Torsions and integrations

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Abstract

The solutions of a number of well known boundary value problems of complex analysis (for instance, the Riemann boundary value problem) can be found in the form of curvilinear integrals over the boundaries of domains under consideration. In this connection the classical results on that problems concern domains with rectifiable boundaries only. On the other hand, the boundary value problems themselves keep their sense for non-rectifiable boundaries. This is the reason for recent development of theory of generalized integration over non-rectifiable plane Jordan curves and arcs. The existence of that generalized integrations over non-rectifiable arcs depends on certain geometry properties of the arcs in neighborhoods of their ends. Here we consider connections of generalized integration and so called torsions of the path of integration at its end points.

Key words. Riemann boundary value problem, jump problem, generalized integration, non-rectifiable arc, torsion, integral of Cauchy type.

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Introduction

Let us consider first a well known boundary-value problem of complex analysis – so called Riemann problem on a simple Jordan arc (see, for instance, [1, 2, 3]).

Given a directed Jordan arc Γ in the complex plane $\mathbb{C}$ with beginning and end at points a_1 and a_2 relatively, and two functions $G(t)$, $g(t)$, $t \in \Gamma$. Find all holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ functions $\Phi(z)$ which vanish at ∞ and have boundary limits $\Phi^{\pm}(t)$ from the left and from the right correspondingly at any point $t \in \Gamma \setminus \{a_1, a_2\}$ such that

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma \setminus \{a_1, a_2\}. \tag{1}$$

In addition, the desired function Φ must satisfy certain conditions on its growth at the end points $a_{1,2}$.

In numerous classical works (see [1, 2, 3] and many other) the solutions of this problem are obtained in terms of Cauchy type integrals. Particularly, a solution of so called jump problem

$$\Phi^+(t) = \Phi^-(t) + g(t), \quad t \in \Gamma \setminus \{a_1, a_2\}, \tag{2}$$

on piecewise - smooth arc Γ is representable as the Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{g(t)\,dt}{t-z}, \quad z \notin \Gamma. \tag{3}$$

As a result, in all classical works on this problem the boundary Γ is assumed rectifiable, although the formulation of the problem does not imply this restriction. It keeps the sense for non-rectifiable arcs, too.

The Riemann boundary-value problem for non-rectifiable boundaries was solved first in the papers [4, 5, 6, 7]. The main result of these studies is the following. Let $H_\nu(\Gamma)$, $\nu \in (0,1]$, stand for the set of all defined on Γ functions $g(t)$ satisfying the Hölder condition

$$h_\nu(g,\Gamma) := \sup\left\{ \frac{|g(t)-g(t')|}{|t-t'|^\nu} : t, t' \in \Gamma, t \neq t' \right\} < +\infty. \tag{4}$$

If $g(t) \in H_\nu(\Gamma)$, then the jump problem on a closed Jordan curve Γ has a solution under restriction

$$\nu > \frac{1}{2} \overline{\mathrm{dm}}\, \Gamma, \tag{5}$$

and this condition cannot be improved. The symbol $\overline{\mathrm{dm}}$ stands here for the upper Minkowskii dimension, which is known also as box dimension and upper metric dimension (see [8]). It is defined for any compact set $K \subset \mathbb{C}$ by formula

$$\overline{\mathrm{dm}}\, K = \limsup_{r \to 0} \frac{\log N(K,r)}{-\log r},$$

where $N(K,r)$ is the least number of disks of radius r covering K. As known, the Minkowskii dimension of any rectifiable curve equals to 1 (see, for instance, [9]), and for rectifiable curves this result turns to the theorems of E.M. Dynkin [10] and T. Salimov [11] on continuity of the Cauchy type integral. The mentioned above exactness of condition (5) means that for any values $\nu \in (0,1]$ and $d \in [2\nu, 2)$ we can find a curve Γ and a function $g(t)$ such that $\overline{\mathrm{dm}}\, \Gamma = d$, $g(t)$ satisfies on Γ the Hölder condition with exponent ν, and the corresponding jump problem has not solution. On the other hand, there exist curves and jumps such that condition (5) is broken, but the corresponding jump problem has a solution. Analogous results are valid for arcs.

Later this result was improved in terms of so called Marcinkiewicz exponents [12]. We will introduce this concept in the next section.

All mentioned results concerning non-rectifiable paths were obtained without using of the Cauchy type integral because that integral is not defined for non-rectifiable paths of integration. But this situation leads to the problem of generalization of concept of curvilinear integral

$$\int\limits_{\Gamma} u\,dz + v\,d\overline{z}$$

for non-rectifiable paths Γ. It seems that the first attempt in this direction was made in article [13]. The jump problem on non-rectifiable arc and the task of integration over that paths turned out to be closely related.

Presently there are published numerous papers on this subject, see, for instance, [14]–[27]. In the next section we formulate the definition of integration over non-rectifiable paths using constructions of these works. This definition is based on the concept of integrator (see below). In section 1 we show that the offered construction of integrator is closely connected with so called torsion of arc Γ. Almost all published until now results on this subject concern arcs of moderate torsion. Then we describe new construction of generalized integration over non-rectifiable paths, which enables us to integrate over arcs of high torsion (see definitions of arcs of moderate and high torsion below). Then we apply this construction for study of the Riemann boundary value problem.

1 Arcs of moderate torsion

1.1 Generalization of curvilinear integral

First we describe a recent version of definition of integral over non-rectifiable arc Γ. Let Γ be a Jordan arc of null plane measure beginning at point a_1 and ending in a_2.

Definition 1 *Let function $f(t)$ be defined on $\Gamma \setminus \{a_1, a_2\}$. A defined in $\overline{\mathbb{C}} \setminus \Gamma$ function $F(z)$ is called an integrator of f on dz, if it has limit values $F^+(t)$ and $F^-(t)$ from the left and from the right correspondingly at every point $t \in \Gamma \setminus \{a_1, a_2\}$, and*

$$F^+(t) - F^-(t) = f(t), \quad t \in \Gamma \setminus \{a_1, a_2\},$$

support of F is compact set, F is continuously differentiable in $\overline{\mathbb{C}} \setminus \Gamma$, and integrable together with its derivative $F_{\overline{z}}$ in $\mathbb{C}$.

A integrator F is called continuous if it is integrable together with $F_{\overline{z}}$ in a power $p > 2$.

If functions $u(t)$ has integrator $U(z)$, then in the case of piecewise-smooth arc Γ the Green formula implies equality

$$\int\limits_{\Gamma} u(z)dz = -\iint\limits_{\mathbb{C}} \frac{\partial U}{\partial \overline{z}} dz\, d\overline{z}.$$

It is source of the following definition of integration over non-rectifiable arc.

