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On Regularization for Magneto-thermal Problems

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Abstract

This paper studies the magneto-thermal problems with the nonlinear material law. The main difficulty is to analysis the Joule heating term $\gamma(u)|\nabla\times\boldsymbol{H}|^2$. First, a regularized model is introduced. By time discretization, the well-posedness of the discrete problem is established, and the convergence of the solution as the time step size $\tau \to 0$ is deduced. Finally, the solution to the regularized problem converges to the original as $\epsilon \to 0$. The mathematical analysis of this paper provide a routine to obtain the well-posedness of the magneto-thermal problems and gives an answer to the open question from the previous work.

KEYWORDS:
magneto-thermal; nonlinear; joule heating; well-posedness; convergence

1 | INTRODUCTION

1.1 | Background and motivation

The magnetic-thermal coupling system commonly arises from induction hardening of steel[1] , large power transformers[2], and magnetohydrodynamics (MHD)[3] etc. Such devices consist of excitation coils applied source current, which generates the varying magnetic field. Then the eddy current is induced in the conductive domain and generates the Joule heating and changes the temperature. Meanwhile, the electric conductivity depends on the thermal field, thus the change of temperature impacts the magnetic field vice versa.

There are many papers dealing with electromagnetic-thermal problems. The early results are on the electrostatic and time dependent thermal equations coupled by the gradient of electric potential[4,5,6,7]. The both diffusion equations of electric potential and thermal distributions are studied, and convergence of approximate solution is achieved[8]. The Maxwell's equations with temperature dependent conductivity were investigated, and some theoretical results including global existence of solution and regularities are established[9,10,11]. Authors provided mathematical analysis of the magneto-thermal coupling model of large power transformers, which consists of a magnetic field coupled with the thermal convection-diffusion[2]. In a magnetohydrodynamical dynamo with turbulent convection zone, authors obtain the solvability and well-posedness of a magneto-thermal coupling model[3].

The works mention above focus on the both linear mathematical models of the magnetic and thermal field. Since nonlinear material is commonly used in the industry, it will better reflect reality that the nonlinear dependency is taken into account. In the series of papers[1,12,13], the authors dedicated to the mathematical models considering the nonlinear relation between the magnetic field and the flux, and cut off the Joule heating to be bounded. Although the cut-off function avoids the unbounded solution, the uniqueness still remains an open problem[1]. Furthermore, without applying the cut-off function, the thermal equation with the uncontrolled Joule heating needs stronger regularity assumptions on the magnetic field, which could not be satisfied, and leaves the existence and uniqueness of the problem to be an open task[12]. This paper is to fulfil the task. There are some totally new theoretical work as follow.

- introduce a regularization model to deal with the uncontrolled Joule heating term $\gamma(u)|\nabla\times\boldsymbol{H}|^2$.
- prove solution of the regularization model converges to the magneto-thermal problem.

The mathematical analysis provide a routine to establish the well-posedness of the magneto-thermal problems and gives an answer to the open question from the previous work.

1.2 | Magnetic-thermal problem

Set the bounded domain $\Omega\subset\mathbb{R}^3$ is occupied by nonlinear electromagnetic material. Its boundary $\partial\Omega$ is Lipschitz continuous, with $\boldsymbol{n}$ is the outer normal unit vector. The equations for eddy current field is deduced from Maxwell's equations neglecting the displacement current term [14,15,16,17,18],

$$\begin{cases}\nabla\times(\boldsymbol{H}+\boldsymbol{H}^s)=\boldsymbol{J}^s+\boldsymbol{J},\\ \nabla\times\boldsymbol{E}=-\partial_t\boldsymbol{B}(\boldsymbol{H}+\boldsymbol{H}^s),\\ \nabla\cdot\boldsymbol{B}=0,\end{cases}\tag{1}$$

where $\boldsymbol{B}$ is the magnetic flux density depended on the total field, which includes $\boldsymbol{H}$ as the induced field, and $\boldsymbol{H}^s$ as the source field generated by the source current $\boldsymbol{J}^s$, and $\boldsymbol{J}$ is the induced current depended on the electric field by the Ohm's law,

$$\boldsymbol{J}=\sigma(u)\boldsymbol{E},\quad \text{in }\Omega,\tag{2}$$

where the conductivity is thermal dependent. For $\sigma(u)>0$ in Ω, denote the electric resistivity $\gamma(u)=1/\sigma(u)$.

The relationship between magnetic field and flux in the ferromagnetic material is based on the magnetization phenomena. Hysteresis curve shows monotone character of the relation between magnetic strength and the magnetization. In general, we denote the constitutive law for the magnetic induction as

$$\boldsymbol{B}:=\boldsymbol{B}(\boldsymbol{H}+\boldsymbol{H}^s),\quad \text{in }\Omega.\tag{3}$$

Using the (1)-(3), we have a nonlinear parabolic equation for the magnetic field,

$$\begin{cases}\partial_t\boldsymbol{B}(\boldsymbol{H}+\boldsymbol{H}^s)+\nabla\times(\gamma(u)\nabla\times\boldsymbol{H})=0, & (\boldsymbol{x},t)\in\Omega\times(0,T),\\ \boldsymbol{H}\times\boldsymbol{n}=\boldsymbol{0}, & (\boldsymbol{x},t)\in\partial\Omega\times(0,T),\\ \boldsymbol{H}(\boldsymbol{x},0)=\boldsymbol{H}_0, & \boldsymbol{x}\in\bar{\Omega}.\end{cases}\tag{4}$$

The varying magnetic field will induce eddy current in the conductor and generate heat, which is called the *Joule heating* described as

$$q=\boldsymbol{J}\cdot\boldsymbol{E}=|\boldsymbol{J}|^2/\sigma(u)=\gamma(u)|\nabla\times\boldsymbol{H}|^2.\tag{5}$$

Let u be the thermal field in the conductor, and it is governed by the following nonlinear parabolic equation [1,12,13],

$$\begin{cases}\partial_t\beta(u)-\nabla\cdot(\lambda\nabla u)=q, & (\boldsymbol{x},t)\in\Omega\times(0,T),\\ -\lambda\dfrac{\partial u}{\partial\boldsymbol{n}}=0, & (\boldsymbol{x},t)\in\partial\Omega\times(0,T),\\ u(\boldsymbol{x},0)=u_0, & \boldsymbol{x}\in\bar{\Omega}.\end{cases}\tag{6}$$

where λ is the thermal conductivity, and $\beta(u)$ is nonlinear function.

The remainder of the paper is organized as follows. In Section 2, we introduce some function spaces and establish the variational problem and the regularization. In Section 3, the time discretization scheme to the regularized problem is given, the existence and uniqueness, boundedness and convergence as the time step size $\tau\to 0$ are discussed. In Section 4, passing limit for the regularization parameter $\epsilon\to 0$, the regularized solution converges the original.

2 | VARIATIONAL PROBLEM

In this section, we introduce some function spaces related to solutions of the problem, and derive the variational. Since the Joule heating source contains $|\nabla\times\boldsymbol{H}|^2\in L^1(\Omega)$, this term brings the main difficulty to the analysis. An regularization model is introduced to overcome this problem. And Some assumptions for the material laws are proposed for later analysis.

2.1 | Function spaces

The $L^p(\Omega)$ denote as the Lebesgue integrable function space with norm

$$\|u\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |u(\boldsymbol{x})|^p \mathrm{d}\boldsymbol{x}\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{\boldsymbol{x}\in\Omega} |u(\boldsymbol{x})|, & p = \infty. \end{cases}$$

Throughout the paper, the vector valued functions and function spaces are denoted in bold (e.g. $\boldsymbol{u}$, $\boldsymbol{L}^p$) while the scalar ones are in normal (e.g. u, L^p). Norm $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ for short.

The curl space for the magnetic field $\boldsymbol{H}$ [19,20],

$$\begin{aligned} \boldsymbol{H}(\boldsymbol{curl},\Omega) &:= \left\{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) : \nabla\times\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)\right\}, \\ \boldsymbol{H}_0(\boldsymbol{curl},\Omega) &:= \{\boldsymbol{u} \in \boldsymbol{H}(\boldsymbol{curl},\Omega) : \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} \text{ on } \partial\Omega\}, \\ \|\boldsymbol{u}\|_{\boldsymbol{H}(\boldsymbol{curl},\Omega)} &:= \left(\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\times\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}\right)^{1/2}. \end{aligned}$$

and the $\boldsymbol{H}^{-1}(\boldsymbol{curl},\Omega)$ is the dual space consisted of all bounded linear functionals on $\boldsymbol{H}_0(\boldsymbol{curl},\Omega)$.

The $H^1(\Omega)$ space for thermal field u,

$$\begin{aligned} H^1(\Omega) &:= \left\{u \in L^2(\Omega) : \nabla u \in \boldsymbol{L}^2(\Omega)\right\}, \\ \|u\|_{H^1(\Omega)} &:= \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{\boldsymbol{L}^2(\Omega)}\right)^{1/2}, \end{aligned}$$

and the $H^{-1}(\Omega)$ is the dual space.

Let X^* denote the dual space of X, then the dual norm could be defined as

$$\|u\|_{X^*} = \sup_{0\neq v\in X} \frac{\langle u, v\rangle}{\|v\|_X},$$

where the dual product $\langle u, v\rangle$ denotes the value of the linear functional $u \in X^*$ at the point $v \in X$.

The $C\big([0,T];X\big)$ denotes for the space of abstract functions continuous in time, i.e.,

$$C\big([0,T];X\big) := \left\{u(\boldsymbol{x},\cdot) : [0,T] \to X, \|u\|_{C([0,T];X)} = \max_{t\in[0,T]} \|u(\boldsymbol{x},t)\|_X < \infty\right\}.$$

And the $L^p\big((0,T);X\big)$ is defined with the norm

$$\|u(\boldsymbol{x},t)\|_{L^p((0,T);X)} = \begin{cases} \left(\int_0^T \|u(\boldsymbol{x},t)\|^p_X \,\mathrm{d}t\right)^{1/p}, & 1 \le p \le \infty, \\ \operatorname*{ess\,sup}_{t\in[0,T]} \|u(\boldsymbol{x},t)\|_X, & p = \infty. \end{cases}$$

2.2 | Variational problem and regularization

With the above function spaces statement, we now establish the Galerkin variational problem for (4) and (6).

Weak form for magnetic field: Given $\boldsymbol{H}(0) = \boldsymbol{H}_0$, find $\boldsymbol{H} \in L^2((0,T);\boldsymbol{H}_0(\boldsymbol{curl},\Omega))$, such that

$$\left(\partial_t \boldsymbol{B}(\boldsymbol{H}+\boldsymbol{H}^s), \bar{\boldsymbol{H}}\right) + \left(\gamma(u)\nabla\times\boldsymbol{H}, \nabla\times\bar{\boldsymbol{H}}\right) = 0, \quad \forall\bar{\boldsymbol{H}} \in \boldsymbol{H}_0(\boldsymbol{curl},\Omega). \tag{7}$$

Weak form for the thermal field: Given $u(0) = u_0$, find $u \in L^2((0,T);H^1(\Omega))$, such that

$$\left(\partial_t \beta(u), \bar{u}\right) + (\lambda\nabla u, \nabla\bar{u}) = \left(q, \bar{u}\right), \quad \forall\bar{u} \in H^1(\Omega)\cap L^\infty(\Omega). \tag{8}$$

Since the Joule heating q contains $|\nabla\times\boldsymbol{H}|^2 \in L^1(\Omega)$, which requires the test function $\bar{u} \in L^\infty(\Omega)$. To overcome this difficulty, we introduce a truncation as follows[2,9]

$$[q]_\epsilon = \frac{q}{1+\epsilon|q|}, \quad \epsilon > 0. \tag{9}$$

Then the Joule heating source term becomes a essentially bounded, i.e. $[q]_\epsilon \in L^\infty(\Omega)$. The more critical point is that for any $q \in L^1(\Omega)$, passing to the limit for $\epsilon \to 0$, we have $\big\|[q]_\epsilon - q\big\|_{L^{1/2}(\Omega)} \to 0$.

In another word, the Joule heating q could be replace by $[q]_\epsilon$ and we obtain a series of solution $\{(\boldsymbol{H}^\epsilon, u^\epsilon)\}$, then by passing to the limit for $\epsilon \to 0$, we will achieve the solution of the origin problem (8), i.e. $(\boldsymbol{H}^\epsilon, u^\epsilon) \to (\boldsymbol{H}, u)$. We call this process the *regularization*, and the number ϵ plays the role as the regularization parameter. Therefore, we turn to solve the regularized problem (7)-(8): Given $\boldsymbol{H}(0) = \boldsymbol{H}_0$, find $\boldsymbol{H} \in L^2((0,T); \boldsymbol{H}_0(\boldsymbol{curl}, \Omega))$, such that

$$\left(\partial_t \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s), \bar{\boldsymbol{H}}\right) + \left(\gamma(u)\nabla \times \boldsymbol{H}, \nabla \times \bar{\boldsymbol{H}}\right) = 0, \quad \forall \bar{\boldsymbol{H}} \in \boldsymbol{H}_0(\boldsymbol{curl}, \Omega). \tag{10}$$

Given $u(0) = u_0$, find $u \in L^2\left((0,T); H^1(\Omega)\right)$, such that

$$\left(\partial_t \beta(u), \bar{u}\right) + (\lambda \nabla u, \nabla \bar{u}) = \left([q]_\epsilon, \bar{u}\right), \quad \forall \bar{u} \in H^1(\Omega). \tag{11}$$

In deed, the solution of (10)-(11) should be denoted by $(\boldsymbol{H}^\epsilon, u^\epsilon)$, without confusion, we reuse the same symbol $(\boldsymbol{H}, u)$.

