Split variational inclusion problem and fixed point problem for asymptotically nonexpansive semigroup with application to optimization problem

Shih-sen Chang[1], ∗, Liang-Cai Zhao[2], Zhao-Li Ma[3], Gang Wang[4]

[1] Center for General Education, China Medical University, Taichung, 40402, Taiwan

[2] College of Mathematics, Yibin University, Yibin, Sichuan 644007, China

[3] College of public foundation, Yunnan Open University, Kunming, Yunnan, 650500, China

[4] Yunnan University of Finance and Economics, Kunming, 650221, Yunnan, China [1]

E-mail: changss2013@163.com, lczhaoyb@163.com, kmszmzl@126.com, wg631208@sina.com

Abstract The purpose of this paper is by using the shrinking projection method to introduce and study an iterative process to approximate a common solution of split variational inclusion problem and fixed point problem for an asymptotically nonexpansive semigroup in real Hilbert spaces. Further, we prove that the sequences generated by the proposed iterative method converge strongly to a common solution of split variational inclusion problem and fixed point problem for an asymptotically nonexpansive semigroup. As applications, we shall utilize the results to study the split optimization problem and the split variational inequality.

MSC: 54E70; 47H25.

Key Words: split variational inclusion problem; asymptotically nonexpansive semigroup; fixed point problem; nonexpansive semigroup.

1 Introduction

Throughout the paper, unless otherwise stated, let $H,\ H_1,\ H_2$ be three real Hilbert spaces, C be a nonempty, closed and convex subset of H.

Recall that a mapping $T : C \to C$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$||T^n x - T^n y|| \leq k_n ||x - y|| \ \ \forall n \geq 1,\ x, y \in C.$$

A family $\mathfrak{T} :\ = \{T(s) : 0 \leq s < \infty\}$ of mappings from C into itself is called *asymptotically nonexpansive semigroup on* C (*respectively, nonexpansive semigroup on* C), if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;

(iii) There exists a sequence $\{k_n\} \subset [1, \infty)$ (respectively, $\{k_n = 1\}$) such that $k_n \to 1$ and satisfying the following condition

$$||T^n(s)x - T^n(s)y|| \leq k_n||x - y|| \ \ \forall x, y \in C\ n \geq 1 \ and \ s \geq 0;$$

[1]Corresponding author S. S. Chang (e-mail: changss2013@163.com)

(iv) for all $x \in C$, $s \mapsto T(s)x$ is continuous.

The set of all the common fixed points of a semigroup $\mathfrak{T}$ is denoted by Fix($\mathfrak{T}$), i.e.,

$$\mathrm{Fix}(\mathfrak{T}) := \{x \in C : T(s)x = x,\ 0 \le s < \infty\} = \bigcap_{0 \le s < \infty} Fix(T(s)),$$

where Fix$(T(s))$ is the set of fixed points of $T(s)$, $s \ge 0$.

Recall that a mapping $T : H_1 \to H_1$ is said to be

(i) monotone, if

$$\langle Tx - Ty, x - y\rangle \ge 0,\ \forall x, y \in H_1;$$

(ii) *α-strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge \alpha ||x - y||^2,\ \forall x, y \in H_1;$$

(iii) *firmly nonexpansive*, if

$$||Tx - Ty||^2 \le \langle Tx - Ty, x - y\rangle,\ \forall x, y \in H_1. \tag{1.1}$$

Remark 1.1 It is easy to see that the definition of firmly nonexpansive mapping is equivalent to the following

(iii)$'$ $T : C \to C$ is said to be *firmly nonexpansive*, if

$$||Tx - Ty||^2 \le ||x - y||^2 - \langle x - y,\ (x - Tx) - (y - Ty)\rangle,\ \forall x, y \in C. \tag{1.2}$$

(iv) Recall that a multi-valued mapping $M : H_1 \to 2^{H_1}$ is said to be *monotone*, if for all $x, y \in H_1$, $u \in Mx$ and $v \in My$ such that

$$\langle x - y, u - v\rangle \ge 0.$$

(v) A monotone mapping $M : H_1 \to 2^{H_1}$ is said to be *maximal*, if the Graph(M) is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping M is maximal if and only if for $(x, u) \in H_1 \times H_1$, $\langle x - y, u - v\rangle \ge 0$, for every $(y, v) \in Graph(M)$ implies that $u \in Mx$.

Let $M : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping. Then, the *resolvent mapping* $J_\lambda^M : H_1 \to H_1$ *associated with* M, is defined by

$$J_\lambda^M(x) := (I + \lambda M)^{-1}(x),\ \forall x \in H_1, \tag{1.3}$$

for some $\lambda > 0$, where I stands identity operator on H_1.

We note that for all $\lambda > 0$ the resolvent operator J_λ^M is single-valued, nonexpansive and firmly nonexpansive.

Recently, Moudafi [1] introduced the following *split variational inclusion problem (in short, SVIP)*: Find $x^* \in H_1$, $y^* = Ax^* \in H_2$ such that

$$0 \in B_1(x^*) \quad and \quad 0 \in B_2(y^*), \tag{1.4}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings.

From the definition of resolvent mapping J_λ^M, we have the following technical lemma.

Lemma 1.2 SVIP (1.4) is equivalent to find $x^* \in H_1$, $y^* = Ax^* \in H_2$ such that

$$x^* \in Fix(J_\lambda^{B_1}) \ \ and \ \ y^* \in Fix(J_\lambda^{B_2}) \ \ for\ some\ \lambda > 0. \tag{1.5}$$

In the sequel, we denote the solution set Ω of problem (1.4) or (1.5) by

$$\begin{aligned}\Omega := &\{x^* \in H_1, y^* = Ax^* \in H_2 \ such\ that\ x^* \in B_1^{-1}(0),\ Ax^* \in B_2^{-1}(0)\}\\ = &\{x^* \in H_1, y^* = Ax^* \in H_2 \ such\ that\ x^* \in Fix(J_\lambda^{B_1}),\ Ax^* \in Fix(J_\lambda^{B_2})\}.\end{aligned} \tag{1.6}$$

Moudafi [1] also introduced an iterative method for solving SVIP (1.4), which can be seen as an important generalization of an iterative method given by Censor et al. [2] for split variational inequality problem. As Moudafi notes in [1], SVIP (1.4) includes as special cases, the split common fixed point problem, split variational inequality problem, split zero problem and split feasibility problem [1 - 6] which have already been studied and used in practice as a model in intensity-modulated radiation therapy treatment planning, see [5, 6]. This formalism is also at the core of modeling of many inverse problems arising for phase retrieval and other real-world problems; for instance, in sensor networks in computerized tomography and data compression; see e.g. [7,8].

