INVESTIGATING GLOBAL BEHAVIOR OF SOME SYSTEMS OF EXPONENTIAL DIFFERENCE EQUATIONS

ABDUL KHALIQ AND MUHAMMAD ZUBAIR

Abstract. This paper focused to the study of the boundedness, the persistence, and the asymptotic behavior of the positive solutions of the system of three difference equations of exponential form:

$$x_{n+1} = \frac{\lambda+\rho e^{-x_n}+\varepsilon e^{-y_n}}{\eta+\omega y_n},\ y_{n+1} = \frac{\lambda+\rho e^{-y_n}+\varepsilon e^{-z_n}}{\eta+\omega z_n},\ z_{n+1} = \frac{\lambda+\rho e^{-z_n}+\varepsilon e^{-x_n}}{\eta+\omega x_n}$$

where $\lambda,\ \rho,\ \varepsilon,\ \eta$ and ω are positive constants and the initial values $x_o,\ y_o,\ z_o$ are positive real values.

2010 Mathematics Subject Classification. 39A10; 39A11; 40A05.

Keywords and phrases. exponential difference equations, , stability, periodic solution, rate of convergence.

1. Introduction

Discrete dynamical structures defined by means of difference equations are great appropriate for population dynamics in comparison to maintains ones. Population fashions incorporate exponential difference equations and their stability evaluation although complex, however interesting. The start of 21^{st} century has witnessed a growing interest inside the population dynamics. Therefore, many works were regarded on difference equations or systems of difference equations associated with exponential terms (see [1-9] and reference referred to therein). "As an instance, Metwally et al. [1] have investigated the dynamics of the subsequent second-order difference equation:

$$z_{n+1} = \sigma + \psi z_{n-1} e^{-z_n} \tag{1}$$

That is the solution of the subsequent logistic equation with piecewise regular arguments:

$$\frac{dz}{dt} = rz(1 - \frac{z}{K}) \tag{2}$$

Wherein σ and ψ and preliminary conditions $z_{-1},\ z_0$ are arbitrary non-negative real numbers. Equation (1) can be considered as a model in Mathematical Biology where σ is immigration rate and ψ is the populace growth rate. Further it's far additionally mentioned in [2] that this model is recommended through the people from the Harvard school of public health, reading the poplution dynamics of single-species z_n.

Further, Papaschinopoulos et al. [2] and Papaschinopoulos and Schinas [3] delivered pleasant outcomes toward this path by exploring the dynamical properties like boundedness and persistence of positive solutions, existence of the unique positive equilibrium, local and global asymptotic stability of two-species model portrayed by frameworks of differecne equations, which is natural extension of single-species population model depicted in (1).

In [4], Grove et al. have researched the global dynamics of the positive solution of the accompanying difference equations:

$$z_{n+1} = \sigma z_n + \psi z_{n-1} e^{-z_n} \tag{3}$$

where $\sigma,\ \psi$ and initial conditions $z_{-1},\ z_0$ are arbitrary non-negative real numbers. This equation can be considered as a biological model, since it arises from models studying the amount of litter in perennial grassland (see [6]). After that Papaschinopoulos et al. [5, 6] have studied the asymptotic conduct of the effective result of two-species model which is also natural extension of single-species model represented in (3). In 2016, Wang and Feng [7] have investigated the dynamics of positive solution for the following difference equation that is clearly a brand new form of single-species model depicted in (1):

$$z_{n+1} = \sigma + \psi z_n e^{-z_{n-1}} \tag{4}$$

where $\sigma,\ \psi$ and initial conditions $z_{-1},\ z_0$ are arbitrary nonnegative real numbers. According to biological point of view σ is immigration rate and ψ is population growth rate.

Ozturk et al. [8] have investigated the global asymptotic stability, boundedness and periodic nature of the following 2^{nd}-order exponential difference equation:

$$z_{n+1} = \frac{\sigma + \psi e^{-z_n}}{\chi + z_{n-1}},\ n = 0, 1, ... \tag{5}$$

where $\sigma,\ \psi,\ \chi$ and $z_{-1},\ z_0$ are arbitrary non-negative numbers.

Equation (5) is likewise viewed as a model in Mathematical Biology wherein σ is immigration rate, ψ is population growth rate and χ is the carrying capacity. Later Papaschinopoulos et al. [9] have investigated boundedness and persistence and local and global asymptotic behavior of two-species model which is natural extension of single-species model (5), represented by way of the subsequent exponential structures of difference equations:

$$\begin{aligned} x_{n+1} &= \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}},\quad y_{n+1} = \frac{\delta + \epsilon e^{-x_n}}{\varsigma + x_{n-1}} \\ x_{n+1} &= \frac{\alpha + \beta e^{-y_n}}{\gamma + x_{n-1}},\ y_{n+1} = \frac{\delta + \epsilon e^{-x_n}}{\varsigma + y_{n-1}} \\ x_{n+1} &= \frac{\alpha + \beta e^{-x_n}}{\gamma + y_{n-1}},\ y_{n+1} = \frac{\delta + \epsilon e^{-y_n}}{\varsigma + x_{n-1}} \end{aligned} \tag{6}$$

where $\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon,\ \varsigma$ and initial conditions $x_{-1},\ x_0,\ y_{-1},\ y_0$ are non-negative real numbers.

Vu Van Khuong and Tran Hang Thai [10], have investigated the boundedness, persistence, and the asymptotic behavior of the positive solutions of the system of two difference equations of exponential form:

$$x_{n+1} = \frac{a + be^{-y_n} + ce^{-x_n}}{d + hy_n},\ y_{n+1} = \frac{a + be^{-x_n} + ce^{-y_n}}{d + hx_n} \tag{7}$$

where $a,\ b,\ c,\ d$ and h are positeve constants and the initial values $x_0,\ y_0$ are positive real values".

Prompted by means of the above study, we can amplify the above difference equation to a system of difference equations; our aim could be to research the boundedness character, persistence, and asymptotic conduct of the positive solutions of the following system of exponential form:

$$x_{n+1}=\frac{\lambda+\rho e^{-x_n}+\varepsilon e^{-y_n}}{\eta+\omega y_n},\ y_{n+1}=\frac{\lambda+\rho e^{-y_n}+\varepsilon e^{-z_n}}{\eta+\omega z_n},\ z_{n+1}=\frac{\lambda+\rho e^{-z_n}+\varepsilon e^{-x_n}}{\eta+\omega x_n} \tag{8}$$

where $\lambda,\ \rho,\ \varepsilon,\ \eta$ and ω are positive constants and the initial values $x_o,\ y_o,\ z_o$ are positive real values.

Difference equations and system of difference equations of exponential form can be discovered in [1, 2, 11, 12, 13]. Furthermore, as difference equations have many programs in applied sciences, there are numerous papers and books that can be determined concerning the theory and applications of difference equations; see [14-16] and the references mentioned therein.

2. Preliminaries

Let us consider three-dimensional discrete dynamical system of the following form:

$$x_{n+1}=f(x_n,\ y_n),\ y_{n+1}=g(y_n,\ z_n),\ z_{n+1}=h(z_n,\ x_n),\ n=0,\ 1,\ 2,\ .... \tag{9}$$

where $f:I\times J\rightarrow I$, $g:J\times K\rightarrow J$ and $h:K\times I\rightarrow K$ are continuously differentiable functions and I, J and K are some intervals of real numbers. Furthermore, a solution $\{x_n,\ y_n,\ z_n\}_{n=0}^{\infty}$ of system (9) is uniquely determined by initial conditions $(x_0,\ y_0,\ z_0)\in I\times J\times K$. We consider the corresponding vector map $F=(f,\ g,\ h)$ along with system (9). An equilibrium point of (9) is a point $(\bar{x},\ \bar{y},\ \bar{z})$ that satisfies

$$\bar{x}=f(\bar{x},\ \bar{y},\ \bar{z}),\ \bar{y}=g(\bar{x},\ \bar{y},\ \bar{z}),\ \bar{z}=h(\bar{x},\ \bar{y},\ \bar{z}).$$

So, this point $(\bar{x},\ \bar{y},\ \bar{z})$ of the vector map F is also called a fixed point.

Definition 1. *(see[24]) "Let $(\bar{x},\ \bar{y},\ \bar{z})$ be an equilibrium point of system(9).*

(i) An equilibrium point $(\bar{x},\ \bar{y},\ \bar{z})$ is said to be stable if for any $\varepsilon>0$ there is $\delta>0$ such that for every initial conditions $(x_0,\ y_0,\ z_0)$, if $\|(x_0,\ y_0,\ z_0)-(\bar{x},\ \bar{y},\ \bar{z})\|<\delta$ implies that

$\|(x_n,y_n,\ z_n)-(\bar{x},\ \bar{y},\ \bar{z})\|<\varepsilon$ for all $n>0$, where $\|.\|$ is usual Euclidean norm in R^2.