2.3 | Assumptions

For a better representation of the paper, all the assumptions are listed here.

$$\begin{aligned}
&\boldsymbol{H}_0(\boldsymbol{x}) \in \boldsymbol{H}_0(\boldsymbol{curl}, \Omega), && && (12)\\
&\boldsymbol{H}^s(\boldsymbol{x}, t) \in H^1\left((0,T); \boldsymbol{H}_0(\boldsymbol{curl}, \Omega)\right), && && (13)\\
&\nabla \cdot \boldsymbol{B}(\boldsymbol{H}_0 + \boldsymbol{H}_0^s) = 0, && \text{for a.e. } \boldsymbol{x} \in \Omega, && (14)\\
&0 < \gamma_* \le \gamma\left(\boldsymbol{x}, t\right) \le \gamma^* < \infty, && \forall (\boldsymbol{x}, t) \in \Omega \times (0,T), && (15)\\
&|\gamma(u_1) - \gamma(u_2)| \le L_\gamma \left\|u_1 - u_2\right\|_{H^{-1}(\Omega)}, && \forall u_1, u_2 \in H^1(\Omega), && (16)\\
&0 < \lambda_* \le \lambda(\boldsymbol{x}, t) \le \lambda^* < \infty, && \forall (\boldsymbol{x}, t) \in \Omega \times (0,T), && (17)\\
&|\lambda(\boldsymbol{x}, t_2) - \lambda(\boldsymbol{x}, t_1)| \le L_\lambda |t_2 - t_1|, && \forall \boldsymbol{x} \in \Omega, \forall t_1, t_2 \in (0,T), && (18)\\
&\boldsymbol{B}(\boldsymbol{0}) = \boldsymbol{0}, && && (19)\\
&|\boldsymbol{B}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{y})| \le L_{\boldsymbol{B}} |\boldsymbol{x} - \boldsymbol{y}|, && L_{\boldsymbol{B}} > 0, \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3, && (20)\\
&\left(\boldsymbol{B}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{y})\right) \cdot (\boldsymbol{x} - \boldsymbol{y}) \ge m_{\boldsymbol{B}} |\boldsymbol{x} - \boldsymbol{y}|^2, && m_{\boldsymbol{B}} > 0, \forall \boldsymbol{x},\ \boldsymbol{y} \in \mathbb{R}^3, && (21)\\
&\beta(0) = 0, && && (22)\\
&0 < \beta'_* \le \beta'(u), && \forall u \in \mathbb{R}, && (23)\\
&|\beta(u) - \beta(v)| \le L_\beta |u - v|, && L_\beta > 0, \forall u, v \in \mathbb{R}, && (24)\\
&\left(\beta(u) - \beta(v)\right) \cdot (u - v) \ge m_\beta |u - v|^2, && m_\beta > 0, \forall u,\ v \in \mathbb{R}. && (25)
\end{aligned}$$

Suppose the magnetic induction $\boldsymbol{B}(\cdot)$ to be potential, i.e., there exists a functional $\Phi_{\boldsymbol{B}} : \boldsymbol{H}_0(\boldsymbol{curl}, \Omega) \to \mathbb{R}$, such that for all $\boldsymbol{H} \in \boldsymbol{H}_0(\boldsymbol{curl}, \Omega)$,

$$\text{Grad } \Phi_{\boldsymbol{B}}(\boldsymbol{H}) = \boldsymbol{B}(\boldsymbol{H}),$$

where Grad $\Phi_{\boldsymbol{B}}$ is called the *gradient of the functional* $\Phi_{\boldsymbol{B}}$. By the (21) and according to the theorem of the monotone potential operator[21, Theorem 5.1], the potential $\Phi_{\boldsymbol{B}}$ is convex, i.e., for any $\boldsymbol{x},\ \boldsymbol{y} \in \boldsymbol{H}_0(\boldsymbol{curl}, \Omega)$ and any $\lambda \in (0,1)$,

$$\Phi_{\boldsymbol{B}}(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \le \lambda \Phi_{\boldsymbol{B}}(\boldsymbol{x}) + (1-\lambda)\Phi_{\boldsymbol{B}}(\boldsymbol{y}).$$

Lemma 1. Let $f(\boldsymbol{x})$ be a convex functional and twice Gâteaux-differential on a convex set E in a normed space then its twice Gâteaux-differential is non-negative, i.e.

$$D^2 f(\boldsymbol{x}; \boldsymbol{h}, \boldsymbol{h}) \ge 0, \quad \boldsymbol{x} \in E.$$

The proof of the Lemma 1 is trivial, and we omit it here. By Lemma 1, for the convexity of potential $\Phi_{\boldsymbol{B}}$, we have

$$D^2 \Phi_{\boldsymbol{B}}(\boldsymbol{x}; \boldsymbol{h}, \boldsymbol{h}) \ge 0. \tag{26}$$

Using generalized Lagrange formula[21] twice and by (26), for $\xi_1,\ \xi_2 \in (0,1)$, we have

$$\begin{aligned}
&\Phi_{\boldsymbol{B}}(\boldsymbol{y}) - \Phi_{\boldsymbol{B}}(\boldsymbol{x}) - D\Phi_{\boldsymbol{B}}(\boldsymbol{x}, \boldsymbol{y} - \boldsymbol{x}) = D\Phi_{\boldsymbol{B}}\left(\boldsymbol{x} + \xi_1(\boldsymbol{y} - \boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x}\right) - D\Phi_{\boldsymbol{B}}(\boldsymbol{x}, \boldsymbol{y} - \boldsymbol{x})\\
=&D^2 \Phi_{\boldsymbol{B}}\left(\boldsymbol{x} + \xi_2(\boldsymbol{y} - \boldsymbol{x}); \boldsymbol{y} - \boldsymbol{x}, \xi_1(\boldsymbol{y} - \boldsymbol{x})\right) \ge 0.
\end{aligned}$$

Hence, $\Phi_{\boldsymbol{B}}(\boldsymbol{y}) - \Phi_{\boldsymbol{B}}(\boldsymbol{x}) \ge D\Phi_{\boldsymbol{B}}(\boldsymbol{x}, \boldsymbol{y} - \boldsymbol{x}) = \boldsymbol{B}(\boldsymbol{x}) \cdot (\boldsymbol{y} - \boldsymbol{x})$, i.e.,

$$\boldsymbol{B}(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \ge \Phi_{\boldsymbol{B}}(\boldsymbol{x}) - \Phi_{\boldsymbol{B}}(\boldsymbol{y}). \tag{27}$$

By (19)-(21),

$$\Phi_{\boldsymbol{B}}(\boldsymbol{x}) = \int_0^1 \boldsymbol{B}(p\boldsymbol{x}) \cdot \boldsymbol{x} \mathrm{d}p \geq \int_0^1 m_{\boldsymbol{B}} |p\boldsymbol{x}|^2 \frac{1}{p} \mathrm{d}p = \frac{m_{\boldsymbol{B}}}{2} |\boldsymbol{x}|^2 . \tag{28}$$

$$\Phi_{\boldsymbol{B}}(\boldsymbol{x}) = \int_0^1 \boldsymbol{B}(p\boldsymbol{x}) \cdot \boldsymbol{x} \mathrm{d}p \leq \int_0^1 L_{\boldsymbol{B}} |p\boldsymbol{x}|^2 \frac{1}{p} \mathrm{d}p = \frac{L_{\boldsymbol{B}}}{2} |\boldsymbol{x}|^2 . \tag{29}$$

Thus, we obtain the boundedness of the potential $\Phi_{\boldsymbol{B}}$,

$$0 \leq \frac{m_{\boldsymbol{B}}}{2}|\boldsymbol{x}|^2 \leq \Phi_{\boldsymbol{B}}(\boldsymbol{x}) \leq \frac{L_{\boldsymbol{B}}}{2}|\boldsymbol{x}|^2. \tag{30}$$

Since $\boldsymbol{B}$ is strongly monotone, we have

$$|\boldsymbol{B}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{y})| \cdot |\boldsymbol{x} - \boldsymbol{y}| \geq \big(\boldsymbol{B}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y}\big) \geq m_{\boldsymbol{B}}|\boldsymbol{x} - \boldsymbol{y}|^2.$$

together with Lipschitz continuity of $\boldsymbol{B}$ in (20), we have,

$$L_{\boldsymbol{B}}|\boldsymbol{x} - \boldsymbol{y}| \geq |\boldsymbol{B}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{y})| \geq m_{\boldsymbol{B}}|\boldsymbol{x} - \boldsymbol{y}|,$$

thus, we have

$$L_{\boldsymbol{B}}^{-1}|\boldsymbol{x} - \boldsymbol{y}| \leq |\boldsymbol{B}^{-1}(\boldsymbol{x}) - \boldsymbol{B}^{-1}(\boldsymbol{y})| \leq m_{\boldsymbol{B}}^{-1}|\boldsymbol{x} - \boldsymbol{y}| := L_{\boldsymbol{B}^{-1}}|\boldsymbol{x} - \boldsymbol{y}|,$$

which concludes that the inverse $\boldsymbol{B}^{-1}$ is also Lipschitz continuous. By (19), we have

$$\boldsymbol{B}^{-1}(\boldsymbol{0}) = \boldsymbol{0}. \tag{31}$$

Since $\boldsymbol{B}$ is strongly monotone, we have $\boldsymbol{B}^{-1}$ is also strongly monotone,

$$\begin{aligned}\big(\boldsymbol{B}^{-1}(\boldsymbol{x}) - \boldsymbol{B}^{-1}(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y}\big) &= \Big(\boldsymbol{B}^{-1}(\boldsymbol{x}) - \boldsymbol{B}^{-1}(\boldsymbol{y}), \boldsymbol{B}\big(\boldsymbol{B}^{-1}(\boldsymbol{x})\big) - \boldsymbol{B}\big(\boldsymbol{B}^{-1}(\boldsymbol{y})\big)\Big) \\ &\geq m_{\boldsymbol{B}}|\boldsymbol{B}^{-1}(\boldsymbol{x}) - \boldsymbol{B}^{-1}(\boldsymbol{y})|^2 \geq m_{\boldsymbol{B}}(L_{\boldsymbol{B}}^{-1})^2|\boldsymbol{x} - \boldsymbol{y}|^2 := m_{\boldsymbol{B}^{-1}}|\boldsymbol{x} - \boldsymbol{y}|^2,\end{aligned}$$

then the potential $\Phi_{\boldsymbol{B}^{-1}}$ of $\boldsymbol{B}^{-1}$ is also convex. Similar in (26)-(30), we have

$$\boldsymbol{B}^{-1}(\boldsymbol{x}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \geq \Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{x}) - \Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{y}), \tag{32}$$

$$0 \leq \frac{m_{\boldsymbol{B}^{-1}}}{2}|\boldsymbol{x}|^2 \leq \Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{x}) \leq \frac{L_{\boldsymbol{B}^{-1}}}{2}|\boldsymbol{x}|^2, \tag{33}$$

$$0 \leq c_{\boldsymbol{B}}|\boldsymbol{x}|^2 \leq \Phi_{\boldsymbol{B}^{-1}}\big(\boldsymbol{B}(\boldsymbol{x})\big) \leq C_{\boldsymbol{B}}|\boldsymbol{x}|^2. \tag{34}$$

3 | TIME DISCRETIZATION

In this section, the regularized problem is discretized in time using backwards Euler method. First, the existence and uniqueness of the discrete problem is proved based on the theorem of monotone operators[21]. Second, the boundedness of the solution is guaranteed. Third, the convergence of the solution are discussed in the framework of the Rothe's method.

3.1 | Time discretization scheme

Let the time interval $[0, T]$ is partitioned into n equidistant subintervals by time step size τ, i.e.,

$$[0, T] = \bigcup_{i=1}^{n} [t_{i-1}, t_i], \text{ where } t_i = i\tau, \text{ for } i = 0, 1, \cdots, n, \text{ and } \tau = T/n.$$

For a general function $f(t)$ with respect to time, denote $f_i := f(t_i)$. Approximate the time derivative of $f(t)$ at t_i by backward Euler method,

$$\partial_t f(t_i) \approx \delta_\tau f_i := \frac{f_i - f_{i-1}}{\tau}.$$

where δ_τ denotes the divided difference operator in time with step size τ.