In 2012 Byrne et al. [4] studied the weak and strong convergence of the following iterative method for SVIP (1.4): For given $x_0 \in H_1$, compute iterative sequence $\{x_n\}$ generated by the following scheme:

$$x_{n+1} = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \tag{1.7}$$

for $\lambda > 0$.

Very recently, Kazmi and Rizvi [9] studied the strong convergence of the following iterative method for split variational inclusion problem and fixed point problem for a nonexpansive mapping S:

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n); \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n, \end{cases} \tag{1.8}$$

for $\lambda > 0$.

Motivated by the work of Moudafi [1], Byrne et al.[4], Kazmi and Rizvi [9], Deepho et al [10] and Sitthithakerngkiet et al [11], the purpose of this paper is by using the shrinking projection method to introduce and study an iterative process to approximate a common solution of split variational inclusion problem and fixed point problem for an asymptotically nonexpansive semigroup in real Hilbert spaces. Further, we prove that the sequences generated by the proposed iterative method converge strongly to a common solution of split variational inclusion problem and fixed point problem for a asymptotically nonexpansive semigroup. The results presented in this paper are an extension and generalization of the previously known results in related topic.

2 Preliminaries

In this section, we recall some concepts and lemmas which will be used in proving our main results.

Let C be a nonempty closed and convex subset of H. For each $x \in H$, *the (metric) projection* $P_C : H \to C$ is defined as the unique element $P_C x \in C$ such that

$$||x - P_C x|| = inf_{y \in C} ||x - y||.$$

It is well known that for any given $x \in H$, $y = P_C(x)$, if and only if

$$\langle y - z,\ x - y \rangle \geq 0, \quad for\ each\ z \in C, \tag{2.1}$$

and P_C is a firmly nonexpansive mapping from H onto C, that is,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle. \tag{2.2}$$

Recall that a mapping $T : C \to H$ is said to be α*-inverse strongly monotone*, if there exists $\alpha > 0$ such that

$$\alpha \|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle, \forall x, y \in C. \tag{2.3}$$

This implies that each firmly nonexpansive mapping is 1-inverse strongly monotone. Also it is easy to prove that the following result holds.

Lemma 2.1 If $T : C \to H$ is α-inverse strongly monotone, then for each $\lambda \in (0, 2\alpha], I - \lambda T$ is a nonexpansive mapping of C into H (see, for example, [12]).

Lemma 2.2 Let H be a real Hilbert space, then the following result holds:

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$$

for all $x, y \in H$ and for all $t \in [0, 1]$.

Lemma 2.3 [13] Let H be a real Hilbert space, C be a nonempty closed convex subset of H, and $S : C \to C$ be an asymptotically nonexpansive mapping. If the set of fixed points $Fix(S)$ of S is nonempty, then it is closed and convex, and the mapping $I - S$ is demiclosed at zero, that is, for any sequence $\{x_n\}$ in C such that $\{x_n\}$ converges weakly to $\bar{x}$ and $||x_n - Sx_n|| \to 0$, then $\bar{x} \in Fix(S)$.

3 Main Results

In this section, we shall prove a strong convergence theorem based on the proposed iterative method for computing the common approximate solution of SVIP (1.4) and fixed point of the asymptotically nonexpansive semigroup $\mathfrak{T} = \{T(s) : 0 \leq s < \infty\}$.

Throughout this section we assume that

(1) H_1 and H_2 are two real Hilbert spaces;

(ii) $A : H_1 \to H_2$ is a bounded linear operator, A^* is the adjoint of A and it is strongly positive, i.e., there exists an constant $\gamma > 0$ such that

$$\langle A^*x,\ y\rangle \geq \gamma||x||||y||,\ \forall y \in H_1,\ and\ x \in H_2.$$

(iii) $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are two maximal monotone mappings;

(iv) $J_\lambda^{B_1} : H_1 \to H_1$ and $J_\lambda^{B_2} : H_2 \to H_2$ are the resolvent mappings associated with B_1 and B_2 defined by (1.3), respectively;

(v) $\mathfrak{T} = \{T(s) : 0 \leq s < \infty\} : H_1 \to H_1$ is an asymptotically nonexpansive semigroup.

First, we give the following lemma.

Lemma 3.1 Let $H_1,\ H_2,\ A,\ A^*, B_1,\ B_2,\ J_\lambda^{B_1},\ J_\lambda^{B_2}$ be the same as above. Let L be the spectral radius of the operator A^*A and $\gamma \in (0, \frac{2}{L})$. Then $(I - \gamma A^*(I - J_\lambda^{B_2})A)$ and $J_\lambda^{B_1}(I - \gamma A^*(I - J_\lambda^{B_2})A)$ both are nonexpansive mappings.