(ii) An equilibrium point $(\bar{x},\ \bar{y},\ \bar{z})$ is said to be unstable if it is not stable.

(iii) An equilibrium point $(\bar{x},\ \bar{y},\ \bar{z})$ is said to be asymptotically stable if there exists $r>0$ such that $(x_n,\ y_n,\ z_n)\rightarrow(\bar{x},\ \bar{y},\ \bar{z})$ as $n\rightarrow\infty$ for all $(x_0,\ y_0,\ z_0)$ that satisfy $\|(x_0,\ y_0,\ z_0)-(\bar{x},\ \bar{y},\ \bar{z})\|<r$.

(iv) An equilibrium point $(\bar{x},\ \bar{y},\ \bar{z})$ is called global attractor if $(x_n,\ y_n,\ z_n)\rightarrow(\bar{x},\ \bar{y},\ \bar{z})$ as $n\rightarrow\infty$.

(v) An equilibrium point $(\bar{x},\ \bar{y},\ \bar{z})$ is called asymptotic global attractor if it is a global attractor and stable.

Definition 2. *(see[24]) Let* $(\bar{x},\ \bar{y},\ \bar{z})$ *be an equilibrium point of a map* $F = (f,\ g,\ h)$ *where* $f,\ g$ *and* h *are continuously differentiable functions at* $(\bar{x},\ \bar{y},\ \bar{z})$*. The linearized system of* (9) *about the equilibrium point* $(\bar{x},\ \bar{y},\ \bar{z})$ *is*

$$X_{n+1} = J_F X_n$$

Where $X_n = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$ *and* J_F *is the Jacobian matrix of system* (9) *about the equilibrium point* $(\bar{x},\ \bar{y},\ \bar{z})$*.*

Lemma 3. *(see[23]) Assume that* $X_{n+1} = F(X_n),\ n = 0,\ 1, ...,$*is a system of difference equations such that* $\overline{X}$ *is a fixed point of* F*. If all eigenvalues of the Jacobian matrix* J_F *about* $\overline{X}$ *lie inside the open unit disk* $|\lambda| < 1$*, then* $\overline{X}$ *is locally asymptotically stable. If one of them has a modulus greater than one, then* $\overline{X}$ *is unstable.*

The following results give the rate of convergence of solutions of a system of difference equations

$$X_{n+1} = [A + B(n)]X_n \tag{10}$$

where X_n *is a m-dimensional vector,* $A \in C^{m\times m}$ *is a constant matrix, and* $B : Z^+ \to C^{m\times m}$ *is a matrix function satisfying*

$$\|B(n)\| \to 0 \text{ when } n \to \infty \tag{11}$$

where $\|.\|$ *denotes any matrix norm which is associated with the vector norm*

$$\|(x,\ y,\ z)\| = \sqrt{x^2 + y^2 + z^2}$$

Proposition 4. *(Perron's theorem [22]) Assume that condition* (11) *holds. If* X_n *is a solution of system* (10)*, then either* $X_n = 0$ *for all large* n *or*

$$\rho = \underset{n\to\infty}{Lim} \sqrt[n]{\|X_n\|} \tag{12}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A*.*

Proposition 5. *(Perron's theorem [22]) Assume that condition* (11) *holds. If* X_n *is a solution of system* (10)*, then either* $X_n = 0$ *for all large* n *or*

$$\rho = \underset{n\to\infty}{Lim} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{13}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A*".*

Definition 6. *(Persistence)***:** *In mathematics, the persistence of a number is the number of times one must apply a given operation to an integer before reaching a fixed point at which the operation no longer alters the number. The persistence of a number is undefined if a fixed point is never reached.*

3. Global Behavior of Solutions of System(8)

Inside the first lemma we take a look at the boundedness and persistence of the positive solutions of (8).

Lemma 7. *Every positive solution of (8) is bounded and persists.*

Proof. Let $(x_n,\ y_n,\ z_n)$ be an arbitrary solution of (8).

from (8) we can see that

$$x_n \leq \frac{\lambda+\rho+\varepsilon}{\eta},\ y_n \leq \frac{\lambda+\rho+\varepsilon}{\eta},\ z_n \leq \frac{\lambda+\rho+\varepsilon}{\eta},\ n = 1, 2, ... \tag{14}$$

In addition from (8) & (9)

$$\begin{aligned}
x_{n+1} &= \frac{\lambda+\rho e^{-x_n}+\varepsilon e^{-y_n}}{\eta+\omega y_n},\ x_n \geq \frac{\lambda+\rho e^{-(\lambda+\rho+\varepsilon)/\eta}+\varepsilon e^{-(\lambda+\rho+\varepsilon)/\eta}}{\eta+\omega((\lambda+\rho+\varepsilon)/\eta)}\\
y_{n+1} &= \frac{\lambda+\rho e^{-y_n}+\varepsilon e^{-z_n}}{\eta+\omega z_n},\ y_n \geq \frac{\lambda+\rho e^{-(\lambda+\rho+\varepsilon)/\eta}+\varepsilon e^{-(\lambda+\rho+\varepsilon)/\eta}}{\eta+\omega((\lambda+\rho+\varepsilon)/\eta)}\\
z_{n+1} &= \frac{\lambda+\rho e^{-z_n}+\varepsilon e^{-x_n}}{\eta+\omega x_n},\ z_n \geq \frac{\lambda+\rho e^{-(\lambda+\rho+\varepsilon)/\eta}+\varepsilon e^{-(\lambda+\rho+\varepsilon)/\eta}}{\eta+\omega((\lambda+\rho+\varepsilon)/\eta)}\\
&n = 2, 3, ..
\end{aligned} \tag{15}$$

Concluding from (14) and (15), the proof is completed. ■

A good way to prove the main result of this phase, we remember the following theorem without its proof (see [17, 18]).

Theorem 8. *(see[17,18]). "Let* $R = [a_1,\ b_1] \times [c_1,\ d_1] \times [e_1,\ f_1]$ *and*

$$f : R \to [a_1,\ b_1],\ g : R \to [c_1,\ d_1],\ t : R \to [e_1,\ f_1] \tag{16}$$

be a continuous funtions such that the following hold:

(a) $f(x,\ y)$, $g(y,\ z)$ *and* $t(z,\ x)$ *are non-increasing in their variables for each* $(x,\ y,\ z) \in R$

(b) If $(m_1,\ M_1,\ m_2,\ M_2,\ m_3,\ M_3) \in R^3$ *is a solution of*

$$\begin{aligned}
M_1 &= f(m_1,\ m_2),\ m_1 = f(M_1,\ M_2)\\
M_2 &= g(m_2,\ m_3),\ m_2 = g(M_2,\ M_3)\\
M_3 &= t(m_3,\ m_1),\ m_3 = t(M_3, M_1)
\end{aligned} \tag{17}$$

Then $m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3$ *then the following system of difference equations.*

$$x_{n+1} = f(x_n,\ y_n),\ y_{n+1} = g(y_n,\ z_n),\ z_{n+1} = t(z_n,\ x_n) \tag{18}$$

has a unique equilibrium $(\overline{x},\ \overline{y},\ \overline{z})$ *and every solution* $(x_n,\ y_n,\ z_n)$ *of the system (18), with* $(x_o,\ y_o,\ z_o) \in R$ *converges to the unique equilibrium* $(\overline{x},\ \overline{y},\ \overline{z})$*. In addition, the equilibrium* $(\overline{x},\ \overline{y},\ \overline{z})$ *is globally asymptotically stable.*

Now, on this phase, we state the main theorem.

Theorem 9. *(see[10]). Assume that the following relation holds true for system (8):*

$$\rho + \varepsilon < \eta \tag{19}$$

then system (8) has a unique positive equilibrium $(\overline{x},\ \overline{y},\ \overline{z})$ *and each positive solution of (8) approaches to the unique positive equilibrium* $(\overline{x},\ \overline{y},\ \overline{z})$ *as* $n \to \infty$*. In addition, the system is globally asymptotically stable on the equilibrium* $(\overline{x},\ \overline{y},\ \overline{z})$*.*

Proof. Let us consider the functions

$$\begin{aligned} f(u,v) &= \frac{\lambda+\rho e^{-u}+\varepsilon e^{-v}}{\eta+\omega v} \\ g(v,w) &= \frac{\lambda+\rho e^{-v}+\varepsilon e^{-w}}{\eta+\omega w} \\ t(w,u) &= \frac{\lambda+\rho e^{-w}+\varepsilon e^{-u}}{\eta+\omega u} \end{aligned} \tag{20}$$

Where

$$u,v,w \in I = \left[\frac{\lambda+(\rho+\varepsilon)e^{-(\lambda+\rho+\varepsilon)/\eta}}{\eta+\omega\{(\lambda+\rho+\varepsilon)/\eta\}},\ \frac{\lambda+(\rho+\varepsilon)e^{-(\lambda+\rho+\varepsilon)/\eta}}{\eta+\omega\{(\lambda+\rho+\varepsilon)/\eta\}},\ \frac{\lambda+\rho+\varepsilon}{\eta}\right] \tag{21}$$