By shifting the resistivity on the right hand side of (36) to be $\gamma(u_{i-1})$, we obtain a decoupled system of the time discrete problem for (10)-(11): Given $\boldsymbol{H}_0$ and u_0, for $i = 1, 2, \cdots, n$, find $\boldsymbol{H}_i \in \boldsymbol{H}_0(\boldsymbol{curl}, \Omega)$ and $u_i \in H^1(\Omega)$, such that

$$\left(\delta_\tau \boldsymbol{B}(\boldsymbol{H}_i + \boldsymbol{H}_i^s), \bar{\boldsymbol{H}}\right) + \left(\gamma(u_{i-1})\nabla \times \boldsymbol{H}_i, \nabla \times \bar{\boldsymbol{H}}\right) = 0, \forall \bar{\boldsymbol{H}} \in \boldsymbol{H}_0(\boldsymbol{curl}, \Omega), \tag{35}$$

$$\left(\delta_\tau \beta(u_i), \bar{u}\right) + (\lambda_i \nabla u_i, \nabla \bar{u}) = \left(\left[\gamma(u_{i-1})|\nabla \times \boldsymbol{H}_i|^2\right]_\epsilon, \bar{u}\right), \forall \bar{u} \in H^1(\Omega). \tag{36}$$

Based on the theorem of monotone operator[21], we prove the existence and uniqueness of the solution to discrete problem (35)-(36) in the following lemma.

Lemma 2. Assume that (12)-(19) holds. Then, for any $i = 1, 2, \cdots, n$, there exist a uniquely determined couple $(\boldsymbol{H}_i, u_i) \in \boldsymbol{H}_0(\boldsymbol{curl}, \Omega) \times H^1(\Omega)$ solving the system (35)-(36).

Proof. Define operator $\mathcal{F}_\gamma : \ \boldsymbol{H}_0(\boldsymbol{curl}, \Omega) \to \boldsymbol{H}^{-1}(\boldsymbol{curl}, \Omega)$,

$$\left\langle \mathcal{F}_\gamma(\boldsymbol{H}), \bar{\boldsymbol{H}} \right\rangle := \frac{1}{\tau}\left(\boldsymbol{B}(\boldsymbol{H}), \bar{\boldsymbol{H}}\right) + \left(\gamma \nabla \times \boldsymbol{H}, \nabla \times \bar{\boldsymbol{H}}\right).$$

Define operator $\mathcal{G} : \ H^1(\Omega) \to H^{-1}(\Omega)$,

$$\langle \mathcal{G}(u), \bar{u} \rangle := \frac{1}{\tau}\left(\beta(u), \bar{u}\right) + (\lambda \nabla u, \nabla \bar{u}).$$

Then (35)-(36) could be rewritten as the following operator equations

$$\left\langle \mathcal{F}_{\gamma(u_{i-1})}(\boldsymbol{H}_i + \boldsymbol{H}_i^s), \bar{\boldsymbol{H}} \right\rangle = \frac{1}{\tau}\left(\boldsymbol{B}(\boldsymbol{H}_{i-1} + \boldsymbol{H}_{i-1}^s), \bar{\boldsymbol{H}}\right), \tag{37}$$

$$\langle \mathcal{G}(u_i), \bar{u} \rangle = \frac{1}{\tau}\left(\beta(u_{i-1}), \bar{u}\right) + \left(\left[\gamma(u_{i-1})|\nabla \times \boldsymbol{H}_i|^2\right]_\epsilon, \bar{u}\right), \tag{38}$$

Then the strictly monotone of $\mathcal{F}_\gamma$ follows from (15) and (21). For any $\boldsymbol{H}_1, \ \boldsymbol{H}_2 \in \boldsymbol{H}_0(\boldsymbol{curl}, \Omega)$,

$$\begin{aligned}\left\langle \mathcal{F}_\gamma(\boldsymbol{H}_1) - \mathcal{F}_\gamma(\boldsymbol{H}_2), \boldsymbol{H}_1 - \boldsymbol{H}_2 \right\rangle &\geq m_{\boldsymbol{B}}/\tau \left\|\boldsymbol{H}_1 - \boldsymbol{H}_2\right\|^2_{\boldsymbol{L}^2(\Omega)} + \gamma_* \left\|\nabla \times (\boldsymbol{H}_1 - \boldsymbol{H}_2)\right\|^2_{\boldsymbol{L}^2(\Omega)} \\ &\geq \min(m_{\boldsymbol{B}}/\tau, \gamma_*) \cdot \left\|\boldsymbol{H}_1 - \boldsymbol{H}_2\right\|^2_{\boldsymbol{H}(\boldsymbol{curl}, \Omega)} > 0.\end{aligned}$$

Strictly monotone of $\mathcal{G}$ follows from (17) and (25). For any $u_1, \ u_2 \in H^1(\Omega)$,

$$\langle \mathcal{G}(u_1) - \mathcal{G}(u_2), u_1 - u_2 \rangle \geq m_\beta/\tau \left\|u_1 - u_2\right\|^2_{L^2(\Omega)} + \lambda_* \left\|\nabla(u_1 - u_2)\right\|^2_{L^2(\Omega)} \geq \min(m_\beta/\tau, \lambda_*) \cdot \left\|u_1 - u_2\right\|^2_{H^1(\Omega)} > 0.$$

Coercivity of $\mathcal{F}_\gamma$ follows from (19) and (21). For any $\boldsymbol{H} \in \boldsymbol{H}_0(\boldsymbol{curl}, \Omega)$,

$$\left\langle \mathcal{F}_\gamma(\boldsymbol{H}), \boldsymbol{H} \right\rangle \geq m_{\boldsymbol{B}}/\tau \left\|\boldsymbol{H}\right\|^2_{\boldsymbol{L}^2(\Omega)} + \gamma_* \left\|\nabla \times \boldsymbol{H}\right\|^2_{\boldsymbol{L}^2(\Omega)} \geq C \left\|\boldsymbol{H}\right\|^2_{\boldsymbol{H}(\boldsymbol{curl}, \Omega)}.$$

Coercivity of $\mathcal{G}$ follows from (22) and (25). For any $u \in H^1(\Omega)$,

$$\langle \mathcal{G}(u), u \rangle \geq m_\beta/\tau \left\|u\right\|^2_{L^2(\Omega)} + \lambda_* \left\|\nabla u\right\|^2_{L^2(\Omega)} \geq C \left\|u\right\|^2_{H^1(\Omega)}.$$

Hemi-continuity of $\mathcal{F}_\gamma$ follows from (20),

$$\left\langle \mathcal{F}_\gamma(\boldsymbol{H} + \varepsilon \boldsymbol{h}) - \mathcal{F}_\gamma(\boldsymbol{H}), \bar{\boldsymbol{H}} \right\rangle \leq L_{\boldsymbol{B}} \varepsilon/\tau \left|\left(\boldsymbol{h}, \bar{\boldsymbol{H}}\right)\right| + \varepsilon \gamma^* \left(\nabla \times \boldsymbol{h}, \nabla \times \bar{\boldsymbol{H}}\right) \leq C\varepsilon \left\|\boldsymbol{h}\right\|_{\boldsymbol{H}(\boldsymbol{curl}, \Omega)} \cdot \left\|\bar{\boldsymbol{H}}\right\|_{\boldsymbol{H}(\boldsymbol{curl}, \Omega)} \to 0, \quad \varepsilon \to 0.$$

Hemi-continuity of $\mathcal{G}$ follows from (24),

$$\left\langle \mathcal{G}_\gamma(u + \varepsilon v) - \mathcal{G}_\gamma(u), \bar{u} \right\rangle \leq L_\beta \varepsilon/\tau \left|(v, \bar{u})\right| + \varepsilon \lambda^* \left(\nabla v, \nabla \bar{u}\right) \leq C\varepsilon \left\|v\right\|_{H^1(\Omega)} \cdot \left\|\bar{u}\right\|_{H^1(\Omega)} \to 0, \quad \varepsilon \to 0.$$

We have shown the strict monotonicity, coercivity and hemi-continuity of operators $\mathcal{F}_\gamma$ and $\mathcal{G}_\tau$. Furthermore, for every time step $i = 1, 2, \cdots, n$, the right hand side of (37) is a bounded linear functionals on $\boldsymbol{H}_0(\boldsymbol{curl}, \Omega)$,

$$\frac{1}{\tau}\left(\boldsymbol{B}(\boldsymbol{H}_{i-1} + \boldsymbol{H}_{i-1}^s), \bar{\boldsymbol{H}}\right) \leq C_\tau \left\|\boldsymbol{B}(\boldsymbol{H}_{i-1} + \boldsymbol{H}_{i-1}^s)\right\|_{\boldsymbol{H}^{-1}(\boldsymbol{curl}, \Omega)} \left\|\bar{\boldsymbol{H}}\right\|_{\boldsymbol{H}_0(\boldsymbol{curl}, \Omega)} \leq C_\tau \left\|\bar{\boldsymbol{H}}\right\|_{\boldsymbol{H}_0(\boldsymbol{curl}, \Omega)},$$

and the right hand side of (38) is also a bounded linear functionals on $H^1(\Omega)$,

$$\frac{1}{\tau}\left(\beta(u_{i-1}), \bar{u}\right) + \left(\left[\gamma(u_{i-1})|\nabla \times \boldsymbol{H}_i|^2\right]_\epsilon, \bar{u}\right) \leq \frac{1}{\tau}\left\|\beta(u_{i-1})\right\|_{H^{-1}(\Omega)} \left\|\bar{u}\right\|_{H^1(\Omega)} + \frac{1}{\epsilon}\left\|\bar{u}\right\|_{L^2(\Omega)} \leq C_{\tau,\epsilon} \left\|\bar{u}\right\|_{H^1(\Omega)}.$$

Therefore, we obtain the existence and uniqueness of the solution to (35) and (36) at every time step[21, Theorem 18.2]. □

3.2 | Boundedness of discrete solution

In this subsection, we establish some bounded estimates of the solution, which will play a key role in the convergence of the solution.

Lemma 3. Assume (12) - (19). Then for any $j = 1, 2, \cdots, n$, there exists a positive constant C, such that

$$(a)\ \sum_{i=1}^{j} \tau \left\|\delta_\tau u_i\right\|^2 + \sum_{i=1}^{j} \left\|\nabla(u_i - u_{i-1})\right\|^2 + \max_{1\le i\le j} \left\|\nabla u_i\right\|^2 \le C, \tag{39}$$

$$(b)\ \max_{1\le i\le j} \left\|u_i\right\|^2 \le C, \tag{40}$$

$$(c)\ \max_{1\le i\le j} \left\|\delta_\tau \beta(u_i)\right\|_{H^{-1}(\Omega)} \le C, \tag{41}$$

$$(d)\ \max_{1\le i\le j} \left\|\delta_\tau u_i\right\|_{H^{-1}(\Omega)} \le C. \tag{42}$$

Proof. (a) Set $\bar{u} = \tau\delta_\tau u_i$ in (36),

$$\left(\delta_\tau \beta(u_i), \tau\delta_\tau u_i\right) + (\lambda_i \nabla u_i, \tau\nabla\delta_\tau u_i) = \left(\left[\gamma(u_{i-1})|\nabla\times \boldsymbol{H}_i|^2\right]_\epsilon, \tau\delta_\tau u_i\right).$$

Lower bound for the first term in the left hand side:

$$\left(\delta_\tau \beta(u_i), \tau\delta_\tau u_i\right) \ge \int_\Omega \frac{m_\beta}{\tau}|u_i - u_{i-1}|^2 = \tau m_\beta \left\|\delta_\tau u_i\right\|^2.$$

For the second term, combine the Abel's summation law as follows

$$(\lambda_i \nabla u_i, \tau\nabla\delta_\tau u_i) = \frac{1}{2}\int_\Omega \left(\lambda_i|\nabla u_i|^2 - \lambda_{i-1}|u_{i-1}|^2 - (\lambda_i - \lambda_{i-1})|u_{i-1}|^2 + \lambda_i|\nabla(u_i - u_{i-1})|^2\right).$$

Upper bound for the right hand side,

$$\left(\left[\gamma(u_{i-1})|\nabla\times \boldsymbol{H}_i|^2\right]_\epsilon, \delta_\tau u_i \tau\right) \le \frac{\tau}{\epsilon}\int_\Omega |\delta_\tau u_i| \mathrm{d}\boldsymbol{x} \le C_\epsilon C_\varepsilon |\Omega|\tau + \tau\varepsilon C_\epsilon \left\|\delta_\tau u_i\right\|^2_{L^2(\Omega)},$$

where C_ϵ is a big number with respect to ϵ^{-1}, and ε denotes an suitable small number, C_ε denotes a big number related to ε^{-1} according to the Young's inequality.