Proof Since $J_\lambda^{B_2}$ is firmly nonexpansive, $(I - J_\lambda^{B_2})$ is also firmly nonexpansive. Hence it is $1-$inverse strongly monotone. So we have

$$\begin{aligned}
&\|(I - J_\lambda^{B_2})Ax - (I - J_\lambda^{B_2})Ay\|^2 \\
&= \|Ax - Ay\|^2 - 2\langle Ax - Ay, J_\lambda^{B_2}Ax - J_\lambda^{B_2}Ay\rangle + \|J_\lambda^{B_2}Ax - J_\lambda^{B_2}Ay\|^2 \\
&\leq \|Ax - Ay\|^2 - \langle Ax - Ay, J_\lambda^{B_2}Ax - J_\lambda^{B_2}Ay\rangle \\
&= \langle Ax - Ay, (I - J_\lambda^{B_2})Ax - (I - J_\lambda^{B_2})Ay\rangle,\ \forall x, y \in H_1.
\end{aligned} \tag{3.1}$$

It follows from (3.1) that

$$\begin{aligned}
&\|A^*(I - J_\lambda^{B_2})Ax - A^*(I - J_\lambda^{B_2})Ay\|^2 \\
&\leq L||(I - J_\lambda^{B_2})Ax - (I - J_\lambda^{B_2})Ay\|^2 \\
&\leq L\langle Ax - Ay, (I - J_\lambda^{B_2})Ax - (I - J_\lambda^{B_2})Ay\rangle \\
&= L\langle x - y, A^*(I - J_\lambda^{B_2})Ax - A^*(I - J_\lambda^{B_2})Ay\rangle\ \forall x, y \in H_1.
\end{aligned} \tag{3.2}$$

This implies that $A^*(I - J_\lambda^{B_2})A$ is a $\frac{1}{L}-$inverse strongly monotone mapping. Since $\gamma \in (0, \frac{2}{L})$, by Lemma 2.1, $I - \gamma A^*(I - J_\lambda^{B_2})A$ is a nonexpansive mapping. So is $J_\lambda^{B_1}(I - \gamma A^*(I - J_\lambda^{B_2})A)$. This completes the proof of Lemma 3.1.

Theorem 3.2 Let $H_1,\ H_2,\ A,\ A^*, B_1,\ B_2,\ J_\lambda^{B_1},\ J_\lambda^{B_2}$ be the same as in Lemma 3.1. Let $\mathfrak{T} = \{T(s) : 0 \leq s < \infty\} : H_1 \to H_1$ be an asymptotically nonexpansive semigroup with sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Denote by $\Gamma :=$Fix$(\mathfrak{T})\bigcap\Omega$, where Ω is the solution set of problem (1.4) defined by (1.6). For an initial point $x_0 \in H_1$, $C_1 = H_1$, $x_1 = P_{C_1}x_0$, generate a sequence $\{x_n\}$ by

$$\begin{cases}
u_n = J_{r_n}^{B_1}(I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n, \\
y_n = \alpha_n x_n + (1 - \alpha_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T^n(s)u_n ds \\
C_{n+1} = \{z \in C_n :\ \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\} \\
x_{n+1} = P_{C_{n+1}}x_0, \forall n \geq 1
\end{cases} \tag{3.3}$$

where $\theta_n = (1-\alpha_n)(k_n^2-1)\sup\{\|x_n-u\|^2 : u \in \Gamma\}$, $\{s_n\}$ is a sequence of positive numbers. $0 < a \le \alpha_n < c < 1$ for all $n \ge 1$, $0 < b \le r_n < +\infty$, $\gamma \in (0, \frac{2}{L})$. L is the spectral radius of the operator A^*A. If the following conditions are satisfied:

(1) $\Gamma :=\mathrm{Fix}(\mathfrak{T})\bigcap\Omega \neq \emptyset$ and is bounded;

(2) $\limsup_{n\to\infty} ||\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds - T(h)(\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds)|| = 0$, for each $h > 0$,

then the sequence $\{x_n\}$ generated by (3.3) strongly converges to a point $x^* \in \mathrm{Fix}(\mathfrak{T})\bigcap\Omega$.

Proof We divide the proof of Theorem 3.2 into five steps.

Step 1. We show that C_n is closed and convex for each $n \ge 1$.

In fact, since the inequality $||y_n - z||^2 \le ||x_n - z||^2 + \theta_n$ is equivalent to

$$2\langle x_n - y_n, z\rangle \le ||x_n||^2 - ||y_n||^2 + \theta_n,$$

and $z \mapsto 2\langle x_n - y_n, z\rangle$ is a continuous and convex function. Therefore for each $n \ge 1$, C_n is a convex and closed subset in H_1.

Step 2. Now we prove that Fix($\mathfrak{T}$)$\bigcap\Omega \subset C_n, \forall n \ge 1$.

Let $p \in \mathrm{Fix}(\mathfrak{T})\bigcap\Omega$, then $p = T(s)p$, $\forall s \ge 0$, $J_{r_n}^{B_1}p = p$, $J_{r_n}^{B_2}Ap = Ap$, and so $(I - \gamma A^*(I - T_{r_n}^{F_2})A)p = p$. It is obvious that $\mathrm{Fix}(\mathfrak{T})\bigcap\Omega \subset C_1$. Let $\mathrm{Fix}(\mathfrak{T})\bigcap\Omega \subset C_n$ for some $n \ge 2$, by mathematical induction, now we prove that $\mathrm{Fix}(\mathfrak{T})\bigcap\Omega \subset C_{n+1}$. In fact, it follows from (3.3) and Lemma 3.1 that

$$\begin{aligned} \|u_n - p\| &= \|J_{r_n}^{B_1}(I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n - J_{r_n}^{B_1}(I - \gamma A^*(I - J_{r_n}^{B_2})A)p\| \\ &\le \|(I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n - (I - \gamma A^*(I - J_{r_n}^{B_2})A)p\| \\ &= \|x_n - p\|. \end{aligned} \tag{3.4}$$

Also it follows from (3.3), (3.4), and Lemma 2.2 that

$$\begin{aligned} \|y_n - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)(\frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds) - p\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|\frac{1}{s_n}\int_0^{s_n}(T^n(s)u_n - p)ds\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - \frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds||^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\frac{1}{s_n}\int_0^{s_n} ||T^n(s)u_n - p||ds)^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - \frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds||^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)k_n^2||u_n - p||^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - \frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds||^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)k_n^2||x_n - p||^2 \\ &= \|x_n - p\|^2 + (1-\alpha_n)(k_n^2 - 1)||x_n - p||^2 \\ &\le \|x_n - p\|^2 + \theta_n. \end{aligned} \tag{3.5}$$

where

$$\theta_n = (1-\alpha_n)(k_n^2 - 1)\sup_{u\in\Gamma}\{||x_n - u||^2\}. \tag{3.6}$$

This implies that $p \in C_{n+1}$, so is $\mathrm{Fix}(\mathfrak{T})\bigcap\Omega \subset C_{n+1}$. The conclusion is proved.