It can be seen that $f(u,\ v),\ g(v,\ w)$ & $t(w,\ u)$ are non-increasing in variables for each $(u,\ v,\ w)\in I\times I\times I$. In addition from (20) and (21) we have $f(u,\ v)\in I,\ g(v,\ w)\in I$ & $t(w,\ u)\in I$ as $(u,\ v,\ w)\in I\times I\times I$ and so $f: I\times I\times I\to I,\ g: I\times I\times I\to I,\ t: I\times I\times I\to I$

Now let $m_1,\ M_1,\ m_2,\ M_2,\ m_3$ & M_3 be positive real numbers such that

$$\begin{aligned} M_1 &= \frac{\lambda+\rho e^{-m_1}+\varepsilon e^{-m_2}}{\eta+\omega m_2},\ m_1 = \frac{\lambda+\rho e^{-M_1}+\varepsilon e^{-M_2}}{\eta+\omega M_2} \\ M_2 &= \frac{\lambda+\rho e^{-m_2}+\varepsilon e^{-m_3}}{\eta+\omega m_3},\ m_2 = \frac{\lambda+\rho e^{-M_2}+\varepsilon e^{-M_3}}{\eta+\omega M_3} \\ M_3 &= \frac{\lambda+\rho e^{-m_3}+\varepsilon e^{-m_1}}{\eta+\omega m_1},\ m_3 = \frac{\lambda+\rho e^{-M_3}+\varepsilon e^{-M_1}}{\eta+\omega M_1} \end{aligned} \tag{22}$$

Furthermore arguing as inside the proof of theorem (7). It suffices to assume that

$$m_1 \le M_1,\ m_2 \le M_2,\ m_3 \le M_3 \tag{23}$$

From (22), we get:

$$\begin{aligned} M_1 &= \frac{\lambda+\rho e^{-m_1}+\varepsilon e^{-m_2}}{\eta+\omega m_2} \\ M_1\left(\eta+\omega m_2\right) &= \lambda+\rho e^{-m_1}+\varepsilon e^{-m_2} \\ \rho e^{-m_1}+\varepsilon e^{-m_2} &= M_1\left(\eta+\omega m_2\right)-\lambda \end{aligned}$$

Similarly

$$\begin{aligned} \rho e^{-M_1}+\varepsilon e^{-M_2} &= m_1\left(\eta+\omega M_2\right)-\lambda \\ \rho e^{-m_2}+\varepsilon e^{-m_3} &= M_2\left(\eta+\omega m_3\right)-\lambda \\ \rho e^{-M_2}+\varepsilon e^{-M_3} &= m_2\left(\eta+\omega M_3\right)-\lambda \\ \rho e^{-m_3}+\varepsilon e^{-m_1} &= M_3\left(\eta+\omega m_1\right)-\lambda \\ \rho e^{-M_3}+\varepsilon e^{-M_1} &= m_3\left(\eta+\omega M_1\right)-\lambda \end{aligned} \tag{24}$$

Which implies that

$$\begin{aligned}
\rho e^{-m_1} + \varepsilon e^{-m_2} - \rho e^{-M_1} - \varepsilon e^{-M_2} &= M_1(\eta + \omega m_2) - \lambda - m_1(\eta + \omega M_2) + \lambda \\
\rho(e^{-m_1} - e^{-M_1}) + \varepsilon(e^{-m_2} - e^{-M_2}) &= \eta M_1 + \omega m_2 M_1 - \eta m_1 - \omega m_1 M_2 \\
\eta(M_1 - m_1) + \omega(m_2 M_1 - m_1 M_2) &= \rho e^{-m_1 - M_1}(e^{M_1} - e^{m_1}) \\
&\quad + \varepsilon e^{-m_2 - M_2}(e^{M_2} - e^{m_2}) \\
\eta(M_1 - m_1) + \omega(m_2 M_1 - m_1 M_2) &= \rho e^{-m_1 - M_1}(e^{M_1} - e^{m_1}) \\
&\quad + \varepsilon e^{-m_2 - M_2}(e^{M_2} - e^{m_2})
\end{aligned}$$

Similarly

$$\begin{aligned}
\eta(M_2 - m_2) + \omega(m_3 M_2 - m_2 M_3) &= \rho e^{-m_2 - M_2}(e^{M_2} - e^{m_2}) \\
&\quad + \varepsilon e^{-m_3 - M_3}(e^{M_3} - e^{m_3}) \\
\eta(M_3 - m_3) + \omega(m_1 M_3 - m_3 M_1) &= \rho e^{-m_3 - M_3}(e^{M_3} - e^{m_3}) \\
&\quad + \varepsilon e^{-m_1 - M_1}(e^{M_1} - e^{m_1}) \qquad (25)
\end{aligned}$$

Moreover, we get

$$\begin{aligned}
e^{M_1} - e^{m_1} &= e^{\alpha}(M_1 - m_1),\ m_1 \le \alpha \le M_1 \\
e^{M_2} - e^{m_2} &= e^{\beta}(M_2 - m_2),\ m_2 \le \beta \le M_2 \qquad (26) \\
e^{M_3} - e^{m_3} &= e^{\gamma}(M_3 - m_3),\ m_3 \le \gamma \le M_3
\end{aligned}$$

Then by adding the two relations (25) we obtained:

$$\begin{aligned}
&\eta(M_1 - m_1) + \eta(M_2 - m_2) + \eta(M_3 - m_3) + \\
&\omega(m_2 M_1 - m_1 M_2) + \omega(m_3 M_2 - m_2 M_3) + \\
&\omega(m_1 M_3 - m_3 M_1) \\
= \;&\rho e^{-m_1 - M_1 + \alpha}(M_1 - m_1) + \varepsilon e^{-m_2 - M_2 + \beta}(M_2 - m_2) \\
&+ \rho e^{-m_2 - M_2 + \beta}(M_2 - m_2) + \varepsilon e^{-m_3 - M_3 + \gamma}(M_3 - m_3) + \\
&\rho e^{-m_3 - M_3 + \gamma}(M_3 - m_3) + \varepsilon e^{-m_1 - M_1 + \alpha}(M_1 - m_1)
\end{aligned}$$

$$\begin{aligned}
&\eta(M_1 - m_1) + \eta(M_2 - m_2) + \eta(M_3 - m_3) + \\
&\omega(m_2 M_1 - m_1 M_2 + m_3 M_2 - m_2 M_3 + m_1 M_3 - m_3 M_1) \\
= \;&(\rho + \varepsilon)e^{-m_1 - M_1 + \alpha}(M_1 - m_1) + (\rho + \varepsilon)e^{-m_2 - M_2 + \beta}(M_2 - m_2) + \\
&(\rho + \varepsilon)e^{-m_3 - M_3 + \gamma}(M_3 - m_3)
\end{aligned}$$

$$\begin{aligned}
&(M_1 - m_1)[\eta - (\rho + \varepsilon)e^{-m_1 - M_1 + \alpha}] + (M_2 - m_2)[\eta - (\rho + \varepsilon)e^{-m_2 - M_2 + \beta}] + \\
&(M_3 - m_3)[\eta - (\rho + \varepsilon)e^{-m_3 - M_3 + \gamma}] + \\
&\omega(m_2 M_1 - m_1 M_2 + m_3 M_2 - m_2 M_3 + m_1 M_3 - m_3 M_1) \\
= \;&0 \qquad (27)
\end{aligned}$$

Therefore from (27) we have:

$$\begin{aligned}
&(M_1 - m_1)[\eta - (\rho + \varepsilon)e^{-m_1 - M_1 + \alpha}] + (M_2 - m_2)[\eta - (\rho + \varepsilon)e^{-m_2 - M_2 + \beta}] \\
&+ (M_3 - m_3)[\eta - (\rho + \varepsilon)e^{-m_3 - M_3 + \gamma}] \\
= \;&0 \qquad (28)
\end{aligned}$$

and

$$\omega(m_2 M_1 - m_1 M_2 + m_3 M_2 - m_2 M_3 + m_1 M_3 - m_3 M_1) = 0 \qquad (29)$$

Then using (19), (23) and (28) gives us

$$m_1 = M_1, m_2 = M_2 \text{ and } m_3 = M_3$$

Hence from theorem (7) system (8) has a unique positive equilibrium $(\overline{x},\ \overline{y},\ \overline{z})$ and each positive solution of (8) approaches to the unique positive equilibrium $(\overline{x},\ \overline{y},\ \overline{z})$ as $n \to \infty$. In addition, the system (8) is globally asymptotically stable on the equilibrium $(\overline{x},\ \overline{y},\ \overline{z})$. The proof of the theorem is completed now. ■

4. Rate of Convergence

On this segment, we provide the rate of convergence of a solution of the system (8) for all values of parametersthat converges to the equilibrium $E = (\overline{x}, \overline{y}, \overline{z})$. In [19, 20], The rate of convergence of solutions that converges to an equilibrium for some three dimensional systems has been obtained.