Definition 2 *If defined on $\Gamma \setminus E$ function $u(t)$ has integrator $U(z)$, then mapping*

$$C^1 \ni \omega \mapsto \int\limits_\Gamma u\omega\, dz := -\iint\limits_{\mathbb{C}} \frac{\partial U\omega}{\partial \overline{z}}\, dz\, d\overline{z} \tag{6}$$

is called a (generalized) integration of u on dz over Γ.

Remark 1 *If we replace in these two definitions derivative $F_{\overline{z}}$ by F_z and mapping* (6) *by*

$$C^1 \ni \omega \mapsto \int\limits_\Gamma u\omega\, d\overline{z} := \iint\limits_{\mathbb{C}} \frac{\partial U\omega}{\partial z}\, dz\, d\overline{z}, \tag{7}$$

then we obtain the definitions of integrator and integration on $d\overline{z}$.

One can show easily, that if $u_{1,2}$ and $v_{1,2}$ belong C^1 and $u_1 = u_2$, $v_1 = v_2$ in a neighborhood of Γ, then

$$\int\limits_\Gamma u_1 \cdot dz = \int\limits_\Gamma u_2 \cdot dz, \quad \int\limits_\Gamma v_1 \cdot d\overline{z} = \int\limits_\Gamma v_2 \cdot d\overline{z},$$

i.e., these mappings are defined on the spaces of $\Gamma-$germs of $C^1(\mathbb{C})$-functions and $(1,1)$ – forms.

1.2 Existence of integrators

Here we obtain the conditions of existence of integrators in terms of Marcinkiewicz exponents.

Let us consider single-valued in $\overline{\mathbb{C}} \setminus \Gamma$ branch of logarithm

$$\frac{1}{2\pi i} \ln \frac{z - a_2}{z - a_1},$$

vanishing at the infinity point. We denote this function $K_\Gamma(z)$, and call it *logarithmic kernel* of arc Γ. Obviously, it has unit jump on Γ. Let $\chi_\Gamma(z) \in C_0^\infty(\mathbb{C})$ be a smooth function with compact support equaling to unit in a neighborhood of Γ. Then product $\chi_\Gamma K_\Gamma$ will be an integrator (a continuous integrator) of unit on arc Γ if K_Γ is integrable (square integrable) in neighborhoods of ends $a_{1,2}$. In this connection we introduce concept of torsion of arc Γ.

Definition 3 *The torsion of arc Γ at point a_j, $j = 1, 2$ is a value*

$$\tau_j := \inf \left\{ p > 0 : \iint |K_\Gamma(z)|^{1/p} dx\, dy < \infty \right\},$$

where integral is taken over a neighborhood of a_j. If $\tau_j < 1$, then we say that the arc has moderate torsion at point a_j, otherwise its torsion is high.

This value characterizes the rate of curling of Γ around the point a_j.

Example 1 *Consider arc Γ such that there exists a smooth arc γ with beginning a_1 which has not other common points with Γ in a neighborhood of a_1. Then the difference $K_\Gamma(z) - K_\gamma(z)$ is bounded near the point a_1, i.e., K_Γ has logarithmic growth at the point a_j. Hence, it is integrable with any positive power near this point, and has zero torsion.*

In particular, if Γ is smooth at point a_j, then it has zero torsion in this point. Logarithmic spiral has zero torsion at its focus, too.

Example 2 *Let*

$$S_p := \{z = r \exp 2\pi i r^{-p} : 0 < r \leq 1\}, \quad p > 0.$$

It is a spiral-like arc with beginning at the origin and end point 1. Denote $I := [0, 1]$ the segment of real axis with the same beginning and end. These arc and segment intersect at points $x_n = n^{-1/p}, n = 1, 2, \ldots$. Let us consider finite domains Δ_n bounded by segments $[x_{n+1}, x_n]$ and arcs $\sigma_n = \{z = r \exp 2\pi r^{-p} : x_{n+1} \leq r \leq x_n\}$, $n = 1, 2, \ldots$. Clearly, $K_{S_p}(z) - K_I(z) = -\sum_{n=1}^{\infty} \chi_n(z)$, where $\chi_n(z)$ is characteristic function of domain Δ_n. In this domain

$$|z|^{-p} - 1 < n < |z|^{-p},$$

and, consequently, near the origin we have

$$K_{S_p}(z) = -|z|^{-p} + \frac{i}{2\pi} \ln |z| + O(1).$$

Clearly, the torsion of this spiral at the origin is $p/2$.

We will seek an integrator $U(z)$ of a defined on Γ function $u(t) \not\equiv 1$ in the form of product

$$U(z) = u^*(z)\chi_\Gamma(z)K_\Gamma(z), \tag{8}$$

where $u^*(z)$ is a continuation of $u(t)$ on the whole complex plane. An universal tool for building of this continuation is the Whitney extension operator $\mathcal{E}_0$ (see, for instance, [32]). If a function $u(t)$ is continuous on a compact set $A \subset \mathbb{C}$, then its Whitney extension $\mathcal{E}_0 u(z)$ from this set is continuous in the whole complex plane function such that its restriction on A coincides with $u(t)$, and in $\mathbb{C} \setminus A$ it has partial derivatives of all orders. In addition, if $u \in H_\nu(A)$, $\nu \in (0, 1]$, then $\mathcal{E}_0 u \in H_\nu(\mathbb{C})$, and its partial derivatives satisfy inequation

$$\left| \frac{\partial^{m+n} \mathcal{E}_0 u(z)}{\partial x^m \partial y^n} \right| \leq \frac{h_\nu(u, A)}{\mathrm{dist}^{m+n-\nu}(z, A)}, \quad z \in \mathbb{C} \setminus A.$$

This estimation implies the mentioned above condition (5) of solvability of the jump problem on a non-rectifiable curve.

In the present paper we will use another characteristics of Γ – so called Marcinkiewicz exponents.

Let $B(t;r) := \{z : |z-t| < r\}$, $r > 0$, $p > 0$, $t \in \Gamma$. We put

$$I_p(t;r) = \iint\limits_{B(t;r)} \frac{dx\,dy}{\operatorname{dist}^p(x+iy,\Gamma)}.$$

If $t \neq a_{1,2}$ and r is sufficiently small, then Γ divides the disk $B(t;r)$ into two parts $B^{\pm}(t;r)$ lying on the left-hand side and on the right-hand side of Γ correspondingly, and we put

$$I_p^{\pm}(t;r) = \iint\limits_{B^{\pm}(t;r)} \frac{dx\,dy}{\operatorname{dist}^p(x+iy,\Gamma)}.$$

As above, we consider that Γ is a simple Jordan arc of null plane measure.