Sum up for $i = 1, 2, \cdots, j \le n$,

$$\begin{aligned}&\sum_{i=1}^{j} C_\epsilon C_\varepsilon|\Omega|\tau + \sum_{i=1}^{j} \tau\varepsilon C_\epsilon \left\|\delta_\tau u_i\right\|^2 = C_\epsilon C_\varepsilon|\Omega| j\tau + \varepsilon C_\epsilon \sum_{i=1}^{j} \tau \left\|\delta_\tau u_i\right\|^2 \\ \ge& m_\beta \sum_{i=1}^{j} \tau \left\|\delta_\tau u_i\right\|^2 + \frac{\lambda_*}{2}\|\nabla u_j\|^2 - \frac{\lambda_0}{2}\|u_0\|^2 + \frac{\lambda_*}{2}\sum_{i=1}^{j}\|\nabla(u_i - u_{i-1})\|^2 - \frac{L_\lambda}{2}\sum_{i=1}^{j}\tau\|u_{i-1}\|^2.\end{aligned}$$

Thus,

$$(m_\beta - \varepsilon C_\epsilon)\sum_{i=1}^{j} \tau \left\|\delta_\tau u_i\right\|^2 + \frac{\lambda_*}{2}\sum_{i=1}^{j}\|\nabla(u_i - u_{i-1})\|^2 + \frac{\lambda_*}{2}\left\|\nabla u_j\right\|^2 \le C + \frac{L_\lambda}{2}\sum_{i=1}^{j}\tau\left\|\nabla u_{i-1}\right\|^2_{L^2(\Omega)}.$$

By fixing a small $\varepsilon < m_\beta/C_\epsilon$ and using the Grönwall's inequality, the result in (a) is concluded.

(b) By (a), for any $j = 1, 2, \cdots, n$,

$$\left\|u_j\right\| = \left\|u_0 + \sum_{i=1}^{j}\delta_\tau u_i \tau\right\| \le \|u_0\| + \sum_{i=1}^{j}\tau\left\|\delta_\tau u_i\right\| \le C.$$

(c) From (36),

$$\left|\left(\delta_\tau \beta(u_i), \bar{u}\right)\right| \le \left|(\lambda_i\nabla u_i, \nabla\bar{u})\right| + \left|\left(\left[\gamma(u_{i-1})|\nabla\times \boldsymbol{H}_i|^2\right]_\epsilon, \bar{u}\right)\right| \le \lambda^*\left\|\nabla u_i\right\|\cdot\|\nabla\bar{u}\| + \frac{1}{\epsilon}\int_\Omega |\bar{u}|\mathrm{d}\boldsymbol{x} \le C\left\|\bar{u}\right\|_{H^1(\Omega)},$$

which concludes that

$$\left\|\delta_\tau \beta(u_i)\right\|_{H^{-1}(\Omega)} \le C, \quad \forall i = 1, 2, \cdots, n.$$

(d) By the mean value theorem and the assumption (23), we have

$$\left\|\delta_\tau \beta(u_i)\right\|_{H^{-1}(\Omega)} = \left\|\tau^{-1}\beta'(\xi_i)\left|u_i - u_{i-1}\right|\right\|_{H^{-1}(\Omega)} \geq \beta'_* \left\|\delta_\tau u_i\right\|_{H^{-1}(\Omega)}.$$

Thus, by the result of (c), we have

$$\left\|\delta_\tau u_i\right\|_{H^{-1}(\Omega)} \leq C, \quad \forall i = 1, 2, \cdots, n.$$

□

Remark 1. For any $i = 1, 2, \cdots, n$, by the weak Lipschitz continuous (16) of the γ and the result (d) of Lemma 3, we have

$$\left|\gamma(u_i) - \gamma(u_{i-1})\right| \leq L_\gamma \left\|u_i - u_{i-1}\right\|_{H^{-1}(\Omega)} = L_\gamma \left\|\delta_\tau u_i\right\|_{H^{-1}(\Omega)} \tau \leq C\tau. \tag{43}$$

Lemma 4. Assume (12) - (19). Then for any $j = 1, 2, \cdots, n$, there exists a positive constant C such that,

$$(a)\ \sum_{i=1}^{j} \tau \left\|\delta_\tau(\boldsymbol{H}_i + \boldsymbol{H}_i^s)\right\|^2 + \sum_{i=1}^{j} \left\|\nabla \times (\boldsymbol{H}_i - \boldsymbol{H}_{i-1})\right\|^2 + \max_{1\leq i\leq j} \left\|\nabla \times \boldsymbol{H}_i\right\|^2 \leq C, \tag{44}$$

$$(b)\ \max_{1\leq i\leq j} \left\|\boldsymbol{H}_i + \boldsymbol{H}_i^s\right\| + \sum_{i=1}^{j} \tau \left\|\nabla \times \boldsymbol{H}_i\right\| \leq C, \tag{45}$$

$$(c)\ \max_{0\leq i\leq j} \left\|\delta_\tau \boldsymbol{B}(\boldsymbol{H}_i + \boldsymbol{H}_i^s)\right\|^2_{\boldsymbol{H}^{-1}(\boldsymbol{curl};\Omega)} \leq C, \tag{46}$$

$$(d)\ \sum_{i=1}^{j} \left\|\boldsymbol{B}(\boldsymbol{H}_i + \boldsymbol{H}_i^s)\right\|^2 \tau \leq C. \tag{47}$$

Proof. (a) Set $\bar{\boldsymbol{H}} = \tau\delta_\tau \boldsymbol{H}_i$ in (35),

$$\left(\delta_\tau \boldsymbol{B}(\boldsymbol{H}_i + \boldsymbol{H}_i^s), \tau\delta_\tau(\boldsymbol{H}_i + \boldsymbol{H}_i^s)\right) + \left(\gamma(u_{i-1})\nabla \times \boldsymbol{H}_i, \nabla \times (\tau\delta_\tau \boldsymbol{H}_i)\right) = \left(\delta_\tau \boldsymbol{B}(\boldsymbol{H}_i + \boldsymbol{H}_i^s), \tau\delta_\tau \boldsymbol{H}_i^s\right).$$

By the monotonicity (21) of the $\boldsymbol{B}(\boldsymbol{H})$, we have the lower bound for the first term of the left hand side,

$$\left(\delta_\tau \boldsymbol{B}(\boldsymbol{H}_i + \boldsymbol{H}_i^s), \tau\delta_\tau(\boldsymbol{H}_i + \boldsymbol{H}_i^s)\right) \geq \tau m_{\boldsymbol{B}} \left\|\delta_\tau(\boldsymbol{H}_i + \boldsymbol{H}_i^s)\right\|^2,$$

For the second term, using the Abel's summation law, we have

$$\begin{aligned}\left(\gamma(u_{i-1})\nabla \times \boldsymbol{H}_i, \nabla \times (\boldsymbol{H}_i - \boldsymbol{H}_{i-1})\right) =& \frac{1}{2}\int_\Omega \Big(\gamma(u_i)|\nabla \times \boldsymbol{H}_i|^2 - \gamma(u_{i-1})|\nabla \times \boldsymbol{H}_{i-1}|^2 + \gamma(u_{i-1})|\nabla \times (\boldsymbol{H}_i - \boldsymbol{H}_{i-1})|^2 \\ &+ \left(\gamma(u_{i-1}) - \gamma(u_i)\right)|\nabla \times \boldsymbol{H}_i|^2\Big).\end{aligned}$$

By (20) and Cauchy's and Young's inequalities, we have the upper bound for the right hand side,

$$\left(\delta_\tau \boldsymbol{B}(\boldsymbol{H}_i + \boldsymbol{H}_i^s), \tau\delta_\tau \boldsymbol{H}_i^s\right) \leq L_{\boldsymbol{B}}\varepsilon\tau \left\|\delta_\tau(\boldsymbol{H}_i + \boldsymbol{H}_i^s)\right\|^2 + L_{\boldsymbol{B}}C_\varepsilon\tau \left\|\delta_\tau \boldsymbol{H}_i^s\right\|^2.$$

Sum up for $i = 1, 2, \cdots, j \leq n$, and by the result (43) in the Remark 1,

$$\begin{aligned}&m_{\boldsymbol{B}} \sum_{i=1}^{j} \tau \left\|\delta_\tau(\boldsymbol{H}_i + \boldsymbol{H}_i^s)\right\|^2 + \frac{\gamma_*}{2}\left\|\nabla \times \boldsymbol{H}_j\right\|^2 - \frac{\gamma^*}{2}\left\|\nabla \times \boldsymbol{H}_0\right\|^2 + \frac{\gamma_*}{2}\sum_{i=1}^{j}\left\|\nabla \times (\boldsymbol{H}_i - \boldsymbol{H}_{i-1})\right\|^2 \\ \leq& L_{\boldsymbol{B}}\varepsilon \sum_{i=1}^{j} \tau \left\|\delta_\tau(\boldsymbol{H}_i + \boldsymbol{H}_i^s)\right\|^2 + L_{\boldsymbol{B}}C_\varepsilon \sum_{i=1}^{j} \tau \left\|\delta_\tau \boldsymbol{H}_i^s\right\|^2 + C\sum_{i=1}^{j} \tau \left\|\nabla \times \boldsymbol{H}_i\right\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&(m_{\boldsymbol{B}} - L_{\boldsymbol{B}}\varepsilon)\sum_{i=1}^{j} \tau \left\|\delta_\tau(\boldsymbol{H}_i + \boldsymbol{H}_i^s)\right\|^2 + \frac{\gamma_*}{2}\left\|\nabla \times \boldsymbol{H}_j\right\|^2 + \frac{\gamma_*}{2}\sum_{i=1}^{j}\left\|\nabla \times (\boldsymbol{H}_i - \boldsymbol{H}_{i-1})\right\|^2 \\ \leq& \frac{\gamma_*}{2}\left\|\nabla \times \boldsymbol{H}_0\right\|^2 + C_\varepsilon L_{\boldsymbol{B}} \sum_{i=1}^{j} \tau \left\|\delta_\tau \boldsymbol{H}_i^s\right\|^2 + C\sum_{i=1}^{j} \tau \left\|\nabla \times \boldsymbol{H}_i\right\|^2.\end{aligned}$$

By fixing a small $\varepsilon < m_{\boldsymbol{B}}/L_{\boldsymbol{B}}$ and assumptions (12), (13), and the Grönwall's inequality, we concludes the result of (a).

(b) Set $\bar{\boldsymbol{H}} = \tau \boldsymbol{H}_i$ in (35),

$$\left(\delta_\tau \boldsymbol{B}(\boldsymbol{H}_i + \boldsymbol{H}_i^s), \tau(\boldsymbol{H}_i + \boldsymbol{H}_i^s)\right) + \left(\gamma(u_{i-1})\nabla \times \boldsymbol{H}_i, \nabla \times (\tau \boldsymbol{H}_i)\right) = \left(\delta_\tau \boldsymbol{B}(\boldsymbol{H}_i + \boldsymbol{H}_i^s), \tau \boldsymbol{H}_i^s\right),$$

For the first term of the left hand side, by the potential inequality (32),

$$\begin{aligned}\left(\delta_\tau \boldsymbol{B}(\boldsymbol{H}_i+\boldsymbol{H}_i^s), \tau(\boldsymbol{H}_i+\boldsymbol{H}_i^s)\right) &=\Big(\boldsymbol{B}(\boldsymbol{H}_i+\boldsymbol{H}_i^s)-\boldsymbol{B}(\boldsymbol{H}_{i-1}+\boldsymbol{H}_{i-1}^s), \boldsymbol{B}^{-1}\big(\boldsymbol{B}(\boldsymbol{H}_i+\boldsymbol{H}_i^s)\big)\Big)\\ &\geq \int_\Omega \Big(\Phi_{\boldsymbol{B}^{-1}}\big(\boldsymbol{B}(\boldsymbol{H}_i+\boldsymbol{H}_i^s)\big)-\Phi_{\boldsymbol{B}^{-1}}\big(\boldsymbol{B}(\boldsymbol{H}_{i-1}+\boldsymbol{H}_{i-1}^s)\big)\Big).\end{aligned}$$

By the boundedness (15) of the γ, the second term is bounded as

$$\left(\gamma(u_{i-1})\nabla\times\boldsymbol{H}_i, \nabla\times(\tau\boldsymbol{H}_i)\right)\geq \gamma_*\tau\left\|\nabla\times\boldsymbol{H}_i\right\|^2.$$

For the right hand side, by Lipschitz continuity (20) of the $\boldsymbol{B}(\boldsymbol{H})$, Cauchy's and Young's inequalities,

$$\left(\delta_\tau \boldsymbol{B}(\boldsymbol{H}_i+\boldsymbol{H}_i^s), \tau\boldsymbol{H}_i^s\right)\leq \tau L_B\left\|\delta_\tau(\boldsymbol{H}_i+\boldsymbol{H}_i^s)\right\|\cdot\left\|\boldsymbol{H}_i^s\right\|\leq C\tau\left\|\delta_\tau(\boldsymbol{H}_i+\boldsymbol{H}_i^s)\right\|^2+C\tau\left\|\boldsymbol{H}_i^s\right\|^2.$$

Sum up for $i=1,2,\cdots,j\leq n$, by the boundedness (34) of the $\Phi_{\boldsymbol{B}^{-1}}(\boldsymbol{B})$,

$$c_{\boldsymbol{B}}\left\|\boldsymbol{H}_j+\boldsymbol{H}_j^s\right\|^2-C_{\boldsymbol{B}}\left\|\boldsymbol{H}_0+\boldsymbol{H}_0^s\right\|^2+\gamma_*\sum_{i=1}^j\tau\left\|\nabla\times\boldsymbol{H}_i\right\|^2\leq C\sum_{i=1}^j\tau\left\|\delta_\tau(\boldsymbol{H}_i+\boldsymbol{H}_i^s)\right\|^2+C\sum_{i=1}^j\tau\left\|\boldsymbol{H}_i^s\right\|^2.$$

By the result of (a) and assumptions (12)-(13), we conclude the result of (b).