The following outcomes provide us the rate of convergence of solutions of a system of difference equations:

$$Z_{n+1} = [A + B(n)]Z_n \tag{30}$$

wherein Z_n is a k-dimensional vector, $A \in C^{k\times k}$ is a constant matrix, and $B : Z^+ \to C^{k\times k}$ is a matrix function that satisfying

$$\|B(n)\| \to 0 \text{ when } n \to \infty \tag{31}$$

Where $\|.\|$ denotes any matrix norm which is associated with the vector norm; $\|.\|$ also denotes the Euclidean norm in R^3 given by

$$\|x\| = \|(x,\ y,\ z)\| = \sqrt{x^2 + y^2 + z^2} \tag{32}$$

Theorem 10. *(See [21]). Assume that condition (31) holds. If x_n is a solution of system (30), then either $x_n = 0$ for all large n or*

$$\rho = \underset{n\to\infty}{Lim} \sqrt[n]{\|x_n\|} \tag{33}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Theorem 11. *(see[21]). Assume that condition (31) holds. If x_n is a solution of system (30), then either $x_n = 0$ for all large n or*

$$\rho = \underset{n\to\infty}{Lim} \frac{\|x_{n+1}\|}{\|x_n\|} \tag{34}$$

exists and equals to the modulus of one of the eignvalues of matrix A.

The following system of equation is satisfied by the equilibrium point of the system (8).

$$\overline{x} = \frac{\lambda + \rho e^{-\overline{x}} + \varepsilon e^{-\overline{y}}}{\eta + \omega\overline{y}},\ \overline{y} = \frac{\lambda + \rho e^{-\overline{y}} + \varepsilon e^{-\overline{z}}}{\eta + \omega\overline{z}},\ \overline{z} = \frac{\lambda + \rho e^{-\overline{z}} + \varepsilon e^{-\overline{x}}}{\eta + \omega\overline{x}} \tag{35}$$

If $\rho + \varepsilon < \eta$, we can easily see that the system (35) has a unique equilibrium $E = (\overline{x},\ \overline{x},\ \overline{x})$.

The system (8) is associated with map T as:

$$T(x,y,z)=\begin{pmatrix} f(x,y)\\ g(y,z)\\ t(z,x)\end{pmatrix}=\begin{pmatrix} \frac{\lambda+\rho e^{-\overline{x}}+\varepsilon e^{-\overline{y}}}{\eta+\omega\overline{y}}\\ \frac{\lambda+\rho e^{-\overline{y}}+\varepsilon e^{-\overline{z}}}{\eta+\omega\overline{z}}\\ \frac{\lambda+\rho e^{-\overline{z}}+\varepsilon e^{-\overline{x}}}{\eta+\omega\overline{x}}\end{pmatrix} \tag{36}$$

The Jacobian matrix T is:

$$J_T=\begin{pmatrix} f_x & f_y & f_z\\ g_x & g_y & g_z\\ t_x & t_y & t_z\end{pmatrix}$$

$$f(x,y)=\frac{\lambda+\rho e^{-x}+\varepsilon e^{-y}}{\eta+\omega y},\ g(y,z)=\frac{\lambda+\rho e^{-y}+\varepsilon e^{-z}}{\eta+\omega z},\ t(z,x)=\frac{\lambda+\rho e^{-z}+\varepsilon e^{-x}}{\eta+\omega x}$$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= f_x=\frac{-\rho e^{-x}}{\eta+\omega y},\ \frac{\partial f}{\partial y}=f_y=\frac{(\eta+\omega y)(-\varepsilon e^{-y})-(\lambda+\rho e^{-x}+\varepsilon e^{-y})\omega}{(\eta+\omega y)^2},\ \frac{\partial f}{\partial z}=f_z=0\\
\frac{\partial g}{\partial x} &= g_x=0,\ \frac{\partial g}{\partial y}=g_y=\frac{-\rho e^{-y}}{\eta+\omega z},\ \frac{\partial g}{\partial z}=g_z=\frac{(\eta+\omega z)(-\varepsilon e^{-z})-(\lambda+\rho e^{-y}+\varepsilon e^{-z})\omega}{(\eta+\omega z)^2}\\
\frac{\partial t}{\partial x} &= t_x=\frac{(\eta+\omega x)(-\varepsilon e^{-x})-(\lambda+\rho e^{-z}+\varepsilon e^{-x})\omega}{(\eta+\omega x)^2},\ \frac{\partial t}{\partial y}=t_y=0,\ \frac{\partial t}{\partial z}=t_z=\frac{-\rho e^{-z}}{\eta+\omega x}
\end{aligned}$$

$$J_T=\begin{bmatrix} \frac{-\rho e^{-x}}{\eta+\omega y} & \frac{(\eta+\omega y)(-\varepsilon e^{-y})-(\lambda+\rho e^{-x}+\varepsilon e^{-y})\omega}{(\eta+\omega y)^2} & 0\\ 0 & \frac{-\rho e^{-y}}{\eta+\omega z} & \frac{(\eta+\omega z)(-\varepsilon e^{-z})-(\lambda+\rho e^{-y}+\varepsilon e^{-z})\omega}{(\eta+\omega z)^2}\\ \frac{(\eta+\omega x)(-\varepsilon e^{-x})-(\lambda+\rho e^{-z}+\varepsilon e^{-x})\omega}{(\eta+\omega x)^2} & 0 & \frac{-\rho e^{-z}}{\eta+\omega x}\end{bmatrix} \tag{37}$$

At the equilibrium point $E=(\overline{x},\overline{y},\overline{z})=(\overline{x},\overline{x},\overline{x})$, the value of Jacobian matrix T from the system (35) is:

$$J_T=\begin{bmatrix} \frac{-\rho e^{-\overline{x}}}{\eta+\omega\overline{x}} & \frac{(\eta+\omega\overline{x})(-\varepsilon e^{-\overline{x}})-(\lambda+\rho e^{-\overline{x}}+\varepsilon e^{-\overline{x}})\omega}{(\eta+\omega\overline{x})^2} & 0\\ 0 & \frac{-\rho e^{-\overline{x}}}{\eta+\omega\overline{x}} & \frac{(\eta+\omega\overline{x})(-\varepsilon e^{-\overline{x}})-(\lambda+\rho e^{-\overline{x}}+\varepsilon e^{-\overline{x}})\omega}{(\eta+\omega\overline{x})^2}\\ \frac{(\eta+\omega\overline{x})(-\varepsilon e^{-\overline{x}})-(\lambda+\rho e^{-\overline{x}}+\varepsilon e^{-\overline{x}})\omega}{(\eta+\omega\overline{x})^2} & 0 & \frac{-\rho e^{-\overline{x}}}{\eta+\omega\overline{x}}\end{bmatrix} \tag{38}$$

Our intention on this segment is to evaluate the rate of convergence of each solution of the system (8) inside the areas in which the factors $\lambda,\ \rho,\ \varepsilon,\ \eta\ \&\ \omega\in(0,\infty)$, $(\rho+\varepsilon<\eta)$ and initial conditions x_o and y_o are arbitrary, non-negative numbers.

Theorem 12. *The error vector* $e_n=\begin{pmatrix} e_n^1\\ e_n^2\\ e_n^3\end{pmatrix}=\begin{pmatrix} x_n-\overline{x}\\ y_n-\overline{y}\\ z_n-\overline{z}\end{pmatrix}$ *of every solution* $x_n\neq 0$ *of (8) satisfies both of the following asymptotic relations.*

$$\begin{aligned}
\underset{n\to\infty}{Lim}\sqrt[n]{\|e_n\|} &= |\lambda_i(J_T(E))| \quad \textit{for some } i=1,2,3....\\
\underset{n\to\infty}{Lim}\frac{\|e_{n+1}\|}{\|e_n\|} &= |\lambda_i(J_T(E))| \quad \textit{for some } i=1,2,3....
\end{aligned} \tag{39}$$

wherein $|\lambda_i(J_T(E))|$ *is equal to modulus of one of the eigenvalues evaluated on the equilibrium* $J_T(E)$ *of the Jacobian matrix.*