Definition 4 *Left and right Marcinkiewicz exponents of an arc Γ at the point $t \in \Gamma \setminus \{a_1, a_2\}$ are equal to*

$$\mathfrak{m}^{\pm}(\Gamma;t) := \sup\{p : \lim_{r\to 0} I_p^{\pm}(t;r) < \infty\}.$$

The value

$$\mathfrak{m}(\Gamma;t) := \max\{\mathfrak{m}^{+}(\Gamma;t), \mathfrak{m}^{-}(\Gamma;t)\}$$

is called its Marcinkiewicz exponent at the point $t \in \Gamma \setminus \{a_1, a_2\}$, and

$$\mathfrak{m}(\Gamma;a_j) := \sup\{p : \lim_{r\to 0} I_p(a_j;r) < \infty\}, \quad j = 1, 2,$$

are Marcinkiewicz exponents at end points a_1 and a_2.

These characteristics were introduced first in [28]. Later (see, for example, [12]) the following their properties were proved:

– all Marcinkiewicz exponents of Γ at any its point does not exceed 1;

– any Marcinkiewicz exponent of Γ at any its point is greater or equal to $2-d$, where $d = \overline{\operatorname{dm}}\, N$ is the upper metric dimension of any neighborhood of this point in Γ. Let us refer an example of calculation of the Marcinkiewicz exponents at end point.

Example 3 *We fix two decreasing and converging to zero sequences of positive values $\{a_n\}$ and $\{b_n\}$, $a_1 = b_1 = 1$. Arc λ begins at point $a_1 \neq 0$, ends at the origin, and consists of successive segments $[a_1, a_1 + ib_1]$, $[a_1 + ib_1, -a_1 + ib_1]$, $[-a_1 + ib_1, -a_1 - ib_1]$, $[-a_1 - ib_1, a_2 - ib_1]$, $[a_2 - ib_1, a_2 + ib_2]$, $[a_2 + ib_2, -a_2 + ib_2]$ and so on. If at least one of series $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ diverges, then length of this broken line is infinite. The convergence of integral $\iint\limits_{|z|<r} \frac{dx\,dy}{\operatorname{dist}^p(z,\lambda)}$ is equivalent to convergence of series of of integrals of the same function over trapezoids with vertices $\pm(a_n + ib_n), \pm(a_{n+1} + ib_{n+1})$ (horizontal trapezoids), with vertices $a_n + ib_n, a_{n+1} + ib_{n+1}, a_{n+1} - ib_n, a_n - ib_{n-1}$ (right vertical trapezoids) and over analogous left vertical trapezoids. In turn, these series of integrals converge if*

and only if there converge series $\sum_{n=1}^{\infty} a_n(b_n - b_{n+1})^{1-p}$ *and* $\sum_{n=1}^{\infty} b_n(a_n - a_{n+1})^{1-p}$. *Particularly, for* $a_n = n^{-\alpha}$, $b_n = n^{-\beta}$, $0 < \alpha, \beta < 1$ *these series converge for* $p > \max\left\{\frac{\alpha+\beta}{1+\alpha}, \frac{\alpha+\beta}{1+\beta}\right\}$, *i.e.,*

$$\mathfrak{m}(\lambda; 0) = \max\left\{\frac{\alpha+\beta}{1+\alpha}, \frac{\alpha+\beta}{1+\beta}\right\}.$$

We will prove a theorem on existence of integrators on arcs of moderate torsion in terms of local Hölder condition. Let us define it.

Definition 5 *Let* $V : \Gamma \mapsto [\nu, 1], \nu > 0$ *be a given mapping. We denote* $H_V(\Gamma)$ *the class of all defined on* Γ *functions* $u(t)$ *satisfying the following assumption:*

– any point $t \in \Gamma$ *has a neighborhood* $N(t)$ *in* $\mathbb{C}$ *such that the restriction of* u *on* $\Gamma(t) := N(t) \cap \Gamma$ *belongs to the Hölder space* $H_{V(t)}(\Gamma(t))$.

Theorem 1 *Let* Γ *be an arc with moderate torsion at both its ends, and* $u \in H_V(\Gamma)$. *If*

$$V(t) > 1 - \mathfrak{m}(\Gamma; t) \tag{9}$$

at any its inner point $t \in \Gamma \setminus \{a_{1,2}\}$ *and*

$$V(a_j) > 1 - \mathfrak{m}(\Gamma; a_j)(1 - \tau_j), \quad j = 1, 2, \tag{10}$$

at its end points, then function $u(t)$ *has an integrator. If*

$$V(t) > 1 - \frac{1}{2}\,\mathfrak{m}(\Gamma; t) \tag{11}$$

for any $t \in \Gamma \setminus \{a_{1,2}\}$ *and*

$$V(a_j) > 1 - (\frac{1}{2} - \tau_j)\,\mathfrak{m}(\Gamma; a_j), \quad j = 1, 2, \tag{12}$$

then it has a continuous integrator.

Proof. We consider without loss of generality that the neighborhoods $N(t)$ are disks: $N(t) = B(t, r)$, and their radii $r = r(t) > 0$ are sufficiently small. These disks cover Γ, and we can select finite covering by disks B_j, $j = 1, 2, \dots, m$, $B_j = B(t_j, r_j)$. We consider that enumeration of these disks corresponds the direction of Γ, $t_1 = a_1$, $t_m = a_2$.

Let ψ_j, $j = 1, 2, \dots, m$, be a smooth partition of unit subordinated to the covering B_j, $\psi_j(t_j) = 1$, $j = 1, 2, \dots, m$. We put $u_j := u\psi_j$, and built integrators for these functions in the following way.

If t_j is not end point of Γ, then Γ divides the disk B_j onto two parts $B_j^{\pm}$ lying from the left and from the right of Γ correspondingly. Let $\chi_j(z)$ be the characteristic function of B_j^+ for $\mathfrak{m}^+(\Gamma; t) \geq \mathfrak{m}^-(\Gamma; t)$ and characteristic function of B_j^- with sign minus otherwise, and put

$$U_j(z) := \psi_j(z)\chi_j(z)\mathcal{E}_0 u(z).$$

Obviously, this product has jump u_j on Γ, and its first partial derivatives are integrable near t_j with any power lesser than $\mathfrak{m}(\Gamma; t_j)/(1-V(t_j))$. Hence, under restriction (9) it is integrable, i.e., U_j is integrator of u_j, and under restriction (11) it is integrable with some power greater 2, and U_j is continuous integrator of u_j.