(c) From (35), by the dual norm definition and result of (a),

$$\left|\left(\delta_\tau\boldsymbol{B}(\boldsymbol{H}_i+\boldsymbol{H}_i^s), \bar{\boldsymbol{H}}\right)\right|=\left|\left(\gamma(u_{i-1})\nabla\times\boldsymbol{H}_i, \nabla\times\bar{\boldsymbol{H}}\right)\right|\leq\gamma^*\left\|\nabla\times\boldsymbol{H}_i\right\|\cdot\left\|\nabla\times\bar{\boldsymbol{H}}\right\|\leq C\left\|\bar{\boldsymbol{H}}\right\|_{\boldsymbol{H}(\boldsymbol{curl},\Omega)}.$$

Thus,

$$\left\|\delta_\tau\boldsymbol{B}(\boldsymbol{H}_i+\boldsymbol{H}_i^s)\right\|_{\boldsymbol{H}^{-1}(\boldsymbol{curl},\Omega)}\leq C,\quad\forall i=1,2,\cdots,n.$$

(d) From (a) and the Lipschitz continuity (20) and coercivity (19) of the $\boldsymbol{B}$,

$$\sum_{i=1}^j\left\|\boldsymbol{B}(\boldsymbol{H}_i+\boldsymbol{H}_i^s)\right\|_{\boldsymbol{L}^2(\Omega)}^2\tau\leq L_{\boldsymbol{B}}^2\sum_{i=1}^j\left\|\boldsymbol{H}_i+\boldsymbol{H}_i^s\right\|_{\boldsymbol{L}^2(\Omega)}^2\tau\leq C.$$

□

3.3 | Convergence of time discrete solution

In this subsection, we prove the convergence of the discrete solution based on the Rothe's framework [22,23].

Define the Rothe's functions: piece-wise constant and piece-wise linear in time for $t\in(t_{i-1},t_i]$, $i=1,2,\cdots,n$,

$$\begin{array}{lll}\bar{\boldsymbol{B}}_\tau(t)=\boldsymbol{B}_i, & \boldsymbol{B}_\tau(t)=\boldsymbol{B}_{i-1}+(t-t_{i-1})\delta_\tau\boldsymbol{B}_i, & \bar{\boldsymbol{B}}_\tau(0)=\boldsymbol{B}_\tau(0)=\boldsymbol{B}_0;\\ \bar{\boldsymbol{H}}_\tau(t)=\boldsymbol{H}_i, & \boldsymbol{H}_\tau(t)=\boldsymbol{H}_{i-1}+(t-t_{i-1})\delta_\tau\boldsymbol{H}_i, & \bar{\boldsymbol{H}}_\tau(0)=\boldsymbol{H}_\tau(0)=\boldsymbol{H}_0;\\ \bar{\beta}_\tau(t)=\beta(u_i), & \beta_\tau(t)=\beta(u_{i-1})+(t-t_{i-1})\beta(u_i), & \bar{\beta}_\tau(0)=\beta_\tau(0)=\beta(u_0);\\ \bar{u}_\tau(t)=u_i, & u_\tau(t)=u_{i-1}+(t-t_{i-1})u_i, & \bar{u}_\tau(0)=u_\tau(0)=u_0;\\ \bar{\lambda}_\tau(t)=\lambda_i, & \bar{\gamma}_\tau(t)=\gamma(u_i), & \bar{\gamma}_\tau(t-\tau)=\gamma(u_{i-1}).\end{array}$$

Rewrite (35)-(36) by the Rothe's functions,

$$\left(\partial_t\boldsymbol{B}_\tau, \bar{\boldsymbol{H}}\right)+\left(\bar{\gamma}_\tau(t-\tau)\nabla\times\bar{\boldsymbol{H}}_\tau, \nabla\times\bar{\boldsymbol{H}}\right)=0,\quad\forall\bar{\boldsymbol{H}}\in\boldsymbol{H}_0(\boldsymbol{curl},\Omega), \tag{48}$$

$$\left(\partial_t\beta_\tau, \bar{u}\right)+(\bar{\lambda}_\tau\nabla\bar{u}_\tau, \nabla\bar{u})=\left(\left[\bar{\gamma}_\tau(t-\tau)|\nabla\times\bar{\boldsymbol{H}}_\tau|^2\right]_\epsilon, \bar{u}\right),\quad\forall\bar{u}\in H^1(\Omega). \tag{49}$$

By setting $\bar{\boldsymbol{H}}=\nabla\varphi$ in (35), where $\varphi\in C_0^\infty(\bar{\Omega})$, we have

$$\left(\delta_\tau\boldsymbol{B}(\boldsymbol{H}_i+\boldsymbol{H}_i^s), \nabla\varphi\right)=-\left(\delta_\tau\nabla\cdot\boldsymbol{B}(\boldsymbol{H}_i+\boldsymbol{H}_i^s), \varphi\right)=0.$$

By the assumption (14),

$$0=\left(\nabla\cdot\boldsymbol{B}(\boldsymbol{H}_0+\boldsymbol{H}_0^s), \varphi\right)=\left(\nabla\cdot\boldsymbol{B}(\boldsymbol{H}_1+\boldsymbol{H}_1^s), \varphi\right)=\cdots=\left(\nabla\cdot\boldsymbol{B}(\boldsymbol{H}_n+\boldsymbol{H}_n^s), \varphi\right),$$

for any $\varphi\in C_0^\infty(\bar{\Omega})$. Since $C_0^\infty(\bar{\Omega})$ is dense in $H_0^1(\Omega)$, we have

$$\left(\nabla\cdot\boldsymbol{B}(\boldsymbol{H}_i+\boldsymbol{H}_i^s), \varphi\right)=0,\quad\forall\varphi\in H_0^1(\Omega),\ \forall i=0,1,...,n.$$

which concludes that $\nabla\cdot\boldsymbol{B}(\boldsymbol{H}_i+\boldsymbol{H}_i^s)=0$ in $H^{-1}(\Omega)$.

Proposition 1. Suppose (12)-(19). Then there exists a vector potential $\boldsymbol{H} \in L^2\big((0,T); \boldsymbol{H}_0(\boldsymbol{curl}, \Omega)\big)$ and sub-sequence of $\boldsymbol{H}_\tau$ (denoted by the same symbol again) such that

$$
\begin{aligned}
&(a)\ \bar{\boldsymbol{B}}_\tau \to \boldsymbol{B}_\tau, && \text{in } L^2\big((0,T); \boldsymbol{H}^{-1}(\boldsymbol{curl}, \Omega)\big), && (50)\\
&(b)\ \bar{\boldsymbol{H}}^s_\tau \to \boldsymbol{H}^s_\tau, && \text{in } L^2\big((0,T); \boldsymbol{L}^2(\Omega)\big), && (51)\\
&(c)\ \bar{\boldsymbol{H}}^s_\tau \to \boldsymbol{H}^s, && \text{in } L^2((0,T); \boldsymbol{H}_0(\boldsymbol{curl}, \Omega)), && (52)\\
&(d)\ \bar{\boldsymbol{H}}_\tau \rightharpoonup \boldsymbol{H}, && \text{in } L^2((0,T); \boldsymbol{L}^2(\Omega)), && (53)\\
&(e)\ \boldsymbol{B}_\tau \rightharpoonup \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s), && \text{in } L^2\big((0,T); \boldsymbol{L}^2(\Omega)\big), && (54)\\
&(f)\ \partial_t \boldsymbol{B}_\tau \rightharpoonup \partial_t \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s), && \text{in } L^2\big((0,T); \boldsymbol{H}^{-1}(\boldsymbol{curl}, \Omega)\big), && (55)\\
&(g)\ \bar{\boldsymbol{H}}_\tau \to \boldsymbol{H}, && \text{in } L^2\big((0,T); \boldsymbol{L}^2(\Omega)\big), && (56)\\
&(h)\ \bar{\boldsymbol{H}}_\tau \rightharpoonup \boldsymbol{H}, && \text{in } L^2\big((0,T); \boldsymbol{H}_0(\boldsymbol{curl}, \Omega)\big). && (57)
\end{aligned}
$$

Proof. (a) By (c) in Lemma 4,

$$\|\bar{\boldsymbol{B}}_\tau - \boldsymbol{B}_\tau\|^2_{L^2((0,T);\boldsymbol{H}^{-1}(\boldsymbol{curl},\Omega))} \le \int_0^T \tau^2 \|\partial_t \boldsymbol{B}_\tau\|^2_{\boldsymbol{H}^{-1}(\boldsymbol{curl},\Omega)} \,\mathrm{d}t \le C\tau^2 \to 0, \quad \tau \to 0.$$

(b) By the assumption (13),

$$\|\bar{\boldsymbol{H}}^s_\tau - \boldsymbol{H}^s_\tau\|^2_{L^2((0,T);\boldsymbol{L}^2(\Omega))} = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_\Omega \Big|(t_i - t)\delta_\tau \boldsymbol{H}^s_i\Big|^2 \,\mathrm{d}t \le \tau^2 \int_0^T \|\partial_t \boldsymbol{H}_\tau\|^2_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}t \le C\tau^2 \to 0, \quad \tau \to 0.$$

(c) By (13) and the mean value theorem,

$$\|\bar{\boldsymbol{H}}^s_\tau - \boldsymbol{H}^s\|^2_{L^2((0,T);\boldsymbol{L}^2(\Omega))} = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_\Omega \big|\boldsymbol{H}^s(t_i) - \boldsymbol{H}^s(t)\big|^2 \,\mathrm{d}t = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_\Omega \big|\partial_t \boldsymbol{H}^s(\xi_i)\big|^2 \cdot \big|t_i - t\big|^2 \,\mathrm{d}t \quad \big(\xi_i \in [t_{i-1}, t_i]\big)$$

$$\le \tau^2 \int_0^T \|\partial_t \boldsymbol{H}^s_\tau\|^2_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}t \le C\tau^2 \to 0, \quad \tau \to 0.$$

Similarly, by (13) again, we have

$$\Big\|\nabla \times \big(\bar{\boldsymbol{H}}^s_\tau - \boldsymbol{H}^s\big)\Big\|^2_{L^2((0,T);\boldsymbol{L}^2(\Omega))} \le \tau^2 \int_0^T \|\nabla \times (\partial_t \boldsymbol{H}^s_\tau)\|^2_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}t \le C\tau^2 \to 0,$$

and concludes the result (52).

(d) By (a) in Lemma 4 and the reflexivity of $L^2((0,T); \boldsymbol{L}^2(\Omega))$, there exists a limit $\boldsymbol{H}$,

$$\bar{\boldsymbol{H}}_\tau \rightharpoonup \boldsymbol{H}, \quad \text{in } L^2((0,T); \boldsymbol{L}^2(\Omega)).$$

Latter we will show that $\boldsymbol{H}$ solves the (7).

(e) Set $\bar{\boldsymbol{H}} \in \boldsymbol{H}_0(\boldsymbol{curl}, \Omega)$, by the Green theorem and result of (d),

$$\lim_{\tau\to 0} \int_0^T \big(\nabla \times \bar{\boldsymbol{H}}_\tau, \bar{\boldsymbol{H}}\big) \,\mathrm{d}t = \lim_{\tau\to 0} \int_0^T \big(\bar{\boldsymbol{H}}_\tau, \nabla \times \bar{\boldsymbol{H}}\big) \,\mathrm{d}t = \int_0^T \big(\boldsymbol{H}, \nabla \times \bar{\boldsymbol{H}}\big) \,\mathrm{d}t = \int_0^T \big(\nabla \times \boldsymbol{H}, \bar{\boldsymbol{H}}\big) \,\mathrm{d}t,$$

which concludes that $\nabla \times \bar{\boldsymbol{H}}_\tau \rightharpoonup \nabla \times \boldsymbol{H}$ in $L^2((0,T); \boldsymbol{L}^2(\Omega))$. And with (d) together, we conclude the result.