Proof. Initially, we can find a system that satisfied by means of error terms. The error terms are given as:

$$
\begin{aligned}
x_{n+1}-\overline{x} &= \frac{\lambda+\rho e^{-x_n}+\varepsilon e^{-y_n}}{\eta+\omega y_n}-\frac{\lambda+\rho e^{-\overline{x}}+\varepsilon e^{-\overline{y}}}{\eta+\omega\overline{y}}\\
x_{n+1}-\overline{x} &= \frac{(\lambda+\rho e^{-x_n}+\varepsilon e^{-y_n})(\eta+\omega\overline{y})-(\lambda+\rho e^{-\overline{x}}+\varepsilon e^{-\overline{y}})(\eta+\omega y_n)}{(\eta+\omega y_n)(\eta+\omega\overline{y})}\\
x_{n+1}-\overline{x} &= \frac{\begin{array}{c}\lambda\eta+\lambda\omega\overline{y}+\rho\eta e^{-x_n}+\rho\omega e^{-x_n}\overline{y}+\varepsilon\eta e^{-y_n}+\varepsilon\omega\overline{y}e^{-y_n}-\lambda\eta-\lambda\omega y_n-\\ \rho\eta e^{-\overline{x}}-\rho\omega y_n e^{-\overline{x}}-\varepsilon\eta e^{-\overline{y}}-\varepsilon\omega y_n e^{-\overline{y}}\end{array}}{(\eta+\omega y_n)(\eta+\omega\overline{y})}\\
x_{n+1}-\overline{x} &= \frac{\begin{array}{c}\lambda\omega(\overline{y}-y_n)+\rho\eta(e^{-x_n}-e^{-\overline{x}})+\rho\omega(\overline{y}e^{-x_n}-y_n e^{-\overline{x}})+\varepsilon\eta(e^{-y_n}-e^{-\overline{y}})+\\ \varepsilon\omega(\overline{y}e^{-y_n}-y_n e^{-\overline{y}})\end{array}}{(\eta+\omega y_n)(\eta+\omega\overline{y})}\\
x_{n+1}-\overline{x} &= \frac{-\rho\eta(e^{x_n}-e^{\overline{x}})}{e^{x_n+\overline{x}}(\eta+\omega y_n)(\eta+\omega\overline{y})}+\frac{-\varepsilon\eta(e^{y_n}-e^{\overline{y}})}{e^{y_n+\overline{y}}(\eta+\omega y_n)(\eta+\omega\overline{y})}+\frac{\rho\omega(\overline{y}e^{-x_n}-y_n e^{-\overline{x}})}{(\eta+\omega y_n)(\eta+\omega\overline{y})}+\\
&\quad \frac{\varepsilon\omega(\overline{y}e^{-y_n}-y_n e^{-\overline{y}})}{(\eta+\omega y_n)(\eta+\omega\overline{y})}+\frac{-\lambda\omega(y_n-\overline{y})}{(\eta+\omega y_n)(\eta+\omega\overline{y})}\\
x_{n+1}-\overline{x} &= \frac{-\rho\eta(e^{x_n}-e^{\overline{x}})}{e^{x_n+\overline{x}}(\eta+\omega y_n)(\eta+\omega\overline{y})}+\frac{-\varepsilon\eta(e^{y_n}-e^{\overline{y}})}{e^{y_n+\overline{y}}(\eta+\omega y_n)(\eta+\omega\overline{y})}+\\
&\quad \frac{\rho\omega(\overline{y}e^{-x_n}-e^{-x_n}y_n+e^{-x_n}y_n-y_n e^{-\overline{x}})}{(\eta+\omega y_n)(\eta+\omega\overline{y})}+\\
&\quad \frac{\varepsilon\omega(\overline{y}e^{-y_n}-e^{-y_n}y_n+e^{-y_n}y_n-y_n e^{-\overline{y}})}{(\eta+\omega y_n)(\eta+\omega\overline{y})}+\frac{-\lambda\omega(y_n-\overline{y})}{(\eta+\omega y_n)(\eta+\omega\overline{y})}\\
x_{n+1}-\overline{x} &= \frac{-\rho\eta(e^{x_n}-e^{\overline{x}})}{e^{x_n+\overline{x}}(\eta+\omega y_n)(\eta+\omega\overline{y})}+\frac{-\varepsilon\eta(e^{y_n}-e^{\overline{y}})}{e^{y_n+\overline{y}}(\eta+\omega y_n)(\eta+\omega\overline{y})}+\\
&\quad \frac{\rho\omega e^{-x_n}(\overline{y}-y_n)}{(\eta+\omega y_n)(\eta+\omega\overline{y})}+\frac{\rho\omega y_n(e^{-x_n}-e^{-\overline{x}})}{(\eta+\omega y_n)(\eta+\omega\overline{y})}+\frac{\varepsilon\omega e^{-y_n}(\overline{y}-y_n)}{(\eta+\omega y_n)(\eta+\omega\overline{y})}\\
&\quad +\frac{\varepsilon\omega y_n(e^{-y_n}-e^{-\overline{y}})}{(\eta+\omega y_n)(\eta+\omega\overline{y})}+\frac{-\lambda\omega(y_n-\overline{y})}{(\eta+\omega y_n)(\eta+\omega\overline{y})}\\
x_{n+1}-\overline{x} &= \frac{-\rho(e^{x_n}-e^{\overline{x}})}{e^{x_n+\overline{x}}(\eta+\omega y_n)(\eta+\omega\overline{y})}\{\eta+\omega y_n\}+\frac{-\varepsilon(e^{y_n}-e^{\overline{y}})}{e^{y_n+\overline{y}}(\eta+\omega y_n)(\eta+\omega\overline{y})}\{\eta+\omega y_n\}+\\
&\quad \frac{-\omega(y_n-\overline{y})}{(\eta+\omega y_n)(\eta+\omega\overline{y})}\{\lambda+\rho e^{-x_n}+\varepsilon e^{-y_n}\}\\
x_{n+1}-\overline{x} &= \frac{-\rho}{e^{x_n+\overline{x}}(\eta+\omega\overline{y})}(e^{x_n}-e^{\overline{x}})+\frac{-\varepsilon}{e^{y_n+\overline{y}}(\eta+\omega\overline{y})}(e^{y_n}-e^{\overline{y}})+\frac{-\omega(\lambda+\rho e^{-x_n}+\varepsilon e^{-y_n})}{(\eta+\omega y_n)(\eta+\omega\overline{y})}(y_n-\overline{y})
\end{aligned}
$$

$$
x_{n+1}-\overline{x}=\frac{-\rho}{e^{x_n}(\eta+\omega\overline{y})}(e^{x_n-\overline{x}}-1)+\frac{-\varepsilon}{e^{y_n}(\eta+\omega\overline{y})}(e^{y_n-\overline{y}}-1)+\frac{-\omega(\lambda+\rho e^{-x_n}+\varepsilon e^{-y_n})}{(\eta+\omega y_n)(\eta+\omega\overline{y})}(y_n-\overline{y})
$$

$$\begin{aligned}
x_{n+1}-\overline{x} &= \frac{-\rho}{e^{x_n}(\eta+\omega\overline{y})}[(x_n-\overline{x})+\Psi_1(x_n-\overline{x})^2]+\frac{-\varepsilon}{e^{y_n}(\eta+\omega\overline{y})}[(y_n-\overline{y})+\Psi_2(y_n-\overline{y})^2]+\\
&\quad \frac{-\omega(\lambda+\rho e^{-x_n}+\varepsilon e^{-y_n})}{(\eta+\omega y_n)(\eta+\omega\overline{y})}(y_n-\overline{y})
\end{aligned}$$

$$\begin{aligned}
x_{n+1}-\overline{x} &= \frac{-\rho}{e^{x_n}(\eta+\omega\overline{y})}(x_n-\overline{x})+\frac{-\varepsilon e^{-y_n}(\eta+\omega y_n)-\omega(\lambda+\rho e^{-x_n}+\varepsilon e^{-y_n})}{(\eta+\omega y_n)(\eta+\omega\overline{y})}(y_n-\overline{y})+\\
&\quad \frac{-\rho}{e^{x_n}(\eta+\omega\overline{y})}\Psi_1(x_n-\overline{x})^2+\frac{-\varepsilon}{e^{y_n}(\eta+\omega\overline{y})}\Psi_2(y_n-\overline{y})^2
\end{aligned}$$

$$\begin{aligned}
x_{n+1}-\overline{x} &= \frac{-\rho}{e^{x_n}(\eta+\omega\overline{y})}(x_n-\overline{x})+\frac{-\varepsilon e^{-y_n}(\eta+\omega y_n)-\omega(\lambda+\rho e^{-x_n}+\varepsilon e^{-y_n})}{(\eta+\omega y_n)(\eta+\omega\overline{y})}(y_n-\overline{y})+\\
&\quad \frac{-\rho}{e^{x_n}(\eta+\omega\overline{y})}\Psi_1(x_n-\overline{x})^2+\frac{-\varepsilon}{e^{y_n}(\eta+\omega\overline{y})}\Psi_2(y_n-\overline{y})^2
\end{aligned} \tag{40}$$