Now let t_j be one of the end points of Γ. Then we put

$$U_j(z) := \psi_j(z) K_\Gamma(z) \mathcal{E}_0 u(z).$$

It also has jump u_j on Γ. The first partial derivatives of product $\psi_j(z)\mathcal{E}_0 u(z)$ are integrable with the same powers as in the previous case. The logarithmic kernel of Γ is integrable near end point a_j with any power lesser than τ_j^{-1}, where τ_j is the torsion. Hence, the first partial derivatives are integrable near a_j with any power less than $\mathfrak{m}(\Gamma; a_j)/(\mathfrak{m}(\Gamma; a_j)\tau_j + 1 - V(t_j))$. Hence, under restriction (10) it is integrable, i.e., U_j is integrator of u_j, and under restriction (12) it is integrable with some power greater 2, and U_j is continuous integrator of u_j. The sum of appropriate integrators u_j gives integrator or continuous integrator of u. The theorem is proved.

Remark 2 *The conditions* (10) *and* (12) *imply that the arc* Γ *has moderate torsion.*

Remark 3 *The obtained in the proof of Theorem 1 integrator* U *has the following additional property:*

– any point $t \in \Gamma \setminus \{a_{1,2}\}$ *has sufficiently small neighborhood* $B(t, r)$ *such that* $U(z)$ *satisfies the Hölder condition with exponent* $V(t)$ *in the both semi-neighborhoods* $B^{\pm}(t, r)$.

Remark 4 *Integration on* $d\overline{z}$ *exists under just the same conditions.*

The assumptions of Theorem are less restrictive than in the theorems on generalized integrability over arcs from works [24]–[26].

1.3 Uniqueness of integration

Let us call to our mind the concepts of Hausdorff dimension and Hausdorff measure. As above, we denote $B(z, r)$ the disk of radius r with center z, and put for a fixed compact set $A \subset \mathbb{C}$ and positive r, λ

$$\mathcal{H}_r^\lambda(A) = \inf\left\{\sum_{k=1}^n r_k^\lambda : A \subset \bigcup_{k=1}^n B(x_k, r_k), x_k \in E, 0 < r_k \le r\right\}.$$

Then

$$\mathcal{H}^\lambda(A) = \lim_{r \downarrow 0} \mathcal{H}_r^\lambda(A)$$

is λ-dimensional Hausdorff measure of A, and

$$\mathrm{dmh}(E) = \inf\{\lambda \ge 0 : \mathcal{H}^\lambda(A) = 0\}$$

is the Hausdorff dimension of this set. Its properties are well-known (see, for instance, [8]).

As known (see, for instance, [30]), if $\operatorname{dmh}\Gamma = \lambda > 1$ and $\mathcal{H}^{\lambda}(\Gamma) > 0$, then there exists a non-trivial continuous in $\overline{\mathbb{C}}$ and holomorphic in $\mathbb{C}\setminus\Gamma$ function $H(z)$. We consider it as integrator of zero, because it has zero jump on the arc Γ. If U is an integrator of u, then $U + H$ is its integrator, too, i.e., in general integrator cannot be unique. But different integrators can determine equal integrations. In this connection we introduce the following two definitions.

Definition 6 *A continuous in $\mathbb{C} \setminus \Gamma$ function $U(z)$ belongs to class $HO_V(\Gamma)$ if any point $t \in \Gamma \setminus \{a_{1,2}\}$ is a center of a small disk $B(t, r)$ such that Γ divides this disk on left-hand and right-hand parts $B^{\pm}(t, r)$, and U satisfies the Hölder condition with exponent $V(t)$ in both these parts.*

Definition 7 *Let any point $t \in \Gamma\setminus\{a_{1,2}\}$ is a center of a small disk $B(t, r)$ such that $\operatorname{dmh} B(t, r)\cap\Gamma = h(t)$. Then function $h(t)$ is the local Hausdorff dimension of arc Γ.*

Theorem 2 *Let Γ be an arc with a local Hausdorff dimension $h(t) < 2$, a function $u(t)$ is defined on this arc, and U_1, U_2 are two its integrators. If $U_1 - U_2 \in HO_V(\Gamma)$, and*

$$V(t) > h(t) - 1, \quad t \in \Gamma \setminus \{a_{1,2}\}, \tag{13}$$

then these integrators determine the same integration.

Proof. Denote $U := U_1 - U_2$. We have to prove that

$$\iint\limits_{\mathbb{C}} \frac{\partial \omega U}{\partial \overline{z}}\, dz\, d\overline{z} = 0$$

for any $\omega \in C^1(\mathbb{C})$. Any point $t \in \Gamma \setminus \{a_{1,2}\}$ is a center of a small disk $B(t, r)$ from the definition 7, and the end points $a_{1,2}$ are centers of small disks where derivative $U_{\overline{z}}$ is integrable. All these disks together cover Γ, and we can select from them the finite covering $B(t_j, r_j)$, $j = 1, 2, \cdots$. Assume that t_1 and t_m are the end points of Γ. We construct a smooth partition of unit $\psi_j, j = 1, 2, \dots, m$, subordinated to this covering, and denote $\Gamma_j := \Gamma \cap B(t_j, r_j)$, $\omega_j := \omega\psi_j$. Obviously,

$$\iint\limits_{\mathbb{C}} \frac{\partial \omega U}{\partial \overline{z}}\, dz\, d\overline{z} = \sum_{j=1}^{m} \iint\limits_{\mathbb{C}} \frac{\partial \omega_j U}{\partial \overline{z}}\, dz\, d\overline{z},$$

and we can select the radii r_1 and r_m such that the first and last terms of this sum are arbitrarily small.

Let $1 < j < m$, i.e., t_j is not the end point of Γ. We fix the value $\alpha > h(t_j)$. Then the Hausdorff measure of order α for Γ_j is zero, and for any $\varepsilon > 0$ we can cover Γ_j by a finite family of disks $B_k = B(z_k, r_{j,k}), j = 1, 2, \dots,$ such that $r_{j,k} < \varepsilon$ and $\sum_{k>0} r_{j,k}^{\alpha} < \varepsilon/m$. Let Λ be a boundary of union $\mathbf{B} = \cup_{k>0} B_k$.

We have

$$\left|\iint\limits_{\mathbb{C}} \frac{\partial U\omega_j}{\partial \overline{z}}\, dz\, d\overline{z}\right| \leq \left|\iint\limits_{\mathbf{B}} \frac{\partial (U\omega_j)}{\partial \overline{z}} dz\, d\overline{z}\right| + \left|\int\limits_{\Lambda} U\omega_j\, dz\right|.$$

The first term of the right-hand side vanishes for $\varepsilon \to 0$, because by assumptions of the theorem Γ has null plane measure and U is integrable together with its first derivative. It remains to show that the second term has null limit, too.