(f) By (d) in Lemma 4 and the reflexivity of $L^2((0,T); \boldsymbol{L}^2(\Omega))$, there exists a limit $\boldsymbol{B}$ such that

$$\boldsymbol{B}_\tau \rightharpoonup \boldsymbol{B}, \quad \text{in } L^2((0,T); \boldsymbol{L}^2(\Omega)). \tag{58}$$

Next, we show that $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s)$. By Lemma 4, and the transient *div-curl* Lemma 3.1 in [24],

$$\lim_{\tau\to 0} \int_0^T \big(\boldsymbol{B}_\tau, (\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}^s_\tau)\varphi\big) \,\mathrm{d}t = \int_0^T \big(\boldsymbol{B}, (\boldsymbol{H} + \boldsymbol{H}^s)\varphi\big)\mathrm{d}t, \quad \forall \varphi \in C_0^\infty(\bar{\Omega}).$$

Choose $\varphi \geq 0$ and any $\boldsymbol{h} \in L^2((0,T);\boldsymbol{L}^2(\Omega))$, and by the monotonicity of $\boldsymbol{B}(\cdot)$, we have

$$
\begin{aligned}
0 \leq \int_0^T \Big(\bar{\boldsymbol{B}}_\tau - \boldsymbol{B}(\boldsymbol{h}), (\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}^s{}_\tau - \boldsymbol{h})\varphi\Big)\mathrm{d}t = \overbrace{\int_0^T \Big(\bar{\boldsymbol{B}}_\tau, (\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}^s_\tau)\varphi\Big)\mathrm{d}t}^{I_1} \\
- \underbrace{\int_0^T \Big(\bar{\boldsymbol{B}}_\tau, \varphi\boldsymbol{h}\Big)\mathrm{d}t}_{I_2} - \underbrace{\int_0^T \Big(\boldsymbol{B}(\boldsymbol{h}), (\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}^s_\tau)\varphi\Big)\mathrm{d}t}_{I_3} + \underbrace{\int_0^T \Big(\boldsymbol{B}(\boldsymbol{h}), \varphi\boldsymbol{h}\Big)\mathrm{d}t}_{I_4}.
\end{aligned}
\tag{59}
$$

For the first term,

$$
I_1 = \int_0^T \Big(\bar{\boldsymbol{B}}_\tau - \boldsymbol{B}_\tau, (\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}^s_\tau)\varphi\Big)\mathrm{d}t + \int_0^T \Big(\boldsymbol{B}_\tau, (\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}^s_\tau)\varphi\Big)\mathrm{d}t.
$$

By Lemma 4 and assumption (13), together with the result of (a),

$$
\left|\int_0^T \Big(\bar{\boldsymbol{B}}_\tau - \boldsymbol{B}_\tau, (\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}^s_\tau)\varphi\Big)\mathrm{d}t\right| \leq C \left\|\bar{\boldsymbol{B}}_\tau - \boldsymbol{B}_\tau\right\|_{L^2((0,T);\boldsymbol{H}^{-1}(\boldsymbol{curl},\Omega))} \cdot \left\|\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}^s_\tau\right\|_{L^2((0,T);\boldsymbol{H}(\boldsymbol{curl},\Omega))} \leq C\tau \to 0, \quad \tau \to 0.
$$

Therefore,

$$
\lim_{\tau\to 0} I_1 = \lim_{\tau\to 0}\int_0^T \Big(\boldsymbol{B}_\tau, (\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}^s_\tau)\varphi\Big)\mathrm{d}t = \int_0^T \Big(\boldsymbol{B}, (\boldsymbol{H} + \boldsymbol{H}^s)\varphi\Big)\mathrm{d}t.
$$

For the second term, by (50), $\bar{\boldsymbol{B}}_\tau$ and $\boldsymbol{B}_\tau$ converge to the same limit,

$$
\lim_{\tau\to 0} I_2 = \lim_{\tau\to 0}\int_0^T \Big(\bar{\boldsymbol{B}}_\tau, \varphi\boldsymbol{h}\Big)\mathrm{d}t = \int_0^T \Big(\boldsymbol{B}, \varphi\boldsymbol{h}\Big)\mathrm{d}t.
$$

For the third term, by (53),

$$
\lim_{\tau\to 0} I_3 = \lim_{\tau\to 0}\int_0^T \Big(\boldsymbol{B}(\boldsymbol{h}), (\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}^s_\tau)\varphi\Big)\mathrm{d}t = \int_0^T \Big(\boldsymbol{B}(\boldsymbol{h}), (\boldsymbol{H} + \boldsymbol{H}^s)\varphi\Big)\mathrm{d}t.
$$

Sum up I_1 to I_4,

$$
\lim_{\tau\to 0}\int_0^T \Big(\bar{\boldsymbol{B}}_\tau - \boldsymbol{B}(\boldsymbol{h}), (\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}^s_\tau - \boldsymbol{h})\varphi\Big)\mathrm{d}t = \int_0^T \Big(\boldsymbol{B} - \boldsymbol{B}(\boldsymbol{h}), (\boldsymbol{H} + \boldsymbol{H}^s - \boldsymbol{h})\varphi\Big)\mathrm{d}t \geq 0.
$$

Set $\boldsymbol{h} = \boldsymbol{H} + \boldsymbol{H}^s \pm \varepsilon\boldsymbol{h}$, and let $\varepsilon \to 0^+$,

$$
\int_0^T \Big(\boldsymbol{B} - \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s), \pm\varphi\boldsymbol{h}\Big)\mathrm{d}t \geq 0.
$$

Thus, for any $0 < \varphi \in C_0^\infty(\bar{\Omega})$, and for all $\boldsymbol{h} \in L^2((0,T);\boldsymbol{L}^2(\Omega))$, we have

$$
\int_0^T \Big(\boldsymbol{B} - \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s), \varphi\boldsymbol{h}\Big)\mathrm{d}t = 0.
$$

which concludes that $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s)$ almost everywhere in $\Omega \times (0,T)$.

Recall (58), we have

$$
\boldsymbol{B}_\tau \rightharpoonup \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s), \quad \text{in } L^2\big((0,T);\boldsymbol{L}^2(\Omega)\big).
$$

(g) By Lemma 4, there exists a $\boldsymbol{B}_t \in L^2((0,T);\boldsymbol{H}^{-1}(\boldsymbol{curl},\Omega))$, such that for all $\bar{\boldsymbol{H}} \in L^2((0,T);\boldsymbol{H}_0(\boldsymbol{curl},\Omega))$,

$$
\Big(\boldsymbol{B}_\tau(t), \bar{\boldsymbol{H}}\Big) - \Big(\boldsymbol{B}_\tau(0), \bar{\boldsymbol{H}}\Big) = \int_0^t \Big(\partial_s\boldsymbol{B}_\tau(s), \bar{\boldsymbol{H}}\Big)\mathrm{d}s \to \int_0^t (\boldsymbol{B}_t, \bar{\boldsymbol{H}})\mathrm{d}s \quad \tau \to 0.
\tag{60}
$$

Next we will show that $\boldsymbol{B}_t = \partial_t\boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s)$. By Lemma 4 again, for all $\bar{\boldsymbol{H}} \in \boldsymbol{H}(\boldsymbol{curl},\Omega)$,

$$
\begin{aligned}
\left|\Big\langle \boldsymbol{B}_\tau(t), \bar{\boldsymbol{H}}\Big\rangle\right| &= \left|\Big(\boldsymbol{B}_\tau(0), \bar{\boldsymbol{H}}\Big) + \int_0^t \Big(\partial_s\boldsymbol{B}_\tau(s), \bar{\boldsymbol{H}}\Big)\mathrm{d}s\right| \\
&\leq \left\|\boldsymbol{B}_\tau(0)\right\| \cdot \left\|\bar{\boldsymbol{H}}\right\| + \left\|\partial_t\boldsymbol{B}_\tau(t)\right\|_{L^2((0,T);\boldsymbol{H}^{-1}(\boldsymbol{curl},\Omega))} \cdot \left\|\bar{\boldsymbol{H}}\right\|_{L^2((0,T);\boldsymbol{H}(\boldsymbol{curl},\Omega))} \leq C\left\|\bar{\boldsymbol{H}}\right\|_{\boldsymbol{H}(\boldsymbol{curl},\Omega)},
\end{aligned}
$$

which concludes $\big\{\boldsymbol{B}_\tau(t)\big\}$ is equi-bounded in $\boldsymbol{H}^{-1}(\boldsymbol{curl},\Omega)$.

Furthermore, by Lemma 4, for all $\bar{\boldsymbol{H}} \in \boldsymbol{H}(\boldsymbol{curl}, \Omega)$,

$$\left|\left\langle \boldsymbol{B}_\tau(t_2) - \boldsymbol{B}_\tau(t_1), \bar{\boldsymbol{H}} \right\rangle\right| = \left| \int_{t_1}^{t_2} \left\langle \partial_s \boldsymbol{B}_\tau(s), \bar{\boldsymbol{H}} \right\rangle \mathrm{d}s \right|$$
$$\leq \left\|\partial_t \boldsymbol{B}_\tau(t)\right\|_{L^2((t_1,t_2);\boldsymbol{H}^{-1}(\boldsymbol{curl},\Omega))} \cdot \left\|\bar{\boldsymbol{H}}\right\|_{L^2((t_1,t_2);\boldsymbol{H}(\boldsymbol{curl},\Omega))} \leq C\sqrt{t_2 - t_1}\,\left\|\bar{\boldsymbol{H}}\right\|_{\boldsymbol{H}(\boldsymbol{curl},\Omega)},$$

which implies $\left\{\boldsymbol{B}_\tau(t)\right\}$ is equi-continuous.

By the Arzelà-Ascoli Theorem[22, Lemma 1.3.10],

$$\boldsymbol{B}_\tau(t) \rightharpoonup \boldsymbol{B}(t), \quad \text{in } \boldsymbol{H}^{-1}(\boldsymbol{curl}, \Omega), \forall t \in [0, T].$$

Recall (60),

$$\int_0^t \left(\partial_s \boldsymbol{B}(s), \bar{\boldsymbol{H}}\right) \mathrm{d}s = \left(\boldsymbol{B}(t) - \boldsymbol{B}(0), \bar{\boldsymbol{H}}\right) = \lim_{\tau\to 0}\left(\boldsymbol{B}_\tau(t) - \boldsymbol{B}_\tau(0), \bar{\boldsymbol{H}}\right) = \lim_{\tau\to 0}\int_0^t \left(\partial_s \boldsymbol{B}_\tau(s), \bar{\boldsymbol{H}}\right) = \int_0^t \left(\boldsymbol{B}_t(s), \bar{\boldsymbol{H}}\right)\mathrm{d}s,$$

which concludes that $\boldsymbol{B}_t = \partial_t \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s)$ a.e. in $\Omega \times (0, T)$. Therefore,

$$\partial_t \boldsymbol{B}_\tau \rightharpoonup \partial_t \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s), \quad \text{in } L^2\big((0, T); \boldsymbol{H}^{-1}(\boldsymbol{curl}, \Omega)\big).$$

(h) Set $0 \leq \forall \varphi \in C_0^\infty(\bar{\Omega})$. By the strong monotonicity of $\boldsymbol{B}(\cdot)$ in (21),

$$\int_0^T \left(\boldsymbol{B}(\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}_\tau^s) - \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s), (\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}_\tau^s - \boldsymbol{H} - \boldsymbol{H}^s)\varphi\right)\mathrm{d}t \geq m_{\boldsymbol{B}} \int_0^T \left(\varphi, |\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}_\tau^s - \boldsymbol{H} - \boldsymbol{H}^s|^2\right)\mathrm{d}t.$$

Follow the similar procedure in (g), we have

$$m_{\boldsymbol{B}} \lim_{\tau\to 0} \int_0^T \left(\varphi, |\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}_\tau^s - \boldsymbol{H} - \boldsymbol{H}^s|^2\right)\mathrm{d}t \leq \lim_{\tau\to 0} \int_0^T \left(\boldsymbol{B}(\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}_\tau^s) - \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s), (\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}_\tau^s - \boldsymbol{H} - \boldsymbol{H}^s)\varphi\right)\mathrm{d}t = 0.$$

Therefore,

$$\lim_{\tau\to 0}\int_0^T \left\|\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}_\tau^s - (\boldsymbol{H} + \boldsymbol{H}^s)\right\|^2 \mathrm{d}t = 0,$$

which concludes the strong convergence of

$$\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}_\tau^s \to \boldsymbol{H} + \boldsymbol{H}^s, \quad \text{in } L^2\big((0, T); \boldsymbol{L}^2(\Omega)\big).$$

By (51), we obain

$$\bar{\boldsymbol{H}}_\tau \to \boldsymbol{H}, \quad \text{in } L^2\big((0, T); \boldsymbol{L}^2(\Omega)\big).$$

□

Proposition 2. Suppose (12)-(19). There exists $u \in C\big([0, T]; L^2(\Omega)\big) \bigcap L^\infty\big((0, T); H^1(\Omega)\big)$ with $\partial_t u \in L^2\big((0, T); L^2(\Omega)\big)$ and a sub-sequence of u_τ (denoted by the same symbol) such that

$$(a)\ u_\tau \to u, \qquad \text{in } C\big([0, T]; L^2(\Omega)\big), \tag{61}$$
$$(b)\ \bar{u}_\tau(t) \rightharpoonup u(t), \qquad \text{in } H^1(\Omega), \forall t \in [0, T], \tag{62}$$
$$(c)\ \bar{u}_\tau \to u, \qquad \text{in } L^2\big((0, T); L^2(\Omega)\big), \tag{63}$$
$$(d)\ \bar{\gamma}_\tau \to \gamma(u), \qquad \text{in } L^2\big((0, T); L^2(\Omega)\big), \tag{64}$$
$$(e)\ \bar{\gamma}_\tau(t - \tau) \to \gamma(u), \qquad \text{in } L^2\big((0, T); L^2(\Omega)\big), \tag{65}$$
$$(f)\ \bar{\beta}_\tau \to \beta_\tau, \qquad \text{in } C\big([0, T]; H^{-1}(\Omega)\big), \tag{66}$$
$$(g)\ \bar{\beta}_\tau \to \beta(u), \qquad \text{in } L^2\big((0, T); L^2(\Omega)\big). \tag{67}$$

Proof. From Lemma 3, $\partial_t u_\tau \in L^2\big((0, T); L^2(\Omega)\big)$ and $\bar{u}_\tau \in C\big([0, T]; H^1(\Omega)\big)$. By Lemma 1.3.13[22], the compact embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ concludes the results (a) and (b).