Similarly, we get

$$\begin{aligned}
y_{n+1}-\overline{y} &= \frac{-\rho}{e^{y_n}(\eta+\omega\overline{z})}(y_n-\overline{y})+\frac{-\varepsilon e^{-z_n}(\eta+\omega z_n)-\omega(\lambda+\rho e^{-y_n}+\varepsilon e^{-z_n})}{(\eta+\omega z_n)(\eta+\omega\overline{z})}(z_n-\overline{z})+\\
&\quad \frac{-\rho}{e^{y_n}(\eta+\omega\overline{z})}\Psi_3(y_n-\overline{y})^2+\frac{-\varepsilon}{e^{z_n}(\eta+\omega\overline{z})}\Psi_4(z_n-\overline{z})^2
\end{aligned} \tag{41}$$

$$\begin{aligned}
z_{n+1}-\overline{z} &= \frac{-\rho}{e^{z_n}(\eta+\omega\overline{x})}(z_n-\overline{z})+\frac{-\varepsilon e^{-x_n}(\eta+\omega x_n)-\omega(\lambda+\rho e^{-z_n}+\varepsilon e^{-x_n})}{(\eta+\omega x_n)(\eta+\omega\overline{x})}(x_n-\overline{x})+\\
&\quad \frac{-\rho}{e^{z_n}(\eta+\omega\overline{x})}\Psi_5(z_n-\overline{z})^2+\frac{-\varepsilon}{e^{x_n}(\eta+\omega\overline{x})}\Psi_6(x_n-\overline{x})^2
\end{aligned} \tag{42}$$

From equations (40), (41) & (42)

$$\begin{aligned}
x_{n+1}-\overline{x} &\approx \frac{-\rho}{e^{x_n}(\eta+\omega\overline{y})}(x_n-\overline{x})+\frac{-\varepsilon e^{-y_n}(\eta+\omega y_n)-\omega(\lambda+\rho e^{-x_n}+\varepsilon e^{-y_n})}{(\eta+\omega y_n)(\eta+\omega\overline{y})}(y_n-\overline{y})\\
y_{n+1}-\overline{y} &\approx \frac{-\rho}{e^{y_n}(\eta+\omega\overline{z})}(y_n-\overline{y})+\frac{-\varepsilon e^{-z_n}(\eta+\omega z_n)-\omega(\lambda+\rho e^{-y_n}+\varepsilon e^{-z_n})}{(\eta+\omega z_n)(\eta+\omega\overline{z})}(z_n-\overline{z})\\
z_{n+1}-\overline{z} &\approx \frac{-\rho}{e^{z_n}(\eta+\omega\overline{x})}(z_n-\overline{z})+\frac{-\varepsilon e^{-x_n}(\eta+\omega x_n)-\omega(\lambda+\rho e^{-z_n}+\varepsilon e^{-x_n})}{(\eta+\omega x_n)(\eta+\omega\overline{x})}(x_n-\overline{x})
\end{aligned} \tag{43}$$

set

$$e_n^1=x_n-\overline{x},\ e_n^2=y_n-\overline{y},\ e_n^3=z_n-\overline{z} \tag{44}$$

Then system (43) can be represented as:

$$\begin{aligned}
e_{n+1}^1 &\approx a_ne_n^1+b_ne_n^2\\
e_{n+1}^2 &\approx c_ne_n^2+d_ne_n^3\\
e_{n+1}^3 &\approx p_ne_n^3+q_ne_n^1
\end{aligned} \tag{45}$$

Where

$$\begin{aligned} a_n &= \frac{-\rho}{e^{x_n}(\eta+\omega\overline{y})}\ ;\ b_n = \frac{-\varepsilon e^{-y_n}(\eta+\omega y_n)-\omega(\lambda+\rho e^{-x_n}+\varepsilon e^{-y_n})}{(\eta+\omega y_n)(\eta+\omega\overline{y})} \\ c_n &= \frac{-\rho}{e^{y_n}(\eta+\omega\overline{z})}\ ;\ d_n = \frac{-\varepsilon e^{-z_n}(\eta+\omega z_n)-\omega(\lambda+\rho e^{-y_n}+\varepsilon e^{-z_n})}{(\eta+\omega z_n)(\eta+\omega\overline{z})} \\ p_n &= \frac{-\rho}{e^{z_n}(\eta+\omega\overline{x})}\ ;\ q_n = \frac{-\varepsilon e^{-x_n}(\eta+\omega x_n)-\omega(\lambda+\rho e^{-z_n}+\varepsilon e^{-x_n})}{(\eta+\omega x_n)(\eta+\omega\overline{x})} \end{aligned} \tag{46}$$

Taking the limits of $a_n,\ b_n,\ c_n,\ d_n,\ p_n$ and q_n as $n\to\infty$, we obtain

$$\begin{aligned} \underset{n\to\infty}{Lim} a_n &= \frac{-\rho}{e^{\overline{x}}(\eta+\omega\overline{y})} = \frac{-\rho}{e^{\overline{x}}(\eta+\omega\overline{x})} \\ \underset{n\to\infty}{Lim} b_n &= \frac{-\varepsilon e^{-\overline{y}}(\eta+\omega\overline{y})-\omega(\lambda+\rho e^{-\overline{x}}+\varepsilon e^{-\overline{y}})}{(\eta+\omega\overline{y})(\eta+\omega\overline{y})} = \frac{-\varepsilon e^{-\overline{x}}(\eta+\omega\overline{x})-\omega[\lambda+(\rho+\varepsilon)e^{-\overline{x}}]}{(\eta+\omega\overline{x})^2} \\ \underset{n\to\infty}{Lim} c_n &= \frac{-\rho}{e^{\overline{y}}(\eta+\omega\overline{z})} = \frac{-\rho}{e^{\overline{x}}(\eta+\omega\overline{x})} \\ \underset{n\to\infty}{Lim} d_n &= \frac{-\varepsilon e^{-\overline{z}}(\eta+\omega\overline{z})-\omega(\lambda+\rho e^{-\overline{y}}+\varepsilon e^{-\overline{z}})}{(\eta+\omega\overline{z})(\eta+\omega\overline{z})} = \frac{-\varepsilon e^{-\overline{x}}(\eta+\omega\overline{x})-\omega[\lambda+(\rho+\varepsilon)e^{-\overline{x}}]}{(\eta+\omega\overline{x})^2} \\ \underset{n\to\infty}{Lim} p_n &= \frac{-\rho}{e^{\overline{z}}(\eta+\omega\overline{x})} = \frac{-\rho}{e^{\overline{x}}(\eta+\omega\overline{x})} \\ \underset{n\to\infty}{Lim} q_n &= \frac{-\varepsilon e^{-\overline{x}}(\eta+\omega\overline{x})-\omega(\lambda+\rho e^{-\overline{z}}+\varepsilon e^{-\overline{x}})}{(\eta+\omega\overline{x})(\eta+\omega\overline{x})} = \frac{-\varepsilon e^{-\overline{x}}(\eta+\omega\overline{x})-\omega[\lambda+(\rho+\varepsilon)e^{-\overline{x}}]}{(\eta+\omega\overline{x})^2} \end{aligned} \tag{47}$$

that is

$$\begin{aligned} a_n &= \frac{-\rho}{e^{\overline{x}}(\eta+\omega\overline{x})}+\alpha_n\ ;\ b_n = \frac{-\varepsilon e^{-\overline{x}}(\eta+\omega\overline{x})-\omega[\lambda+(\rho+\varepsilon)e^{-\overline{x}}]}{(\eta+\omega\overline{x})^2}+\beta_n \\ c_n &= \frac{-\rho}{e^{\overline{x}}(\eta+\omega\overline{x})}+\gamma_n\ ;\ d_n = \frac{-\varepsilon e^{-\overline{x}}(\eta+\omega\overline{x})-\omega[\lambda+(\rho+\varepsilon)e^{-\overline{x}}]}{(\eta+\omega\overline{x})^2}+\delta_n \\ p_n &= \frac{-\rho}{e^{\overline{x}}(\eta+\omega\overline{x})}+\mu_n\ ;\ q_n = \frac{-\varepsilon e^{-\overline{x}}(\eta+\omega\overline{x})-\omega[\lambda+(\rho+\omega)e^{-\overline{x}}]}{(\eta+\omega\overline{x})^2}+\upsilon_n \end{aligned} \tag{48}$$

where $\alpha_n\to 0,\ \beta_n\to 0,\ \gamma_n\to 0,\ \delta_n\to 0,\ \mu_n\to 0,\ \&\ \upsilon_n\to 0$ as $n\to\infty$