We consider without loss of generality that any disk of the last covering is not covered by union of other disks. Number these disks in order of increasing radii, and put $\Delta_1 = B_1$, $\Delta_2 = B_2 \setminus \Delta_1$, $\Delta_3 = B_3 \setminus \bigcup\limits_{k=1}^{2} \Delta_k$, $\Delta_4 = B_4 \setminus \bigcup\limits_{k=1}^{3} \Delta_k$, and so on. As a result, we represent $\mathbf{B}$ as a union of finite number of non-overlapping simply connected domains $\Delta_k \subset B_k$ such that their boundaries $\lambda_k = \partial\Delta_k$ consist of arcs of circles of radii $\geq r_{j,k}$ lying inside $B_{j,k}$. Therefore, length of λ_k does not exceed $2\pi r_{j,k}$. Obviously,

$$\int_{\Lambda} U\omega_j\, dz = \sum_k \int_{\lambda_k} U\omega_j\, dz,$$

and

$$\int_{\lambda_k} U\omega_j\, dz = \int_{\lambda_k} U(z_k)\omega_j(z)\, dz + \int_{\lambda_k} (U(z) - U(z_k))\omega_j(z)\, dz.$$

We have

$$\int_{\lambda_k} U(z_k)\omega_j(z)\, dz = -\iint_{\Delta_k} U(z_k)\frac{\partial\omega_j}{\partial\overline{z}}\, dz\, d\overline{z},$$

and by virtue of boundedness of the function $U\frac{\partial\omega_j}{\partial\overline{z}}$

$$\left|\sum_k \int_{\lambda_k} U(z_k)\omega_j(z)\, dz\right| \leq c|\mathbf{B}|,$$

where $|\cdot|$ stands for the plane Jordan measure. Thus, this sum vanishes together with $\varepsilon \to 0$. Finally,

$$\left|\sum_k \int_{\lambda_k} (U(z) - U(z_k)\omega_j(z)\, dz\right| \leq c\sum_k r_k^{V(t_j)+1} \leq c\sum_k r_{j,k}^{\alpha}$$

for $V(t_j) > \alpha - 1$, and the last sum also tends to zero for $\varepsilon \to 0$. The theorem is proved.

Remark 5 *Under assumptions of the theorem the integrators U_1 and U_2 generate equal integrations on the differential $d\overline{z}$, too.*

1.4 Cauchy type integral over arc of moderate torsion

Let us introduce a generalization of the Cauchy type integral (3) for a non-rectifiable arc Γ of moderate torsion. We consider first a smooth version Ω of the Cauchy kernel.

If $z \in \mathbb{C} \setminus \Gamma$, $0 < \rho < \operatorname{dist}(z, \Gamma)$, then we assume that $\Omega(t, z)$ is smooth as function of $t \in \mathbb{C}$, vanishes in the disk $B(z, \rho)$, and equals to the Cauchy kernel $(2\pi i(t - z))$ for $t \in \Gamma$.

Definition 8 *If a defined on arc Γ function u has integrator U, then its generalized Cauchy type integral is a result of application of mapping (6) to the kernel $\Omega(t, z)$ as a function of variable t.*

The Green formula immediately leads to the following representation for the Cauchy type integral over arc Γ:

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{u(t)\, dt}{t - z} = U(z) - \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial U}{\partial \overline{\tau}} \frac{d\tau\, d\overline{\tau}}{\tau - z}. \tag{14}$$

The double integral in the right-hand side is a so-called Cauchy potential. Its properties are well-known (see, for instance, [31]). In particular, if derivative $\frac{\partial F}{\partial \overline{\tau}}$ is integrable in $\mathbb{C}$ in a power $p > 2$ (i.e., if the integrator U is continuous), then this integral determines continuous in the whole complex plane function of variable z, and, consequently, the generalized Cauchy integral (14) has jump u on the curve Γ. Thus, by means of Theorem 1 we obtain the following result.

Theorem 3 *Let Γ be an arc with moderate torsion at both its ends, and $u \in H_V(\Gamma)$. If there are valid conditions (11) and (12), then the Cauchy type integral (14) is defined, and has the following properties:*

1. it is holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function;

2. this function vanishes at the point at infinity;

3. at any point $t \in \Gamma \setminus \{a_{1,2}\}$ it has limit values from the left and from the right, and difference of these values equals to $u(t)$;

4. any point $t \in \Gamma \setminus \{a_{1,2}\}$ has a neighborhood $N(t)$ such that restriction of the mentioned in previous property limit values from the left and from the right belong to the Hölder space $H_{W(t)}(\Gamma(t))$, $\Gamma(t) := N(t) \cap \Gamma$, where $W(t)$ is any value satisfying inequality

$$W(t) < 1 - 2(1 - V(t))/\,\mathfrak{m}(\Gamma; t); \tag{15}$$

5. at the ends of Γ the Cauchy type integral equals to $K_\Gamma(z)\mathcal{E}_0 u(z) + O(1)$.

Proof. The properties 1 and 2 are obvious. According to Theorem 1, under restrictions (11) and (12) the function u has continuous integrator U, and the Cauchy potential in formula (14) is continuous in the whole complex plane, what implies the property 3. As we have seen in the proof of this theorem, any point $t \in \Gamma \setminus \{a_{1,2}\}$ has a neighborhood $N(t)$ such that the derivative

$U_{\overline{z}}$ is integrable near t_j with any power less than $p_j := \mathfrak{m}(\Gamma; t_j)/(1 - V(t_j))$. Hence (see, for instance, [31]), the Cauchy potential in (14) satisfies near t_j the Hölder condition with any exponent less than $W(t_j) = 1 - 2/p_j$, i.e., this value is meaning of function (15) at the point t_j. This assertion implies the property 4. Finally, for continuous integrator the Cauchy potential is continuous at the points $a_{1,2}$. Therefore, at these points the Cauchy type integral equals to $U(z) + O(1) = K_\Gamma(z)\mathcal{E}_0 u(z) + O(1)$. The theorem is proved.

1.5 Riemann problem on arc of moderate torsion

Here we solve certain cases of the Riemann boundary value problem on non-rectifiable arc Γ of moderate torsion.

We consider first the jump problem in the following statement.

Given jump $g \in H_V(\Gamma)$. Find holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ such that it vanishes at the infinity, satisfies boundary-value condition (2), and is integrable with a power greater than 2 near the end points of the arc.

The previous theorem implies immediately the following condition on solvability of this problem.