(c) It is equivalent to show that u_τ and $\bar{u}_\tau$ converge to the same limit in $L^2\big((0, T); L^2(\Omega)\big)$,

$$\int_0^T \left\|\bar{u}_\tau - u_\tau\right\|^2 \mathrm{d}t = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_\Omega |\delta_\tau u_i(\tau - t + t_{i-1})|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t \leq \tau^2 \sum_{i=1}^n \left\|\delta_\tau u_i\right\|_{L^2(\Omega)}^2 \tau \leq C\tau^2 \to 0, \quad \tau \to 0.$$

(d) By (61) in (a) and (16),

$$\int_0^T \left\|\bar{\gamma}_\tau(t) - \gamma\big(t\big)\right\|^2_{L^2(\Omega)} \mathrm{d}\boldsymbol{x}\mathrm{d}t \le \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_\Omega L_\gamma^2 \cdot \left|\bar{u}_\tau(t) - u(t)\right|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t \le C \int_0^T \|\bar{u}_\tau - u\|^2_{L^2(\Omega)} \mathrm{d}t \to 0, \quad \tau \to 0.$$

Thus, $\bar{\gamma}_\tau(t) \to \gamma(t)$ in $L^2\big((0,T); L^2(\Omega)\big)$.

(e) It is equivalent to show that $\bar{\gamma}_\tau(t-\tau)$ and $\bar{\gamma}_\tau(t)$ converge to the same limit in $L^2\big((0,T); L^2(\Omega)\big)$,

$$\int_0^T \|\bar{\gamma}_\tau(t) - \bar{\gamma}_\tau(t-\tau)\|^2_{L^2(\Omega)} \mathrm{d}t \le \tau^2 L_\gamma^2 \sum_{i=1}^n \tau \|\delta_\tau u_i\|^2_{L^2(\Omega)} \le C\tau^2 \to 0, \quad \tau \to 0.$$

(f) From the result of (c) in Lemma 3,

$$\left|(\bar{\beta}_\tau - \beta_\tau, \bar{u})\right| \le \tau \left\|\partial_t \beta_\tau\right\|_{H^{-1}(\Omega)} \cdot \|\bar{u}\|_{H^1(\Omega)} \le C\tau \|\bar{u}\|_{H^1(\Omega)},$$

which concludes that $\left\|\bar{\beta}_\tau - \beta_\tau\right\|_{H^{-1}(\Omega)} \le C\tau \to 0, \quad \tau \to 0.$

(g) By the Lipschitz continuous of β in (24), and strong convergence of $u_\tau \to u$ in $L^2\big((0,T); \boldsymbol{L}^2(\Omega)\big)$,

$$\int_0^T \|\bar{\beta}_\tau - \beta(u)\|^2_{L^2(\Omega)} \mathrm{d}t \le L_\beta^2 \int_0^T \|\bar{u}_\tau - u\|^2_{L^2(\Omega)} \mathrm{d}t \to 0, \quad \tau \to 0.$$

□

Proposition 3. Suppose (12)-(19). Then there exists a vector field $\boldsymbol{H} \in L^2\big((0,T); \boldsymbol{H}_0(\boldsymbol{curl}, \Omega)\big)$ and sub-sequence of $\boldsymbol{H}_\tau$ (denoted by the same symbol) such that

$$\bar{\boldsymbol{H}}_\tau \to \boldsymbol{H}, \quad \text{in } L^2\big((0,T); \boldsymbol{H}_0(\boldsymbol{curl}, \Omega)\big). \tag{68}$$

Proof. By the Proposition 1, it is left to prove the strong convergence of $\nabla \times \bar{\boldsymbol{H}}_\tau \to \nabla \times \boldsymbol{H}$ in $L^2((0,T); \boldsymbol{L}^2(\Omega))$. By the convergence results in the Proposition 1, Proposition 2 and the equation (7),

$$\begin{aligned}
&\gamma_* \left\|\nabla \times (\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}_\tau^s) - \nabla \times (\boldsymbol{H} + \boldsymbol{H}^s)\right\|^2_{L^2((0,T);\boldsymbol{L}^2(\Omega))} \\
\le& \int_0^T \int_\Omega \bar{\gamma}_\tau(t-\tau) \cdot \big(\nabla \times (\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}_\tau^s)\big)^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_0^T \int_\Omega \bar{\gamma}_\tau(t-\tau) \cdot \big(\nabla \times (\boldsymbol{H} + \boldsymbol{H}^s)\big)^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\quad - 2 \int_0^T \big(\bar{\gamma}_\tau(t-\tau) \cdot \nabla \times (\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}_\tau^s), \nabla \times (\boldsymbol{H} + \boldsymbol{H}^s)\big)\mathrm{d}t \\
=& -\lim_{\tau \to 0} \int_0^T \big(\partial_t \boldsymbol{B}_\tau, \bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}_\tau^s\big)\mathrm{d}t + \int_0^T \big(\partial_t \boldsymbol{B}, \boldsymbol{H} + \boldsymbol{H}^s\big)\mathrm{d}t \\
\le& -\lim_{n \to \infty} \sum_{i=1}^n \int_\Omega \Big(\Phi_{\boldsymbol{B}^{-1}}\big(\boldsymbol{B}(\boldsymbol{H}_i + \boldsymbol{H}_i^s)\big) - \Phi_{\boldsymbol{B}^{-1}}\big(\boldsymbol{B}(\boldsymbol{H}_{i-1} + \boldsymbol{H}_{i-1}^s)\big)\Big)\mathrm{d}\boldsymbol{x} + \int_0^T \big(\partial_t \boldsymbol{B}, \boldsymbol{H} + \boldsymbol{H}^s\big)\mathrm{d}t \\
=& -\int_0^T \int_\Omega \frac{\mathrm{d}\Phi_{\boldsymbol{B}^{-1}}\big(\boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s)\big)}{\mathrm{d}t} \mathrm{d}\boldsymbol{x}\mathrm{d}t + \int_0^T \big(\partial_t \boldsymbol{B}, \boldsymbol{H} + \boldsymbol{H}^s\big)\mathrm{d}t \\
=& -\int_0^T \big(\partial_t \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s), \boldsymbol{H} + \boldsymbol{H}^s\big)\mathrm{d}t + \int_0^T \big(\partial_t \boldsymbol{B}, \boldsymbol{H} + \boldsymbol{H}^s\big)\mathrm{d}t = 0.
\end{aligned}$$

Thus,

$$\bar{\boldsymbol{H}}_\tau + \bar{\boldsymbol{H}}_\tau^s \to \boldsymbol{H} + \boldsymbol{H}^s, \quad \text{in } L^2((0,T); \boldsymbol{H}_0(\boldsymbol{curl}, \Omega)).$$

By (52) and (56), we conclude the result. □

3.4 | Existence of the solution

In this subsection, we prove an important theorem on the existence of weak solution to the regularized problem (10)-(11).

Theorem 1. Suppose (12)-(19). Then the limit $(\boldsymbol{H}, u)$ solves (10)-(11).

Proof. (a) Set $\bar{\boldsymbol{H}} \in \boldsymbol{C}_0^\infty(\Omega)$ in (48), and integrate in time,

$$\int_0^t (\partial_t \boldsymbol{B}_\tau, \bar{\boldsymbol{H}})\mathrm{d}t + \int_0^t \left(\bar{\gamma}_\tau(t-\tau)\nabla \times \bar{\boldsymbol{H}}_\tau, \nabla \times \bar{\boldsymbol{H}}\right) = 0.$$

By the Proposition 1 and the Proposition 2, passing the limit for $\tau \to 0$,

$$\int_0^t \left(\partial_t \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s), \bar{\boldsymbol{H}}\right)\mathrm{d}s + \int_0^t \left(\gamma(u)\nabla \times \boldsymbol{H}, \nabla \times \bar{\boldsymbol{H}}\right)\mathrm{d}s = 0.$$

Since $\boldsymbol{C}_0^\infty(\Omega)$ is dense in $\boldsymbol{H}_0(\boldsymbol{curl}, \Omega)$, differentiating in time, $(\boldsymbol{H}, u)$ solves (10).

(b) Set $\bar{u} \in H^1(\Omega)$, and integrate (49) in time for any $t \in [0, T]$,

$$\int_0^t \left(\partial_t \beta_\tau, \bar{u}\right)\mathrm{d}s + \int_0^t (\bar{\lambda}_\tau \nabla \bar{u}_\tau, \nabla \bar{u})\mathrm{d}s = \int_0^t \left(\left[\bar{\gamma}_\tau(t-\tau)|\nabla \times \bar{\boldsymbol{H}}_\tau|^2\right]_\epsilon, \bar{u}\right)\mathrm{d}s.$$

For the first term on the LHS, by the Proposition 2,

$$\int_0^t \left(\partial_t \beta_\tau, \bar{u}\right)\mathrm{d}s = \left(\beta_\tau(t) - \beta_\tau(0), \bar{u}\right) \to \left(\beta\big(u(t)\big) - \beta(u_0), \bar{u}\right), \quad \tau \to 0.$$

For the second term, by (18) and (62),

$$\int_0^t (\bar{\lambda}_\tau \nabla \bar{u}_\tau, \nabla \bar{u})\mathrm{d}s \to \int_0^t (\lambda \nabla u, \nabla \bar{u})\mathrm{d}s, \quad \tau \to 0.$$

For the RHS, by (65) and (68),

$$\int_0^t \int_\Omega \left[\bar{\gamma}_\tau(t-\tau)|\nabla \times \bar{\boldsymbol{H}}_\tau|^2\right]_\epsilon \to \int_0^t \int_\Omega \left[\gamma(u)|\nabla \times \boldsymbol{H}|^2\right]_\epsilon, \quad \tau \to 0.$$

Given ϵ, $\left[\bar{\gamma}_\tau(t-\tau)|\nabla \times \bar{\boldsymbol{H}}_\tau|^2\right]_\epsilon$ is uniformly bounded with respect to τ. Thus,

$$\int_0^t \left(\left[\bar{\gamma}_\tau(t-\tau)|\nabla \times \bar{\boldsymbol{H}}_\tau|^2\right]_\epsilon, \bar{u}\right)\mathrm{d}s \to \int_0^t \left(\left[\gamma(u)|\nabla \times \boldsymbol{H}|^2\right]_\epsilon, \bar{u}\right)\mathrm{d}s, \quad \tau \to 0.$$

Then, we have

$$\left(\beta\big(u(t)\big) - \beta(u_0), \bar{u}\right) + \int_0^t (\lambda \nabla u, \nabla \bar{u})\mathrm{d}s = \int_0^t \left(\left[\gamma(u)|\nabla \times \boldsymbol{H}|^2\right]_\epsilon, \bar{u}\right)\mathrm{d}s.$$

Finally, differentiate with respect to time to conclude the result. □

4 | CONVERGENCE AND WELL-POSEDNESS

Given a series of ϵ in (10)-(11), we obtain a series of solution $\{(\boldsymbol{H}^\epsilon, u^\epsilon)\}$. The critical point is whether $(\boldsymbol{H}^\epsilon, u^\epsilon)$ converges to the solution of the original problem (7)-(8) as $\epsilon \to 0$. This section dedicates to the convergence and the well-posedness.

Rewrite solution in (10)-(11) with the superscript ϵ to emphasize the solution depends on ϵ.