Now, in accordance to the system of the form (30), we have:

$$e_{n+1} = [A+B(n)]e_n \tag{49}$$

$$A = \begin{bmatrix} \frac{-\rho e^{-\overline{x}}}{(\eta+\omega\overline{x})} & \frac{-\varepsilon e^{-\overline{x}}(\eta+\omega\overline{x})-\omega[\lambda+(\rho+\varepsilon)e^{-\overline{x}}]}{(\eta+\omega\overline{x})^2} & 0 \\ 0 & \frac{-\rho e^{-\overline{x}}}{(\eta+\omega\overline{x})} & \frac{-\varepsilon e^{-\overline{x}}(\eta+\omega\overline{x})-\omega[\lambda+(\rho+\varepsilon)e^{-\overline{x}}]}{(\eta+\omega\overline{x})^2} \\ \frac{-\varepsilon e^{-\overline{x}}(\eta+\omega\overline{x})-\omega[\lambda+(\rho+\varepsilon)e^{-\overline{x}}]}{(\eta+\omega\overline{x})^2} & 0 & \frac{-\rho e^{-\overline{x}}}{(\eta+\omega\overline{x})} \end{bmatrix} \tag{50}$$

$$B(n) = \begin{bmatrix} \alpha_n & \beta_n & 0 \\ 0 & \gamma_n & \delta_n \\ \upsilon_n & 0 & \mu_n \end{bmatrix}$$

$$\|B(n)\| \to 0, \text{ as } n\to\infty$$

Thus,the limiting system of error terms can be written as

$$\begin{bmatrix} e^1_{n+1} \\ e^2_{n+1} \\ e^3_{n+1} \end{bmatrix} = A \begin{bmatrix} e^1_n \\ e^2_n \\ e^3_n \end{bmatrix} \tag{51}$$

The system (8) which evaluated at the equilibrium $E = (\overline{x},\ \overline{y},\ \overline{z}) = (\overline{x},\ \overline{x},\ \overline{x})$ is perfectly linearized system. Then Theorems 9 and 10 follow the result. ■

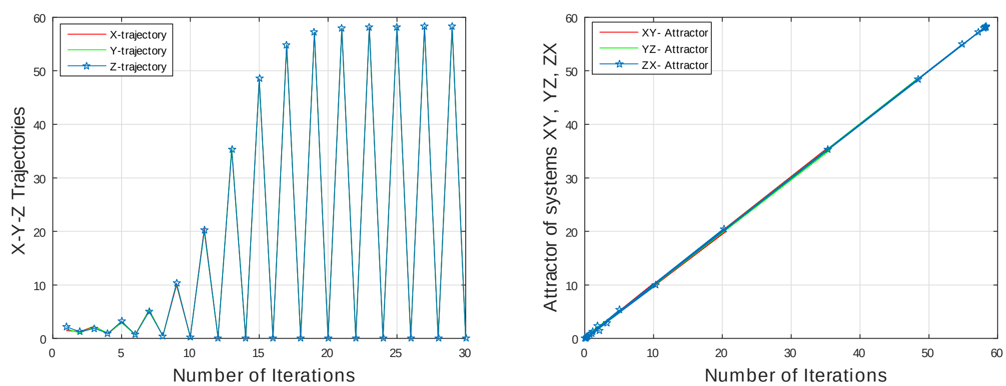
5. Numerical Simulations

If you want to affirm our theoretical discussion, we keep in mind numerous thrilling numerical examples on this phase. These examples constitute one of a kind forms of qualitative behavior of solutions to the system (8) of nonlinear difference equations. The first example suggests that positive equilibrium of system (8) is unstable with suitable parametric choices. Moreover, from the remaining examples it is clear that unique positive equilibrium point of system (8) is globally asymptotically stable with different parametric values. All plots on this phase are drawn with MATLAB.

Example 13. *Let $\lambda = 7.6$, $\rho = 9.2$, $\varepsilon = 3.8$, $\eta = 5.2$ and $\omega = 1.9$ then system can be written as*

$$x_{n+1} = \frac{7.6 + 9.2e^{-x_n} + 3.8e^{-y_n}}{5.2 + 1.9y_n},\ y_{n+1} = \frac{7.6 + 9.2e^{-y_n} + 3.8e^{-z_n}}{5.2 + 1.9z_n},\ z_{n+1} = \frac{7.6 + 9.2e^{-z_n} + 3.8e^{-x_n}}{5.2 + 1.9x_n} \tag{52}$$

with initial condition $x_o = 1.5$, $y_o = 1.8$ and $z_o = 2.2$.



(*a*) *Plot of x_n, y_n & z_n for system* (52) (*b*) *Phase portrait of system* (52)

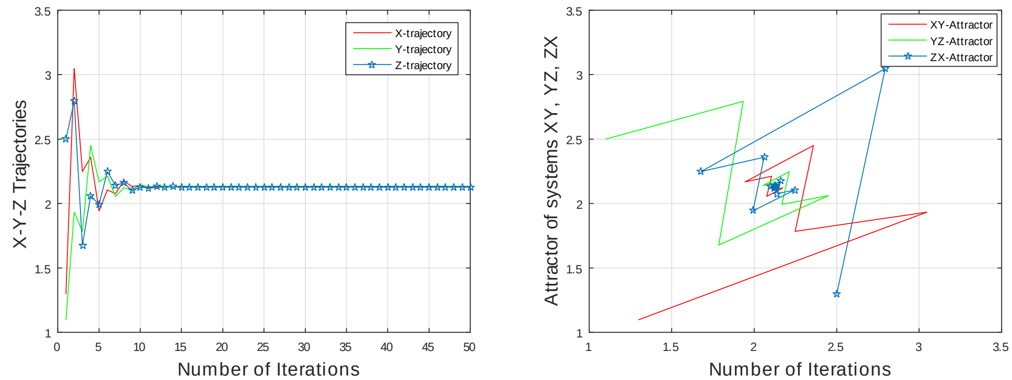
Figure 1: Plots for the system (52)

In this case, the positive equilibrium point of the system (52) *is unstable. Moreover, in Figure* 1 *the plot of x_n, y_n & z_n are shown in Figure* 1(*a*) *and a phase portrait of the system* (52) *is shown in Figure* 1(*b*).

Example 14. *Let $\lambda = 6.6$, $\rho = 0.2$, $\varepsilon = 0.8$, $\eta = 1.2$ and $\omega = 0.9$ then system can be written as*

$$x_{n+1} = \frac{6.6 + 0.2e^{-x_n} + 0.8e^{-y_n}}{1.2 + 0.9y_n},\ y_{n+1} = \frac{6.6 + 0.2e^{-y_n} + 0.8e^{-z_n}}{1.2 + 0.9z_n},\ z_{n+1} = \frac{6.6 + 0.2e^{-z_n} + 0.8e^{-x_n}}{1.2 + 0.9x_n} \tag{53}$$

with initial condition $x_o = 1.3$, $y_o = 1.1$ *and* $z_o = 2.5$.



(a) *Plot of* x_n, y_n & z_n *for system* (53) (b) *Attractors of system* (53)

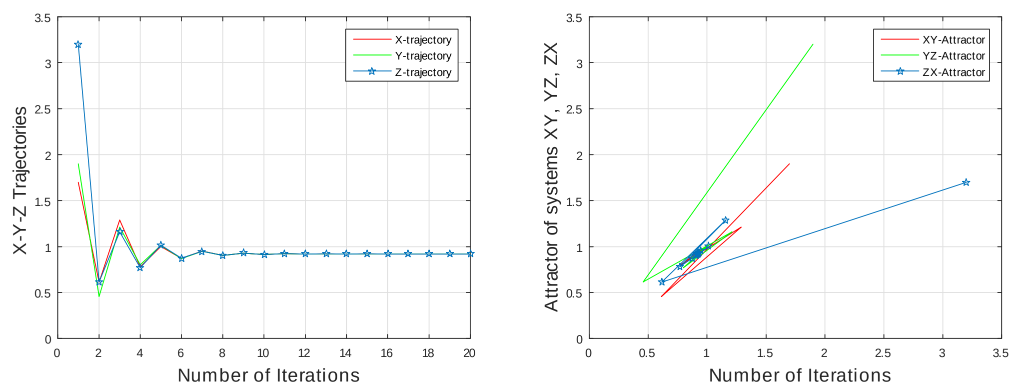
Figure 2: Plots for the system (53)

In this case, the unique positive equilibrium point of the system (53) *is given by* $(\overline{x}, \ \overline{y}, \ \overline{z}) = (2.14542, \ 2.14542, \ 2.14542)$. *Moreover, in Figure* 2, *the plot of* x_n, y_n & z_n *are shown in Figure* 2(*a*), *and XY, YZ & ZX attractors of the system* (53) *is shown in Figure* 2(*b*).

Example 15. *Let* $\lambda = 4.5$, $\rho = 1.2$, $\varepsilon = 1.8$, $\eta = 4.2$ *and* $\omega = 1.9$ *then system can be written as*

$$x_{n+1} = \frac{4.5 + 1.2e^{-x_n} + 1.8e^{-y_n}}{4.2 + 1.9y_n}, \ y_{n+1} = \frac{4.5 + 1.2e^{-y_n} + 1.8e^{-z_n}}{4.2 + 1.9z_n}, \ z_{n+1} = \frac{4.5 + 1.2e^{-z_n} + 1.8e^{-x_n}}{4.2 + 1.9x_n} \tag{54}$$

with initial condition $x_o = 1.7$, $y_o = 1.9$ *and* $z_o = 3.2$.