Corollary 1 *Let Γ be an arc with moderate torsion at both its ends, and $g \in H_V(\Gamma)$. If there are valid inequalities* (11) *and* (12)*, then the Cauchy type integral of g is a solution of the formulated jump problem.*

As known, if Γ is rectifiable, then the solution of the jump problem is unique. According to the cited above E.P. Dolzhenko's theorem [30], the uniqueness is not valid for non-rectifiable arcs. But we can ensure it by means of the following result.

We refer a function $\Phi(z)$ to class $HC_V(\Gamma)$ if any point $t \in \Gamma \setminus \{a_{1,2}\}$ has neighborhood $B(t,r)$ such that $\Phi(z)$ satisfies the Hölder condition with exponent $V(t)$ in the both its parts $B^{\pm}(t,r)$. As above, $B^{\pm}(t,r)$ are parts on which Γ divides $B(t,r)$

Theorem 4 *Let Γ be an arc with moderate torsion at both its ends with local Hausdorff dimension $h(t)$. If a function $\Phi(z)$ is continuous in a domain $D \supset \Gamma$, holomorphic in $D \setminus \Gamma$, belongs to the class $HC_V(\Gamma)$, and exponent $V(t)$ satisfies condition* (13)*, then $\Phi(z)$ is holomorphic in D.*

Proof is analogous to the proof of E.P. Dolzhenko's theorem [30]; it contains also considerations from the proof of Theorem 2 of the present paper.

The last theorem means, that the jump problem has a unique solution of the class $HC_V(\Gamma)$ if

$$1 - 2(1 - V(t))/\,\mathfrak{m}(\Gamma; t); > h(t) - 1. \qquad (16)$$

Hence, there is valid

Corollary 2 *Let Γ be arc with moderate torsion at both its ends with local Hausdorff dimension $h(t)$, and $g \in H_V(\Gamma)$. If there are valid inequalities* (11)*,*

(12) *and* (16)*, then the Cauchy type integral of g is a unique solution of the formulated above jump problem.*

Now we study the homogeneous Riemann boundary value problem on an arc of moderate torsion in the following formulation.

Given function $G \in H_V(\Gamma)$. Find holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ such that it vanishes at the infinity, satisfies boundary value condition

$$\Phi^+(t) = G(t)\Phi^-(t), \quad t \in \Gamma \setminus \{a_{1,2}\}, \tag{17}$$

and is either bounded or integrable with a power greater than two near the end points of the arc.

Assume that $G(t)$ does not vanish on Γ. Then $G(t) = \exp f(t)$, $f \in H_V(\Gamma)$. Let $F(z)$ be an integrator of f. We consider the generalized Cauchy type integral with density f

$$\digamma(z) := \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{f(t)\,dt}{t-z} = F(z) - \frac{1}{2\pi i}\iint\limits_{\mathbb{C}} \frac{\partial F}{\partial \overline{\tau}} \frac{d\tau\, d\overline{\tau}}{\tau - z}, \tag{18}$$

and a function $X(z) := \exp \digamma(z)$. Obviously, under restrictions of Theorem 3 it has the following properties:

i. it is holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function, and equals to 1 at the point at infinity;

ii. at any point $t \in \Gamma \setminus \{a_{1,2}\}$ it has limit values from the left and from the right, and ratio of these values equals to $G(t)$;

iii. any point $t \in \Gamma \setminus \{a_{1,2}\}$ has a neighborhood $N(t)$ such that restriction of the mentioned in previous property limit values from the left and from the right belong to the Hölder space $H_{W(t)}(\Gamma(t))$, $\Gamma(t) := N(t) \cap \Gamma$, where $W(t)$ is any value satisfying inequality (15);

iv. near the ends of Γ it satisfies inequations

$$C^{-1} \exp \operatorname{Re} K_\Gamma(z)\mathcal{E}_0 f(z) \le |X(z)| < C \exp \operatorname{Re} K_\Gamma(z)\mathcal{E}_0 f(z), \quad C > 1.$$

Clearly, under restriction (16) any solution of the homogeneous Riemann boundary value problem in the class $HC_V(\Gamma)$ is representable as

$$\Phi(z) = (\phi_1(z) + \phi_2(z))X(z), \tag{19}$$

where $\phi_j(z)$ is holomorphic in $\overline{\mathbb{C}} \setminus \{a_j\}$, $j = 1, 2$, and $\phi_1(\infty) = \phi_2(\infty) = 0$. This solution is bounded at the end points of the arc if and only if

$$|\phi_j(z)| \le C \exp(-\operatorname{Re} K_\Gamma(z)\mathcal{E}_0 f(z)), \quad j = 1, 2.$$

Obviously,

$$K_\Gamma(z) = A_\Gamma(z) + \frac{1}{2\pi i} \ln \left| \frac{z - a_2}{z - a_1} \right|,$$

where branch of argument

$$A_\Gamma(z) := \frac{1}{2\pi} \arg \frac{z - a_2}{z - a_1}$$

satisfy the condition $A_\Gamma(\infty)=0$. Let us restrict ourself here by arcs satisfying inequations
$$A_\Gamma(z)=O(|z-a_j|^{-q_j}),\quad z\to a_j,\quad j=1,2. \tag{20}$$
We will say in this case that order of Γ at the end a_j does not exceed q_j. Clearly, then the torsion of Γ at the point a_j is greater or equal the half of its order, and the order of arc with moderate torsion is less than 2. Vice versa, if the order is less than 2, then the arc has moderate torsion.

We assume that the orders of Γ at the points a_j is less than $V(a_j)$, $j=1,2$. Then $K_\Gamma(z)\mathcal{E}_0 f(z)=f(a_j)K_\Gamma(z)+O(1)$, and the boundedness of product (19) is equivalent to condition
$$|\phi_j(z)|\leq C\exp(-\operatorname{Re} f(a_j)K_\Gamma(z)),\quad j=1,2.$$
Denote $f(a_j)=u_j+v_j$, where u_j and v_j are real. Then the last condition turns into
$$|\phi_j(z)|\leq C\exp(-u(a_j)A_\Gamma(z)-(2\pi)^{-1}(-1)^j v(a_j)\ln|z-a_j|),\quad j=1,2.$$
We put
$$\beta_j:=\liminf_{z\to a_j}\frac{u(a_j)A_\Gamma(z)}{\ln|z-a_j|},\quad j=1,2. \tag{21}$$
Easy considerations enable us to prove

Corollary 3 *Let Γ be an arc of orders q_j at its ends with local Hausdorff dimension $h(t)$, and function $G\in H_V(\Gamma)$ does not vanish on Γ. If there are valid relations* (11), (12), (16), *$q_j\leq V(a_j)$, $j=1,2$, and at least one of limits* (21) *equals to $+\infty$, then the homogeneous Riemann boundary value problem* (17) *has infinite family of linearly independent bounded solutions in class $HC_V(\Gamma)$.*

Remark 6 *If both limits β_j, $j-1,2$, are finite, then the number of linearly independent bounded solutions in class $HC_V(\Gamma)$ of problem* (17) *is finite, too.*

Remark 7 *These results keep validity for the homogeneous Riemann boundary value problem* (17) *for functions, which are integrable with power greater than two at the end points of the arc.*

It is of interest to research inhomogeneous Riemann boundary value problem (1) under assumptions of the last corollary. It seems that problem is more sophisticated.