Step 3. Now we prove that $\{x_n\}$ is a Cauchy sequence.

In fact, it follows from (3.3) that $x_{n+1} = P_{C_{n+1}}x_0$, $x_n = P_{C_n}x_0$ and $C_{n+1} \subset C_n$. By (2.1) we have

$$\langle x_0 - x_{n+1}, x_{n+1} - y\rangle \geq 0, \forall y \in C_{n+1}.$$

Since $\Gamma = \mathrm{Fix}(\mathfrak{T})\bigcap\Omega \subset C_{n+1}$, we have

$$\langle x_0 - x_{n+1}, x_{n+1} - p\rangle \geq 0, \forall p \in \Gamma.$$

This shows that

$$\begin{aligned} 0 &\leq \langle x_0 - x_{n+1}, x_{n+1} - x_0 + x_0 - p\rangle \\ &\leq -||x_{n+1} - x_0||^2 + ||x_{n+1} - x_0||||x_0 - p||. \end{aligned}$$

Simplifying we have

$$||x_{n+1} - x_0|| \leq ||x_0 - p||,$$

i.e., $\{x_n\}$ is bounded, so are $\{u_n\}$ and $\{y_n\}$.

Also since

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0,$$

we have

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &\leq -||x_n - x_0||^2 + ||x_{n+1} - x_0||||x_0 - x_n||, \end{aligned}$$

i.e., $||x_n - x_0|| \leq ||x_{n+1} - x_0||$. Since $\{x_n\}$ is bounded, this implies that the limit $\lim_{n\to\infty} ||x_n - x_0||$ exists

Hence for any positive integers n, m, it follows from (3.3) that $x_m = P_{C_m}x_0$ and $x_n = P_{C_n}x_0$. By the well known property of projection, we have

$$||x_n - x_m||^2 + ||x_m - x_0||^2 \leq ||x_n - x_0||^2.$$

Since the limit $\lim_{n\to\infty} ||x_n - x_0||$ exists, we have

$$||x_n - x_m||^2 \leq ||x_n - x_0||^2 - ||x_m - x_0||^2 \to 0 \ \ (as\ n,\ m \to \infty).$$

This implies that $\{x_n\}$ is a Cauchy sequence. Without loss of generality, we can assume that

$$\lim_{n\to\infty} x_n = x^* \ (some\ point\ in\ C_n\ \forall n \geq 1). \tag{3.7}$$

By the way, since $\{x_n\}$ is bounded, and Γ is bounded, it follows from (3.6) that

$$\theta_n \to 0\ (as\ n \to \infty). \tag{3.8}$$

Step 4. Next we prove that

$$\lim_{n\to\infty} \|T(h)x_n - x_n\| = 0,\ \forall h \geq 0. \tag{3.9}$$

In fact, since $x_{n+1} \in C_{n+1} \subset C_n$, by the construction of C_{n+1}, we have

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \theta_n.$$

Hence

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| + \sqrt{\theta_n}. \tag{3.10}$$

This together with (3.7) and (3.8) shows that

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0.$$

Therefore we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \to 0 \ (as\ n \to \infty). \tag{3.11}$$

Since $J_{r_n}^{B_1}$ is firmly nonexpansive, by (3.2) $A^*(I - J_\lambda^{B_2})A$ is a $\frac{1}{L}-$inverse strongly monotone mapping. If $p \in \Gamma$, then we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|J_{r_n}^{B_1}(x_n - \gamma A^*(I - J_{r_n}^{B_2})Ax_n) - J_{r_n}^{B_1}(p - \gamma A^*(I - J_{r_n}^{B_2})Ap)\|^2 \\
&\leq \|(I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n - (I - \gamma A^*(I - J_{r_n}^{B_2})A)p\|^2 \\
&- \|(I - J_{r_n}^{B_1})(I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n - (I - J_{r_n}^{B_1})(I - \gamma A^*(I - J_{r_n}^{B_2})A)p\|^2 \\
&= \|x_n - p - \gamma(A^*(I - J_{r_n}^{B_2})Ax_n - A^*(I - J_{r_n}^{B_2})Ap)\|^2 - \|z_n - J_{r_n}^{B_1}z_n\|^2 \\
&= \|x_n - p\|^2 - 2\gamma\langle x_n - p, A^*(I - J_{r_n}^{B_2})Ax_n - A^*(I - J_{r_n}^{B_2})Ap\rangle \\
&\quad + \gamma^2\|A^*(I - J_{r_n}^{B_2})Ax_n - A^*(I - J_{r_n}^{B_2})Ap\|^2 - \|z_n - J_{r_n}^{B_1}z_n\|^2 \ (by(3.2)) \\
&\leq \|x_n - p\|^2 + \gamma(\gamma - \frac{2}{L})\|A^*(I - J_{r_n}^{B_2})Ax_n\|^2 - \|z_n - J_{r_n}^{B_1}z_n\|^2
\end{aligned}$$

where $z_n = (I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n$. This together with (3.5) shows that

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)(\frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds) - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)k_n^2\|u_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)k_n^2\{\|x_n - p\|^2 \\
&\quad + \gamma(\gamma - \frac{2}{L})\|A^*(I - J_{r_n}^{B_2})Ax_n\|^2 - \|z_n - J_{r_n}^{B_1}z_n\|^2\}
\end{aligned}$$

After simplifying, and by using the condition $0 < a \leq \alpha_n < c < 1$, we have

$$\begin{aligned}
&(1 - c)k_n^2[\gamma(\frac{2}{L} - \gamma)\|A^*(I - J_{r_n}^{B_2})Ax_n\|^2 + \|z_n - J_{r_n}^{B_1}z_n\|^2] \\
&\leq (1 - \alpha_n)k_n^2[\gamma(\frac{2}{L} - \gamma)\|A^*(I - J_{r_n}^{B_2})Ax_n\|^2 + \|z_n - J_{r_n}^{B_1}z_n\|^2] \\
&\leq (\alpha_n + (1 - \alpha_n)k_n^2)\|x_n - p\|^2 - \|y_n - p\|^2 \\
&= \alpha_n\|x_n - p\|^2 - \|y_n - p\|^2 + (1 - \alpha_n)k_n^2\|x_n - p\|^2 \\
&\leq (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\| + (1 - \alpha_n)(k_n^2 - 1)\|x_n - p\|^2.
\end{aligned} \tag{3.12}$$