Given $\boldsymbol{H}(0) = \boldsymbol{H}_0$, find $\boldsymbol{H}^\epsilon \in L^2((0, T); \boldsymbol{H}_0(\boldsymbol{curl}, \Omega))$, such that

$$\left(\partial_t \boldsymbol{B}(\boldsymbol{H}^\epsilon + \boldsymbol{H}^s), \bar{\boldsymbol{H}}\right) + \left(\gamma(u^\epsilon)\nabla \times \boldsymbol{H}^\epsilon, \nabla \times \bar{\boldsymbol{H}}\right) = 0, \quad \forall \bar{\boldsymbol{H}} \in \boldsymbol{H}_0(\boldsymbol{curl}, \Omega). \tag{69}$$

Given $u(0) = u_0$, find $u \in L^2\big((0, T); H^1(\Omega)\big)$, such that

$$\left(\partial_t \beta(u^\epsilon), \bar{u}\right) + (\lambda \nabla u^\epsilon, \nabla \bar{u}) = \left([q]_\epsilon, \bar{u}\right), \quad \forall \bar{u} \in H^1(\Omega), \tag{70}$$

where $q = \gamma(u^\epsilon)|\nabla \times \boldsymbol{H}^\epsilon|^2$, and $[q]_\epsilon = \dfrac{q}{1 + \epsilon|q|}$ for any $\epsilon > 0$.

4.1 | Boundedness

The following lemmas ensure the boundedness of the solution to (69)-(70), which are essential for the well-posedness of (7)-(8).

Lemma 5. Assume (12)-(19). Then there exists a positive constant C such that,

$$\int_0^T \|\boldsymbol{H}^\epsilon + \boldsymbol{H}^s\|^2_{L^2(\Omega)}\,\mathrm{d}t + \left\|\int_0^T \nabla\times\boldsymbol{H}^\epsilon \mathrm{d}t\right\|^2_{\boldsymbol{L}^2(\Omega)} \le C. \tag{71}$$

Lemma 6. Assume (12)-(19). Then there exists a positive constant C such that,

$$\int_0^T \|u^\epsilon\|^2_{L^2(\Omega)}\,\mathrm{d}t + \left\|\int_0^T \nabla u^\epsilon \mathrm{d}t\right\|^2_{L^2(\Omega)} \le C. \tag{72}$$

The proof of Lemma 5-6 is similar to Lemma 3-4.

4.2 | Convergence

By Lemma 5-6, we have the convergence results.

Corollary 1. By Lemma 5, there exists a subsequence of $\{\boldsymbol{H}^\epsilon\}$ (denoted by the same symbol) and $\boldsymbol{H} \in L^2((0,T);\boldsymbol{H}_0(\boldsymbol{curl},\Omega))$, such that

$$\boldsymbol{H}^\epsilon \rightharpoonup \boldsymbol{H}, \qquad \text{in } L^2((0,T);\boldsymbol{H}_0(\boldsymbol{curl},\Omega)), \tag{73}$$

$$\boldsymbol{H}^\epsilon \to \boldsymbol{H}, \qquad \text{in } L^2((0,T);\boldsymbol{L}^2(\Omega)), \tag{74}$$

$$\nabla\times\boldsymbol{H}^\epsilon \to \nabla\times\boldsymbol{H}, \qquad \text{in } L^2((0,T);\boldsymbol{L}^2(\Omega)), \tag{75}$$

$$\boldsymbol{B}(\boldsymbol{H}^\epsilon+\boldsymbol{H}^s) \to \boldsymbol{B}(\boldsymbol{H}+\boldsymbol{H}^s), \qquad \text{in } L^2((0,T);\boldsymbol{L}^2(\Omega)). \tag{76}$$

Corollary 2. By Lemma 6, there exists a subsequence of $\{u^\epsilon\}$ (denoted by the same symbol) and $u \in L^2\big((0,T);H^1(\Omega)\big)$, such that

$$u^\epsilon \rightharpoonup u, \qquad \text{in } L^2\big((0,T);H^1(\Omega)\big), \tag{77}$$

$$u^\epsilon \to u, \qquad \text{in } L^2\big((0,T);L^2(\Omega)\big), \tag{78}$$

$$\nabla u^\epsilon \to \nabla u, \qquad \text{in } L^2\big((0,T);L^2(\Omega)\big), \tag{79}$$

$$\beta(u^\epsilon) \to \beta(u), \qquad \text{in } L^2\big((0,T);L^2(\Omega)\big), \tag{80}$$

$$\gamma(u^\epsilon) \to \gamma(u), \qquad \text{in } L^2\big((0,T);L^2(\Omega)\big). \tag{81}$$

Now, we are in the position to prove the convergence of the solution.

Theorem 2. Suppose (12)-(19). Let $(\boldsymbol{H}^\epsilon, u^\epsilon)$ be the solution to (69)-(70), and $(\boldsymbol{H}, u)$ be to solution to (7)-(8). Then $(\boldsymbol{H}^\epsilon, u^\epsilon) \to (\boldsymbol{H}, u)$ as $\epsilon \to 0$.

Proof. (1) Set $\bar{\boldsymbol{H}} \in \boldsymbol{C}_0^\infty(\Omega)$ in (69), and integrate in time for any $t \in [0,T]$,

$$\big(\boldsymbol{B}(\boldsymbol{H}^\epsilon+\boldsymbol{H}^s) - \boldsymbol{B}(\boldsymbol{H}_0+\boldsymbol{H}_0^s), \bar{\boldsymbol{H}}\big) + \int_0^t \big(\gamma(u^\epsilon)\nabla\times\boldsymbol{H}^\epsilon, \nabla\times\bar{\boldsymbol{H}}\big) = 0,$$

By Lemma 5 and Lemma 6, take the limit for $\epsilon \to 0$,

$$\big(\boldsymbol{B}(\boldsymbol{H}+\boldsymbol{H}^s) - \boldsymbol{B}(\boldsymbol{H}_0+\boldsymbol{H}_0^s), \bar{\boldsymbol{H}}\big) + \int_0^t \big(\gamma(u)\nabla\times\boldsymbol{H}, \nabla\times\bar{\boldsymbol{H}}\big) = 0,$$

that is,

$$\int_0^t \big(\partial_t \boldsymbol{B}(\boldsymbol{H}+\boldsymbol{H}^s), \bar{\boldsymbol{H}}\big)\mathrm{d}s + \int_0^t \big(\gamma(u)\nabla\times\boldsymbol{H}, \nabla\times\bar{\boldsymbol{H}}\big)\mathrm{d}s = 0.$$

Since $\boldsymbol{C}_0^\infty(\Omega)$ is dense in $\boldsymbol{H}_0(\boldsymbol{curl},\Omega)$, differentiating in time, we obtain $(\boldsymbol{H}, u)$ solves (7).

(2) Set $\bar{u} \in H^1(\Omega)$, and integrate (70) in time for any $t \in [0,T]$,

$$\int_0^t \big(\partial_t \beta(u^\epsilon), \bar{u}\big) + \int_0^t (\lambda\nabla\bar{u}^\epsilon, \nabla\bar{u}) = \int_0^t \Big(\Big[\bar{\gamma}(u^\epsilon)|\nabla\times\boldsymbol{H}^\epsilon|^2\Big]_\epsilon, \bar{u}\Big).$$

For the first term on the LHS, by (80), we have

$$\int_0^t \big(\partial_t \beta(u^\epsilon), \bar{u}\big) = \big(\beta(u^\epsilon) - \beta(u_0), \bar{u}\big) \to \big(\beta(u) - \beta(u_0), \bar{u}\big), \quad \epsilon \to 0.$$

For the second term, by (77), we have

$$\int_0^t (\lambda \nabla u^\epsilon, \nabla \bar{u}) \to \int_0^t (\lambda \nabla u, \nabla \bar{u}), \quad \epsilon \to 0.$$

For the RHS, we need

$$\int_0^t \int_\Omega \Big| \gamma(u^\epsilon)|\nabla \times \boldsymbol{H}^\epsilon|^2 - \gamma(u)|\nabla \times \boldsymbol{H}|^2 \Big| \to 0, \quad \epsilon \to 0,$$

which could be deduced as follows. Setting both test functions $\bar{\boldsymbol{H}} \in \boldsymbol{C}_0^\infty(\Omega)$ in (7) and (69), subtracting and integrating in time for any $t \in [0, T]$, and setting $\bar{\boldsymbol{H}} = \boldsymbol{H}^\epsilon - \boldsymbol{H}$, we have

$$\big(\boldsymbol{B}(\boldsymbol{H}^\epsilon + \boldsymbol{H}^s) - \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s), \boldsymbol{H}^\epsilon - \boldsymbol{H}\big) + \int_0^t (\gamma(u^\epsilon)\nabla \times (\boldsymbol{H}^\epsilon - \boldsymbol{H}), \nabla \times (\boldsymbol{H}^\epsilon - \boldsymbol{H})) = \int_0^t \Big(\big(\gamma(u) - \gamma(u^\epsilon)\big)\nabla \times \boldsymbol{H}, \nabla \times (\boldsymbol{H}^\epsilon - \boldsymbol{H})\Big).$$

For the first term on the LHS, by (20) and (74), we have

$$\big(\boldsymbol{B}(\boldsymbol{H}^\epsilon + \boldsymbol{H}^s) - \boldsymbol{B}(\boldsymbol{H} + \boldsymbol{H}^s), \boldsymbol{H}^\epsilon - \boldsymbol{H}\big) \le L_{\boldsymbol{B}} \, \|\boldsymbol{H}^\epsilon - \boldsymbol{H}\|^2 \to 0, \; \epsilon \to 0.$$

For the RHS, by (24) and (78), we have

$$\int_0^t \Big(\big(\gamma(u) - \gamma(u^\epsilon)\big)\nabla \times \boldsymbol{H}, \nabla \times (\boldsymbol{H}^\epsilon - \boldsymbol{H})\Big) \le \int_0^t \int_\Omega \Big|\gamma(u) - \gamma(u^\epsilon)\Big| \cdot |\nabla \times \boldsymbol{H}|^2 + \int_0^t \int_\Omega \Big|\gamma(u) - \gamma(u^\epsilon)\Big| \cdot |\nabla \times \boldsymbol{H}^\epsilon|^2 \to 0.$$

Thus, the second term on the LHS should converge to zero, that is

$$\int_0^t \big(\gamma(u^\epsilon)\nabla \times (\boldsymbol{H}^\epsilon - \boldsymbol{H}), \nabla \times (\boldsymbol{H}^\epsilon - \boldsymbol{H})\big) = \int_0^t \int_\Omega \gamma(u^\epsilon)|\nabla \times (\boldsymbol{H}^\epsilon - \boldsymbol{H})|^2 \to 0, \; \epsilon \to 0.$$

Then we have

$$\int_0^t \int_\Omega \Big| \gamma(u^\epsilon)|\nabla \times \boldsymbol{H}^\epsilon|^2 - \gamma(u)|\nabla \times \boldsymbol{H}|^2 \Big| \le \int_0^t \int_\Omega \gamma(u^\epsilon) \cdot |\nabla \times (\boldsymbol{H}^\epsilon - \boldsymbol{H})|^2 + \int_0^t \int_\Omega |\gamma(u^\epsilon) - \gamma(u)| \cdot |\nabla \times \boldsymbol{H}|^2 \to 0,$$

that is

$$\int_0^t \Big(\Big[\gamma(u^\epsilon)|\nabla \times \bar{\boldsymbol{H}}^\epsilon|^2\Big]_\epsilon, \bar{u} \Big) \to \int_0^t \Big(\gamma(u)|\nabla \times \boldsymbol{H}|^2, \bar{u}\Big), \quad \epsilon \to 0.$$

Therefore,

$$\Big(\beta\big(u(t)\big) - \beta(u_0), \bar{u}\Big) + \int_0^t (\lambda \nabla u, \nabla \bar{u}) = \int_0^t \big(\gamma(u)|\nabla \times \boldsymbol{H}|^2, \bar{u}\big).$$

which concludes the result by differentiating respect to time. □

Finally, we state the main result of this paper.

Theorem 3. Suppose (12)-(19). Let $(\bar{\boldsymbol{H}}_\tau^\epsilon, \bar{u}_\tau^\epsilon)$ be the solution to (48)-(49).

(1) Let $(\boldsymbol{H}^\epsilon, u^\epsilon)$ be the solution to (69)-(70), then $(\bar{\boldsymbol{H}}_\tau^\epsilon, \bar{u}_\tau^\epsilon) \to (\boldsymbol{H}^\epsilon, u^\epsilon)$ as $\tau \to 0$ given by (53)-(57).

(2) Let $(\boldsymbol{H}, u)$ be the solution to (7)-(8), then $(\boldsymbol{H}^\epsilon, u^\epsilon) \to (\boldsymbol{H}, u)$ as $\epsilon \to 0$ given by (73)-(75) and (77)-(79).

Furthermore, there exists a positive constant C, such that

$$\int_0^T \|\boldsymbol{H}\|^2 \, \mathrm{d}t + \left\| \int_0^T \nabla \times \boldsymbol{H} \mathrm{d}t \right\|^2 + \int_0^T \|u\|^2 \, \mathrm{d}t + \left\| \int_0^T \nabla u \mathrm{d}t \right\|^2 \le C,$$

where C depends only on the source magnetic field $\boldsymbol{H}^s$ and the domain Ω and the time T and the material parameters γ, β, λ.

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