1.6 J-integrations

Let us consider one more approach to generalization of curvilinear integral on non-rectifiable arcs.

In subsection 1.4 we have seen that the Cauchy type integral can be generalized on non-rectifiable arc of moderate torsion by formula (14). The left-hand side of this formula is (formally) an integral of function $u(t)((2\pi i)(t-z))^{-1}$ of variable t over arc Γ. This fact leads to the following construction. If a function $u(t)$ is defined on Γ, then we fix a point z, and put

$$f(t) := \frac{1}{2\pi i}\frac{u(t)}{t-z}.$$

According to formula (14), we have

$$\int\limits_{\Gamma} f(t)\,dt = U(z) - \frac{1}{2\pi i}\iint\limits_{\mathbb{C}} \frac{\partial U}{\partial \overline{\tau}}\frac{d\tau\,d\overline{\tau}}{\tau - z}. \tag{22}$$

Here $u(t) = 2\pi i(t-z)f(t)$, and U is its integrator. We obtain a new version of integration over non-rectifiable arc. Clearly, it is closely connected with the previous one: the value (22) is meaning at the point z of the result of application of mapping (6) to the generalization of the Cauchy kernel from subsection 1.4.

Let the point z be fixed outside of support of integrator U. Then the function of variable $\zeta \in \mathbb{C}\setminus\Gamma$

$$F(\zeta) := \frac{1}{2\pi i}\frac{U(t)}{\zeta - z}$$

is an integrator for $f(t)$. Clearly, $U(\zeta) = 2\pi i(\zeta - z)F(\zeta)$, and we obtain formula

$$\int\limits_{\Gamma} f(t)\,dt = -\iint\limits_{\mathbb{C}} \frac{\partial F}{\partial \overline{\zeta}}\,d\zeta\,d\overline{\zeta} \tag{23}$$

for J-integration. Thus, for arcs with moderate torsion J-integration is a result of application of mapping (6) to function $\omega \equiv 1$. Particularly, it is independent on the point z.

The next step of the generalization is a replacement of the Cauchy type integral by any solution of the jump problem. If $\Phi(z)$ is a holomorphic in $\overline{\mathbb{C}}\setminus\Gamma$ function satisfying boundary value condition (2) and $\Phi(\infty) = 0$, then its meaning at a fixed point $z \in \mathbb{C}\setminus\Gamma$ can be considered as integral of function $\frac{1}{2\pi i}\frac{g(t)}{t-z}$ of variable t over arc Γ. We will call this operation $J-$interation. Clearly, the formulation of the jump problem must be supplied by conditions ensuring uniqueness of its solution. In the preceding subsection we have described a version of that conditions for arcs of moderate torsion.

Unlike integrations defined in subsection 1.1, J-integration is functional.

For closed non-rectifiable curves the idea of generalization of curvilinear integral in terms of the Cauchy type integral is offered by Liu Hua [35].

2 Arcs of high torsion

The results of subsections 1.1 – 1.5 loose their validity for arcs of high torsion, because we do not know analogs of Theorem 1 for that paths. In this section we introduce other integrations in terms of restrictions of mapping (6) on certain subspaces of C^1 and $J-$integrations.

2.1 Integration of functions, vanishing at the end points.

We consider here integration of functions with nulls of sufficiently large orders at the ends of arc of high torssion.

Let weight $w(z)$ be a differentiable in $D := \mathbb{C}\setminus\{a_{1,2}\}$ function, which vanishes at the points $\{a_1\}$ and $\{a_2\}$ only, and $wH_V(\Gamma)$ stands for set of all defined on Γ functions u representable as products $u = wu_0$, $u_0 \in H_V(\Gamma)$. Theorem 1 implies the following result.

Corollary 4 *Let Γ be an arc with high torsion, and $u \in wH_V(\Gamma)$. If the product $w(z)K_\Gamma(z)$ is integrable together with its derivative $\frac{\partial w}{\partial \overline{z}}K_\Gamma(z)$ near Γ, then under conditions* (9) *and* (10) *function $u(t)$ has an integrator. If this product and its derivative on $\overline{z}$ are integrable with power greater than 2 near Γ, then under conditions* (11) *and* (12) *function $u(t)$ has a continuous integrator.*

There is valid analogous condition of integrability on $d\overline{z}$.

Proof. We repeat considerations from the proof of Theorem 1 for the factor u_0, and obtain function U_0. Generally speaking, it is not integrator for u_0, because the conditions of integrability can be broken near the ends of Γ. But the product wU_0 is integrator for u. The corollary is proved.

We consider also the Cauchy type integral over arc of high torsion with density from the class $wH_V(\Gamma)$. There is valid

Corollary 5 *Let Γ be an arc with high torsion, and the product $w(z)K_\Gamma(z)$ is integrable together with its derivative $\frac{\partial w}{\partial \overline{z}}K_\Gamma(z)$ near Γ with power greater than 2. If conditions* (11) *and* (12) *are fulfilled, then the Cauchy type integral* (14) *with density $u \in wH_V(\Gamma)$ exists and preserves the properties 1–4 from Theorem* (3)*. The property 5 from this theorem must be replaced by*

5'. at the ends of Γ this Cauchy type integral equals to $w(z)K_\Gamma(z)\mathcal{E}_0 u(z) + O(1)$.

Obviously, the analogs of corollaries 1 and 2 are valid for the jump problem on arcs with high torsion and jump of class $wH_V(\Gamma)$.

Example 4 *In Example 2 we define arc S_p with end points 0 and 1, and show that near the origin $K_{S_p}(z) = -|z|^{-p} + \frac{i}{2\pi}\ln|z| + O(1)$. The torsion of this spiral at the origin is $p/2$, and it is an arc of high torsion for $p \geq 2$. At the end point 1 its torsion is 0.*

We consider a weight $w(z) = z^n$ with positive integer n. The product wK_{S_p} is integrable with power greater than two for $p - n < 2$. We put $n = [p] - 1$, where brackets $[\cdot]$ stand for the entire part. Thus, the results of this subsection are valid for functions $u(t) = t^{[p]-1}u_0(t), u_0 \in H_V(S_p), p \geq 2$.