This together with (3.11) shows that

$$\lim_{n\to\infty} \|A^*(I - J_{r_n}^{B_2})Ax_n\| = 0, \quad \lim_{n\to\infty} \|z_n - J_{r_n}^{B_1}z_n\| = 0. \tag{3.13}$$

By the assumption that A^* is a strongly positive linear bounded operator, we can get that

$$\lim_{n\to\infty} \|(I - J_{r_n}^{B_2})Ax_n\| = 0. \tag{3.14}$$

Therefore it follows from (3.3) and (3.13) that

$$\begin{aligned} \|u_n - x_n\| &= \|J_{r_n}^{B_1} z_n - x_n\| \\ &\le \|J_{r_n}^{B_1} z_n - z_n\| + \|z_n - x_n\| \\ &= \|J_{r_n}^{B_1} z_n - z_n\| + \|(I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n - x_n\| \\ &= \|J_{r_n}^{B_1} z_n - z_n\| + \gamma\|A^*(I - J_{r_n}^{B_2})Ax_n\| \to 0 \ (as\ n \to \infty). \end{aligned} \tag{3.15}$$

Now we prove that

$$||\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds - x_n|| \to 0 \ (as\ n \to \infty).$$

Indeed, it follows from (3.3) that

$$\begin{aligned} \|y_n - x_n\| &= \|\alpha_n x_n + (1-\alpha_n)(\frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds) - x_n\| \\ &= (1-\alpha_n)\|\frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds - x_n\|. \end{aligned}$$

Hence from (3.11) we have that

$$\|\frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds - x_n\| = \frac{1}{1-\alpha_n}\|y_n - x_n\| \to 0 \ (as\ n \to \infty) \tag{3.16}$$

This together with (3.15) shows that

$$\begin{aligned} &\|\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds - x_n\| \\ &\le \|\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds - \frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds + ||\frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds - x_n|| \\ &\le \frac{1}{s_n}\int_0^{s_n} ||T^n(s)x_n - T^n(s)u_n||ds + ||\frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds - x_n|| \\ &\le \frac{1}{s_n}k_n\int_0^{s_n} ||x_n - u_n||ds + ||\frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds - x_n|| \ \to 0 \ (as\ n \to \infty). \end{aligned} \tag{3.17}$$

By condition (2) and (3.17), for any $h > 0$ we have

$$\begin{aligned} \limsup_{n\to\infty} ||x_n - T(h)x_n|| &\le \limsup_{n\to\infty} ||x_n - \frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds|| \\ &+ \limsup_{n\to\infty} ||\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds - T(h)(\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds)|| \\ &+ \limsup_{n\to\infty} ||T(h)(\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds) - T(h)x_n|| \\ &\le \limsup_{n\to\infty}(1+k_1)||x_n - \frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds|| \\ &+ \limsup_{n\to\infty} ||\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds - T(h)(\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds)|| = 0 \end{aligned} \tag{3.18}$$

This implies that for each $h \geq 0$

$$\lim_{n\to\infty} \|T(h)x_n - x_n\| = 0.$$

The conclusion (3.9) is proved.

Step 5. Finally, we prove that the limit x^* in (3.7) is a solution of SVIP (1.4) and it is also a fixed point of the asymptotically nonexpansive semigroup $\mathfrak{T} = \{T(s) : 0 \leq s < \infty\}$, i.e., $x^* \in$ Fix($\mathfrak{T}$)$\bigcap \Omega$.

In fact, since $x_n \to x^*$ and $||x_n - T(h)x_n|| \to 0$ for each $h \geq 0$, it follows from Lemma 2.3 that $x^* \in Fix(T(h))$ for each $h \geq 0$, i.e., $x^* \in \mathrm{Fix}(\mathfrak{T})$.

Now we show $x^* \in \Omega$.

In fact, by (3.3), $u_n = J_{r_n}^{B_1}(I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n$, hence we have

$$x_n - \gamma A^*(I - J_{r_n}^{B_2})A)x_n \in (I + r_n B_1)(u_n). \tag{3.19}$$

Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that $u_{n_k} \rightharpoonup w$ (some point in H_1). Since $||x_n - u_n|| \to 0$ and $x_n \to x^*$, this implies that $x^* = w$. Simplifying (3.19), we have

$$\frac{1}{r_{n_k}}(x_{n_k} - u_{n_k} - \gamma A^*(I - J_{r_{n_k}}^{B_2})A)x_{n_k} \in B_1(u_{n_k}). \tag{3.20}$$

By passing to limit $k \to \infty$ in (3.20) and by taking into account (3.13), (3.15) and the fact that the graph of a maximal monotone operator is weakly-strongly closed, we obtain $0 \in B_1(x^*)$, i.e., $x^* \in Fix(J_\lambda^{B_1})$. Furthermore, since $\{x_n\}$ and $\{u_n\}$ have the same asymptotical behavior, $\{Ax_{n_k}\}$ weakly converges to Ax^*. Again, by (3.14), Lemma 2.3 and the fact that the resolvent $J_\lambda^{B_2}$ is nonexpansive, we obtain that $0 \in B_2(Ax^*)$, i.e., $Ax^* \in Fix(J_\lambda^{B_2})$. Thus $x^* \in \mathrm{Fix}(\mathfrak{T})\bigcap \Omega$, i.e., x^* is not only a solution of SVIP (1.4) but also a fixed point of the asymptotically nonexpansive semi-group $\mathfrak{T} = \{T(s) : 0 \leq s < \infty\}$.

This completes the proof of Theorem 3.2.

Remark [14] Next we give an example of asymptotically nonexpansive semigroup which satisfies the condition (2) in Theorem 3.2.