(a) *Plot of* x_n, y_n & z_n *for system* (54) (b) *Attractors of system* (54)

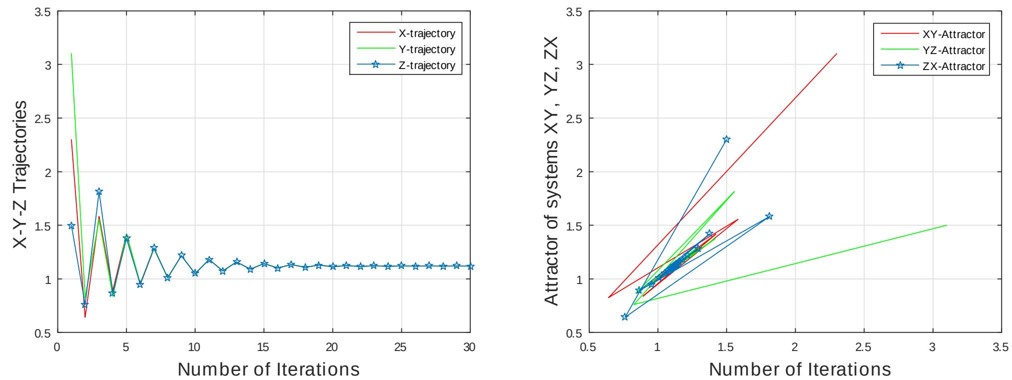
Figure 3: Plots for the system (54)

In this case, the unique positive equilibrium point of the system (54) *is given by* $(\overline{x}, \ \overline{y}, \ \overline{z}) = (0.945045, \ 0.945045, \ 0.945045)$. *Moreover, in Figure* 3, *the plot of* x_n, y_n & z_n *are shown in Figure* 3(*a*), *and XY, YZ & ZX attractors of the system* (54) *is shown in Figure* 3(*b*).

Example 16. *Let* $\lambda = 8.4$, $\rho = 2.2$, $\varepsilon = 3.8$, $\eta = 7.6$ *and* $\omega = 1.9$ *then system can be written as*

$$x_{n+1} = \frac{8.4 + 2.2e^{-x_n} + 3.8e^{-y_n}}{7.6 + 1.9y_n}, \; y_{n+1} = \frac{8.4 + 2.2e^{-y_n} + 3.8e^{-z_n}}{7.6 + 1.9z_n}, \; z_{n+1} = \frac{8.4 + 2.2e^{-z_n} + 3.8e^{-x_n}}{7.6 + 1.9x_n} \tag{55}$$

with initial condition $x_o = 2.3$, $y_o = 3.1$ *and* $z_o = 1.5$.



(a) *Plot of* x_n, y_n & z_n *for system* (55) (b) *Attractors of system* (55)

Figure 4: Plots for the system (55)

In this case, the unique positive equilibrium point of the system (55) *is given by* $(\overline{x}, \overline{y}, \overline{z}) = (1.08099, 1.08099, 1.08099)$. *Moreover, in Figure* 4, *the plot of* x_n, y_n & z_n *are shown in Figure* 4(a), *and XY, YZ & ZX attractors of the system* (55) *is shown in Figure* 4(b).

5.1. **Conclusion.** This work is associated with qualitative conduct of a system of exponential difference equations. We have investigated the existence and uniqueness of positive steady state of system (8). The boundedness character and persistence of positive solutions are verified. Moreover, we have got proven that unique positive equilibrium point of system (8) is locally in addition to globally asymptotically stable under certain parametric conditions. The primary goal of dynamical structures theory is to explore the global conduct of a system based on the knowledge of its present state. An approach to this problem consists of determining the possible global conduct of the system and determining which parametric conditions lead to these long-term behaviors. Furthermore, the rate of convergence of positive solutions of (8) which converges to its unique positive equilibrium point is established. In the end, a few illustrative numerical examples are furnished to help our theortical discussion.

References

[1] H. El-Metwally, E. A. Grove, G. Ladas, R. Levins, and M. Radin, "On the difference equation $z_{n+1} = \sigma + \psi z_{n-1}e^{-z_n}$," Nonlinear Analysis, vol. 47, pp. 4623 – 4634, 2001.

[2] G. Papaschinopoulos, M. A. Radin, and C. J. Schinas, "On the system of two difference equations of exponential form: $x_{n+1} = a + bx_{n-1}e^{-y_n}$, $y_{n+1} = c + dy_{n-1}e^{-x_n}$," Mathematical and Computer Modelling, vol. 54, no. 11, pp. 2969 – 2977, 2011.

[3] G. Papaschinopoulos and C. J. Schinas, "On the dynamics of two exponential type systems of difference equations," Computers and Mathematics with Applications, vol. 64, no. 7, pp. 2326 – 2334, 2012.

[4] E. A. Grove, G. Ladas, N. R. Prokup, and R. Levins, "On the global behavior of solutions of a biological model," Communications on Applied Nonlinear Analysis, vol. 7, no. 2, pp. $33-46$, 2000.

[5] G. Papaschinopoulos, N. Fotiades, and C. J. Schinas, "On a system of difference equations including negative exponential terms," Journal of Difference Equations and Applications, vol. 20, no. $5-6$, pp. $717-732$, 2014.

[6] G. Papaschinopoulos, G. Ellina, and K. B. Papadopoulos, "Asymptotic behavior of the positive solutions of an exponential type system of difference equations," Applied Mathematics and Computation, vol. 245, pp. $181-190$, 2014.

[7] W. Wang and H. Feng, "On the dynamics of positive solutions for the difference equation in a new population model," Journal of Nonlinear Sciences and Applications, vol. 9, no. 4, pp. $1748-1754$, 2016.

[8] I. Ozturk, F. Bozkurt, and S. Ozen, "On the difference equation $z_{n+1}=\sigma+\psi e^{-z_n}/\chi+z_{n-1}$," Applied Mathematics and Computation, vol. 181, no. 2, pp. $1387-1393$, 2006.

[9] G. Papaschinopoulos, M. A. Radin, and C. J. Schinas, "Study of the asymptotic behavior of the solutions of three systems of difference equations of exponential form," Applied Mathematics and Computation, vol. 218, no. 9, pp. $5310-5318$, 2012.

[10] Vu Van Khuong and Tran Hong Thai, "Asymptotic behavior of the solutions of the system of Difference Equations of Exponential Form," Journal of Difference Equations, Vol. 2014, Article ID 936302, 6 pages, doi: 10.1155/2014/936302.

[11] D. C. Zhang and B. Shi, "Oscillation and global asymptotic stability in a discrete epidemic model," Journal of Mathematical Analysis and Applications, vol. 278, no. 1, pp. $194-202$, 2003.

[12] G. Stefanidou, G. Papaschinopoulos, and C. J. Schinas, "On a system of two exponential type difference equations," Communications on Applied Nonlinear Analysis, vol. 17, no. 2, pp. $1-13$, 2010.

[13] S. Stevic, "On a discrete epidemic model," Discrete Dynamics in Nature and Society, vol. 2007, Article ID 87519, 10 pages, 2007.

[14] E. A. Grove and G. Ladas, Periodicities in Nonlinear Difference Equations, Chapman & Hall/CRC, 2005.

[15] R. P. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, NY, USA, 2nd edition, 2000.

[16] V. L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic, Dordrecht, The Netherlands, 1993.

[17] M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman and Hall/CRC, Boca Raton, Fla, USA, 2001.

[18] D. Burgic and Z. Nurkanovic, "An example of a globally asymptotically stable anti- monotonic system of rational difference equations in the plane," Sarajevo Journal of Mathematics, vol. 5, no. 18, pp. $235-245$, 2009.

[19] M. R. S. Kulenovic and Z. Nurkanovic, "Asymptotic behavior of a competitive system of linear fractional difference equations," Advances in Difference Equations, vol. 2006, Article ID 019756, 2006.

[20] M. R. S. Kulenovic and Z. Nurkanovic, "The rate of convergence of solution of a three dimensional linear fractional systems of difference equations," Zbornik radova PMF Tuzla-Svezak Mathematika, vol. 2, pp. $1-6$, 2005.

[21] M. Pituk, "More on Poincare's and Peron's theorems for difference equations," Journal of Difference Equatios and Applications, vol. 8, pp. $201-216$, 2002.

[22] M. Pituk, More on Poincare's and Peron's theorems for difference equations, J. Difference Equ. Appl. 8(3) (2002), $201-216$.

[23] H. Sedaghat, Nonlinear Difference Equations: Theory with Applications to Social Science Models, Kluwer Academic Publishers, Dordreacht, The Netherlands (2003).

[24] Tran Hong Thai, Asymptotic behavior of the solution of a system of difference equations, Department of Mathematics, Hung yen University of Technology and Education, Hung Yen 393008, Vietnam.

[1]Department of Mathematics, Riphah International university,, Lahore, Pakistan.
E-mail address: khaliqsyed@gmail.com, muhammadzubair9299@gmail.com.