1}^2\right)}{{b_1}^2},\\
g_2 &= \frac{2\mu^2\left(3b_3cN + 12b_2b_3c + 8b_1b_3c - 3b_1\right)}{{b_1}^2},\\
g_3 &= -\frac{10b_3c\mu^2}{{b_1}^2},\\
g_4 &= 2\lambda b_1,\\
g_5 &= -2\lambda b_1,\\
c_0 &= \frac{\left(\left(6b_3+12\right){b_4}^2 + \left(8{b_3}^2+8b_3\right)b_4 + 2{b_3}^3\right)\mu^2 - b_1b_4 + b_2{b_3}^2}{{b_3}^2},\\
c_1 &= -\frac{12b_4\mu^2}{b_3} - 8\mu^2 + \frac{b_1}{b_3},\\
c_2 &= \frac{6\mu^2}{b_3},\\
c_3 &= -2\lambda b_3,
\end{aligned}
\tag{46}
$$

we obtain a solution

$$H(z) = b_1 \tanh^2 z + b_2, \tag{47}$$

$$F(z) = b_3 \tanh^2 z + b_4 \tag{48}$$

and for the system (7) we have the soliton solution

$$u(t,x) = b_1 \tanh^2(\mu x - \lambda t) + b_2, \tag{49}$$

$$v(t,x) = b_3 \tanh^2(\mu x - \lambda t) + b_4. \tag{50}$$

In Fig.3 we show the density u given by (49) for different values of x.

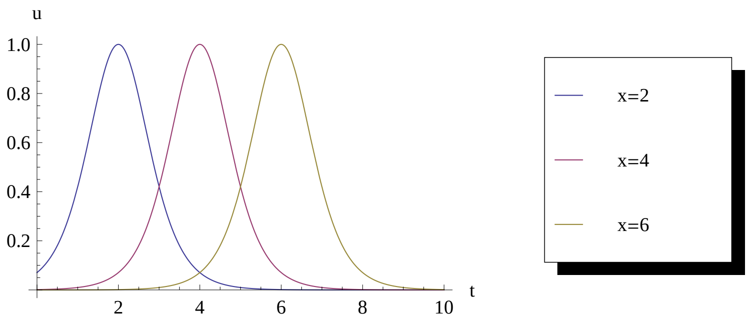

Figure 3: The solution $u(t,x)$ given by (49) with $b_1 = -1$, $b_2 = 0$, $\lambda = 1$, $\mu = 1$.

6 Conclusions

In this short paper a wide class of generalized chemotaxis systems, characterized by suitable constitutive relations, has been considered. In the growth term of the first equation an arbitrary function $g(u)$ appears while in the second equation assuming function $h(u,v) = -f(v) + u$ we introduce an arbitrary positive function $f(v)$. The complete symmetry classification has been performed with respect to the arbitrary functions $g(u)$ and $f(v)$. Once obtained reduced systems we focuse our attention in forms of $g(u)$ of the logistic type. Wide classes of solutions have been derived. Finally, by choosing $g(u)$ and $f(v)$ of a more general polynomial form, solutions of the kink type and of the soliton type are shown.

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