Let H be a real Hilbert space and $L(H)$ be the space of all bounded linear operators on H. For $\psi \in L(H)$, define $\mathfrak{T} = \{T(s) : 0 \leq s < \infty\}$ of bounded linear operators by using the following exponential expression:

$$T(t) = e^{-t\psi} : \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k \psi^k.$$

Then the family $\mathfrak{T} = \{T(s) : 0 \leq s < \infty\}$ satisfies the asymptotically nonexpansive semigroup properties. Moreover, this family forms a one-parameter semigroup of self-mappings of H satisfying the condition (2) in Theorem 3.2.

Now we consider the cases of nonexpansive semi-group. First we give the following lemma.

Lemma 3.3 ([15]) Let C be a nonempty bounded closed and convex subset of a real Hilbert H and let $\mathfrak{T} = \{T(s) : 0 \le s < \infty\}$ be a nonexpansive semigroup on C. Then for any $h \ge 0$,

$$\lim_{t\to\infty} \sup_{x\in C} ||\frac{1}{t}\int_0^t T(t)xds - T(h)(\frac{1}{t}\int_0^t T(t)xds)|| = 0. \tag{3.21}$$

By using Lemma 3.3 we can obtain the following result.

Theorem 3.4 Let $H_1,\ H_2,\ A,\ A^*, B_1,\ B_2,\ J_\lambda^{B_1},\ J_\lambda^{B_2}$ be the same as in Lemma 3.1. Let $\mathfrak{T}_1 = \{T(s) : 0 \le s < \infty\} : H_1 \to H_1$ be an nonexpansive semigroup. Denote by $\Gamma_1 :=$Fix$(\mathfrak{T}_1)\bigcap \Omega$, where Ω is the solution set of problem (1.4) defined by (1.6). For an initial point $x_0 \in H_1$, $C_1 = H_1$, $x_1 = P_{C_1}x_0$, generate a sequence $\{x_n\}$ by

$$\begin{cases} u_n = J_{r_n}^{B_1}(I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T(s)u_n ds \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2\} \\ x_{n+1} = P_{C_{n+1}}x_0, \forall n \ge 1 \end{cases} \tag{3.22}$$

where $\{s_n\}$ is a sequence of positive real numbers with $s_n \to \infty$. $0 < a \le \alpha_n < c < 1$ for all $n \ge 1$, $0 < b \le r_n < +\infty$, $\gamma \in (0, \frac{2}{L})$, where L is the spectral radius of the operator A^*A. If $\Gamma_1 \ne \emptyset$, then the sequence $\{x_n\}$ generated by (3.22) strongly converges to a point $x^* \in$ Fix$(\mathfrak{T}_1)\bigcap \Omega$.

Proof In fact, since $\mathfrak{T}_1 = \{T(s) : 0 \le s < \infty\}$ is a nonexpansive group, the sequence $\{k_n = 1\}$. Hence $\theta_n = (1-\alpha_n)(k_n^2 - 1)\sup\{\|x_n - u\|^2 : u \in \Gamma_1\} = 0$. The condition "$\Gamma_1$ being bounded" is no use. On the other hand, it follows from Lemma 3.3 that

$$\limsup_{n\to\infty} ||\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds - T(h)(\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds)|| = 0 \quad for\ each\ h \ge 0.$$

By the same way as given in the proof of Theorem 3.2, we can prove that the conclusion of Theorem 3.4 is true.

This completes the proof of Theorem 3.4.

4 Applications

4.1 Applications to split optimization problems

Let $H_1,\ H_2$ be two real Hilbert space and $A : H_1 \to H_2$ be a bounded and linear operator. The "so-called" *split optimization problem* (SOP) with respect to the functions $f : H_1 \to \mathbb{R}$ and $g : H_2 \to \mathbb{R}$ is to find [16, 17]: $x^* \in H_1$, $Ax^* \in H_2$ such that

$$f(x^*) \ge f(x) \quad for\ all \quad x \in H_1, \quad g(Ax^*) \ge g(y), \ for\ all \quad y \in H_2. \tag{4.1}$$

We denote by Ω_1 the solution set of the split optimization problem (4.1).

Let $f : H_1 \to \mathbb{R}$ and $g : H_2 \to R$ be two proper convex and lower semi-continuous functions. Denote by $B_1 = \partial f$ and $B_2 = \partial g$. Then $\partial f : H_1 \to H_1$ and $\partial g : H_2 \to H_2$ both are maximal monotone mappings. Denoting by $J_\lambda^{\partial f}$ and $J_\lambda^{\partial g}$ the resolvents associated with ∂f and ∂g defined by (1.3), respectively, then the (SOP) (4.1) ie equivalent to the following split variational inclusion problem: Find $x^* \in H_1$, $y^* = Ax^* \in H_2$ such that

$$0 \in \partial f(x^*) \quad \textit{and} \;\; 0 \in \partial g(Ax^*), \tag{4.2}$$

Therefore, by Theorem 3.2, we have the following.

Theorem 4.1 Let $H_1,\ H_2,\ A,\ A^*, f,\ g, \partial f,\ \partial g,\ J_\lambda^{\partial f},\ J_\lambda^{\partial g}$ be the same as above. Let $\mathfrak{T} = \{T(s) : 0 \le s < \infty\} : H_1 \to H_1$ be an asymptotically nonexpansive semigroup with sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Denote by $\Gamma_2 : = \mathrm{Fix}(\mathfrak{T}) \bigcap \Omega_2$, where Ω_2 is the solution set of problem (4.2). For an initial point $x_0 \in H_1$, $C_1 = H_1$, $x_1 = P_{C_1}x_0$, generate a sequence $\{x_n\}$ by

$$\begin{cases} u_n = J_{r_n}^{\partial f}(I - \gamma A^*(I - J_{r_n}^{\partial g})A)x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T^n(s)u_n ds \\ C_{n+1} = \{z \in C_n : \ \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\} \\ x_{n+1} = P_{C_{n+1}}x_0, \forall n \ge 1 \end{cases} \tag{4.3}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)\sup\{\|x_n - u\|^2 : u \in \Gamma_2\}$, $\{s_n\}$ is a sequence of positive numbers. $0 < a \le \alpha_n < c < 1$ for all $n \ge 1$, $0 < b \le r_n < +\infty$, $\gamma \in (0, \frac{2}{L})$. L is the spectral radius of the operator A^*A. If the following conditions are satisfied:

(1) $\Gamma_2 \ne \emptyset$ and is bounded;

(2) $\limsup_{n\to\infty} ||\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds - T(h)(\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds)|| = 0$, for each $h > 0$,

then the sequence $\{x_n\}$ generated by (4.3) strongly converges to a point $x^* \in \mathrm{Fix}(\mathfrak{T}) \bigcap \Omega_2$.

Theorem 4.2 Let $H_1,\ H_2,\ A,\ A^*, f,\ g, \partial f,\ \partial g,\ J_\lambda^{\partial f},\ J_\lambda^{\partial g}$ be the same as in Theorem 4.1. Let $\mathfrak{T}_3 = \{T(s) : 0 \le s < \infty\} : H_1 \to H_1$ be an nonexpansive semigroup. Denote by $\Gamma_3 : = \mathrm{Fix}(\mathfrak{T}_3) \bigcap \Omega_3$, where Ω_3 is the solution set of problem (4.2). For an initial point $x_0 \in H_1$, $C_1 = H_1$, $x_1 = P_{C_1}x_0$, generate a sequence $\{x_n\}$ by

$$\begin{cases} u_n = J_{r_n}^{\partial f}(I - \gamma A^*(I - J_{r_n}^{\partial g})A)x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T(s)u_n ds \\ C_{n+1} = \{z \in C_n : \ \|y_n - z\|^2 \le \|x_n - z\|^2\} \\ x_{n+1} = P_{C_{n+1}}x_0, \forall n \ge 1 \end{cases} \tag{4.4}$$

where $\{s_n\}$ is a sequence of positive real numbers with $s_n \to \infty$. $0 < a \le \alpha_n < c < 1$ for all $n \ge 1$, $0 < b \le r_n < +\infty$, $\gamma \in (0, \frac{2}{L})$, where L is the spectral radius of the operator A^*A. If $\Gamma_3 \ne \emptyset$, then the sequence $\{x_n\}$ generated by (4.4) strongly converges to a point $x^* \in \mathrm{Fix}(\mathfrak{T}_3) \bigcap \Omega_3$.

4.2 Applications to split variational inequality problems

In [2], Censor et al. proposed the following *split variational inequality problem (shortly, SVIP)*: to find a point $x^* \in C$, $y^* = Ax^* \in Q$ such that

$$\langle f(x^*), x - x^*\rangle \ge 0 \ \ \forall \ \ x \in C \ \ \textit{and} \ \ \ \langle g(y^*), y - y^*\rangle \ge 0 \ \ \forall \ \ y \in Q. \tag{4.5}$$

where $A : C \to Q$ is a bounded linear operator, $f : C \to C$ and $g : Q \to Q$ are α-inverse strongly monotone mappings, where α is a positive constant. The solution set of split variational inequality problem (4.5) is denoted by Ω_4.

It is obvious that the SVIP(4.5) is equivalent to the following fixed point problem: to find a point $x^* \in C,\ \ y^* = Ax^* \in Q$ such that

$$x^* \in Fix(P_C(I-\lambda f)), \quad Ax^* \in Fix(P_Q(I-\lambda g),\ \lambda \in (0, 2\alpha). \tag{4.6}$$

Next we prove that $P_C(I-\lambda f)$ and $P_Q(I-\lambda g),\ \lambda \in (0, 2\alpha)$ both are firmly nonexpansive. In fact, since P_C is firmly nonexpansive, by (1.2) we have

$$\begin{aligned} ||P_C(I-\lambda f)x - P_C(I-\lambda f)y||^2 &\leq ||(I-\lambda f)x - (I-\lambda f)y||^2 \\ &- ||(I-P_C(I-\lambda f))x - (I-P_C(I-\lambda f))y||^2. \end{aligned} \tag{4.7}$$

Also since

$$\begin{aligned} ||(I-\lambda f)x - (I-\lambda f)y||^2 &= ||x-y||^2 + \lambda^2||fx-fy||^2 - 2\lambda\langle x-y, fx-fy\rangle \\ &= ||x-y||^2 + \lambda^2||fx-fy||^2 - 2\lambda\alpha||fx-fy||^2 \\ &\leq ||x-y||^2 + \lambda(\lambda-2\alpha)||fx-fy||^2 \\ &\leq ||x-y||^2 \ (since\ \lambda \in (0, 2\alpha)). \end{aligned} \tag{4.8}$$

Substituting (4.8) into (4.7), we have

$$||P_C(I-\lambda f)x - P_C(I-\lambda f)y||^2 \leq ||x-y||^2 - ||(I-P_C(I-\lambda f))x - (I-P_C(I-\lambda f))y||^2. \tag{4.9}$$

This shows that $P_C(I-\lambda f),\ \lambda \in (0, 2\alpha)$ is firmly nonexpansive.

Similarly, we can also prove that $P_Q(I-\lambda g),\ \lambda \in (0, 2\alpha)$ is firmly nonexpansive.

These shows that the mappings $P_C(I-\lambda f)$ and $P_Q(I-\lambda g)$ in split variational inequality problem (4.6) have the similar properties as mappings $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ in split variational inclusion problem (1.5). Consequently, by Theorem 3.2 we have the following result.

Theorem 4.3 Let $H_1,\ H_2,\ A,\ A^*, f,\ g,$ be the same as above. Let $\mathfrak{T} = \{T(s) : 0 \leq s < \infty\} : H_1 \to H_1$ be an asymptotically nonexpansive semigroup with sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Denote by Γ_4: =Fix($\mathfrak{T}$)$\bigcap \Omega_4$, where Ω_4 is the solution set of split variational inequality problem (4.6). For an initial point $x_0 \in H_1$, $C_1 = H_1$, $x_1 = P_{C_1}x_0$, generate a sequence $\{x_n\}$ by

$$\begin{cases} u_n = P_C(I-\lambda_n f)(I-\gamma A^*(I-P_Q(I-\lambda_n g))A)x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T^n(s)u_n ds \\ C_{n+1} = \{z \in C_n :\ \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\} \\ x_{n+1} = P_{C_{n+1}}x_0, \forall n \geq 1 \end{cases} \tag{4.10}$$

where $\theta_n = (1-\alpha_n)(k_n^2-1)\sup\{\|x_n - u\|^2 : u \in \Gamma_4\}$, $\{s_n\}$ is a sequence of positive numbers. $0 < a \leq \alpha_n < c < 1$ for all $n \geq 1$, $\lambda_n \in (0, 2\alpha)$, $\gamma \in (0, \frac{2}{L})$. L is the spectral radius of the operator A^*A. If the following conditions are satisfied:

(1) $\Gamma_2 \neq \emptyset$ and is bounded;

(2) $\limsup_{n\to\infty} ||\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds - T(h)(\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds)|| = 0$, for each $h > 0$, then the sequence $\{x_n\}$ generated by (4.10) strongly converges to a point $x^* \in \mathrm{Fix}(\mathfrak{T})\bigcap\Omega_4$.

Especially, if $\mathfrak{T} = \{T(s) : 0 \le s < \infty\} : H_1 \to H_1$ be a nonexpansive semigroup, then we have the following

Theorem 4.4 Let $H_1,\ H_2,\ A,\ A^*, f,\ g$, be the same as in Theorem 4.3. Let $\mathfrak{T} = \{T(s) : 0 \le s < \infty\} : H_1 \to H_1$ be a nonexpansive semigroup. Denote by Γ_5: $=\mathrm{Fix}(\mathfrak{T})\bigcap\Omega_5$, where Ω_5 is the solution set of split variational inequality problem (4.6). For an initial point $x_0 \in H_1$, $C_1 = H_1$, $x_1 = P_{C_1}x_0$, generate a sequence $\{x_n\}$ by

$$\begin{cases} u_n = P_C(I-\lambda_n f)(I-\gamma A^*(I-P_Q(I-\lambda_n g))A)x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T(s)u_n ds \\ C_{n+1} = \{z \in C_n :\ \|y_n - z\|^2 \le \|x_n - z\|^2\} \\ x_{n+1} = P_{C_{n+1}}x_0, \forall n \ge 1 \end{cases} \tag{4.11}$$

where $\{s_n\}$ is a sequence of positive numbers with $s_n \to \infty$, $0 < a \le \alpha_n < c < 1$ for all $n \ge 1$, $\lambda_n \in (0, 2\alpha)$, $\gamma \in (0, \frac{2}{L})$. L is the spectral radius of the operator A^*A. If $\Gamma_5 \ne \emptyset$, then the sequence $\{x_n\}$ generated by (4.11) strongly converges to a point $x^* \in \mathrm{Fix}(\mathfrak{T})\bigcap\Omega_5$.

Acknowledgements The authors would like to express their thanks to the Editors and the Referees for their helpful comments and advises. This work was supported by the Natural Science Foundation of China Medical University, Taiwan.

References

[1] Moudafi, A.: Split monotone variational inclusions. J. Optim. Theory Appl. 150, 275 - 283 (2011).

[2] Censor, Y., Gibali, A., Reich, S.: Algorithms for the split variational inequality problem. Numer. Algorithms 59, 301-323 (2012)

[3] Moudafi, A.: The split common fixed point problem for demicontractive mappings. Inverse Probl. 26 055007 (6pp) (2010)

[4] Byrne, C., Censor, Y., Gibali, A., Reich, S.: Weak and strong convergence of algorithms for the split common null point problem. J. Nonlinear Convex Anal. 13, 759-775 (2012)

[5] Censor,Y., Bortfeld, T., Martin,B., Trofimov, A.:Aunified approach for inversion problems in intensity modulated radiation therapy. Phys. Med. Biol. 51, 2353-2365 (2006)

[6] Censor, Y., Elfving, T.: A multiprojection algorithm using Bregman projections in product space. Numer. Algorithms 8, 221-239 (1994)

[7] Byrne, C.: Iterative oblique projection onto convex sets and the split feasibility problem. Inverse probl. 18, 441-453 (2002)

[8] Combettes, P.L.: The convex feasibility problem in image recovery. Adv. Imaging Electron Phys. 95, 155-453 (1996)

[9] Kazmi, K. R., Rizvi, S. H.: An iterative method for split variational inclusion problem and fixed point problem for a nonexpansive mapping, Optim Lett 8: 1113-1124 (2014).

[10] Deepho, J. and Kumam, P., The Hybrid Steepest Desent Method for Split Variational Inclusion and Constrain Convex Minimization Problems, Ab- stract and Applied Analysis, Volume 2014, Article ID 365203, 13 pages.

[11] Sitthithakerngkiet, K., Deepho, J. and Kumam, P., A hybrid viscosity al- gorithm via modify the hybrid steepest descent method for sloving the split variational inclusion and fixed

point problems, Applied Mathematics and Computation, Volume 250, 1 January 2015, Pages 9861001.

[12] Iiduka, H, Takahashi, W: Strong convergence theorems for nonexpansive mappings and inverse strongly monotone mappings. Nonlinear Anal.61, 341-350 (2005)

[13] Lin, PK, Tan, KK, Xu, HK: Demiclosedness principle and asymptotic behavior for asymptotically nonexpansive mappings. Nonlinear Anal.24, 929 - 946 (1995).

[14] Sunthrayuth, P, Kumam, P: Fixed point solutions of variational inequalities for a semigroup of asymptotically nonexpansive mappings in Banach spaces, Fixed Point Theory Appl. 2012, Article ID 177 (2012)

[15] T. Shimizu, T., and Takahashi, W.: Strong convergence to common fixed points of families of nonexpansive mappings, J. Math. Anal. Appl. 211 (1997), no. 1, 71-83.

[16] Chang, S.S., Quan, J., Liu, J. A.: Feasible iterative algorithms and strong convergence theorems for bi-level fixed point problems, J. Nonlinear Sci. Appl. 9, 1515-1528 (2016)

[17] Chang, S. S., Wang, L., Tang, Y. K., Wang, G. : Moudafis open question and simultaneous iterative algorithm for general split equality variational inclusion problems and general split equality optimization problems, Fixed Point Theory and Applications, 2014:215 (2014)