2.2 Holomorphic polynomials.

We pass to the case where $u(t)$ does not vanish at the end points of Γ, and begin from holomorphic polynomials of degree m, i.e.,

$$u(t) = c_m t^m + c_{m-1}t^{m-1} + \cdots + c_1 t + c_0.$$

Clearly, the product $u(z)K_\Gamma(z)$ has jump $u(t)$ on $\Gamma \setminus \{a_{1,2}\}$. We have at the infinity point

$$u(z)K_\Gamma(z) = \frac{u(z)}{2\pi i}\sum_{k=1}^{\infty}\frac{a_1^k - a_2^k}{z^k} = u_\Gamma(z) + U_\Gamma(z),$$

where $u_\Gamma(z)$ is a holomorphic polynomial of degree $m-1$

$$u_\Gamma(z) = \frac{1}{2\pi i}\sum_{k=1}^{m} z^{m-k}\sum_{j=1}^{k}(a_1^j - a_2^j)c_{m-k+j},$$

and $U_\Gamma(z)$ vanishes at the infinity and is holomorphic near it. Thus, there is valid

Proposition 1 *The function*

$$U_\Gamma(z) = u(z)K_\Gamma(z) - u_\Gamma(z)$$

is holomorphic in $\overline{\mathbb{C}}\setminus\Gamma$, *vanishes at the point* ∞, *and satisfies the boundary-value condition*

$$U_\Gamma^+(t) - U_\Gamma^-(t) = u(t), \quad t \in \Gamma \setminus \{a_{1,2}\}. \tag{24}$$

Any point $t \in \Gamma \setminus \{a_{1,2}\}$ *has semi-neighborhoods, where* $U_\Gamma(z)$ *satisfies the Hölder condition with exponent* 1, *and at the end points* $a_{1,2}$ *we have* $U_\Gamma(z) = u(a_j)K_\Gamma(z) + O(1)$.

Strictly speaking, U_Γ is not a solution of the jump problem with jump u, because it can have singularities of high order at the end points of the arc. Let us consider its connection with the jump problem in the following statement. We can use it as generalization of the Cauchy type integral with polynomial density, and as a source of j-integration with polynomial density. Immediate calculation shows that J-integral of holomorphic polinomial $u(z)$ equals to difference $P(a_2) - P(a_1)$, where $P(z)$ is a holomorphic polynomial such that $P'(z) = u(z)$.

2.3 General polynomials.

Here we are dealing with a non-holomorphic polynomials

$$u(z) = \sum_{0\le k+l\le m} c_{k,l} z^k \overline{z}^l. \tag{25}$$

We seek a holomorphic in $\overline{\mathbb{C}}\setminus\Gamma$ function $\Phi(z)$ satisfying on $\Gamma\setminus\{a_1, a_2\}$ boundary-value condition

$$\Phi^+(t) - \Phi^-(t) = u(t), \tag{26}$$

but first let us detail our restrictions on the arc Γ.

A point $t \in \Gamma$ is a point of smooth touch if there exists a smooth arc Λ with end t such that $\Gamma \cup \Lambda = t$. The set of points of smooth touch is dense in Γ (see, for instance, [6]. Therefore, we can fix one of that points a_0 dividing Γ on two arcs: Γ_1 beginning at the point a_1 and ending at a_0, and Γ_2 beginning at a_0 and ending at a_2. The point a_0 is point of smooth touch for arcs $\Gamma_{1,2}$, and both these arcs have null torsion at this point.

We assume that arcs $\Gamma_{1,2}$ are determined by equations

$$\Gamma_{1,2} = \{z = a_{1,2} + r(\tau)\exp i\theta_{1,2}(\tau) : 0 < \tau \leq 1\}, \tag{27}$$

where $r_{1,2}(\tau)$ and $\theta_{1,2}(\tau)$ are real functions, $r_{1,2}(\tau)$ are continuous on $[0,1]$, $r_{1,2}(0) = 0$, $\theta_{1,2}(\tau)$ are continuous on $(0,1]$, and $a_{1,2} + r_{1,2}(1)\exp i\theta_{1,2}(1) = a_0$. Similar arcs are studied in the articles [33, 34], but these arcs have monotone angular functions $\theta(r)$ such that $\lim\limits_{r\to 0}\theta(r)$ is either $+\infty$ or $-\infty$. Here we study more extensive class of arcs; in particular, we do not require the monotonicity of angle function.

In what follows we need certain regularity conditions on the arcs.

Let $I_{1,2}$ be the segments of straight lines connecting the points $a_{1,2}$ and a_0, and $B_{1,2} := \Gamma_{1,2} \cap I_{1,2}$. Let $\mathcal{T}(a_1)$ be the classes of arcs Γ_1 such that

i. the intersection B_1 is a countable set of points $x_0 = a_0, x_1, x_2, \dots$, condensing at the point a_1;

ii. every point $x_n \in B$ is a unique point of intersection of the circle $C_n := \{z : |z - a_1| = |x_n|\}$ with the arc Γ_1.

We number the points of the set B in the order of traversal from a_0 to a_1.

We also define the analogous class $\mathcal{T}(a_2)$. Obviously, classes $\mathcal{T}(a_{1,2})$ are more extensive than classes of arcs considered in [33, 34]. Let class $\mathcal{T}$ consist of arcs Γ such that $\Gamma_j \in \mathcal{T}(a_j)$, $j = 1, 2$.

Let us find a holomorphic in $\overline{\mathbb{C}} \setminus \Gamma_1$ function $\Phi_1(z)$ satisfying boundary-value condition (26) on $\Gamma_1 \setminus \{a_1, a_0\}$.

The points of set B_1 divide arc Γ_1 and segment I_1 onto arcs γ_n and segments I_n correspondingly, $n = 1, 2, 3, \dots$. The arc γ_n and segment I_n begin at point x_n and end at x_{n-1}, and both these paths belong to the ring $R_n := \{z : |x_n| \leq |z - a_1| \leq |x_{n-1}|\}$. We denote Δ_n finite domain bounded by union of γ_n and I_n, and put $s_n = +1$ if subsegment I_n is positively oriented with respect to Δ_n, and $s_n = -1$ otherwise. Then

$$K_{\Gamma_1}(z) - K_{I_1}(z) = \sum_{n=1}^{+\infty} s_n \chi_n(z),$$

where $\chi_n(z)$ is the characteristic function of domain Δ_n.

Some domains Δ_n does not contain the point a_1. That domains does not overlap, and the sum of their characteristic functions is bounded. Let us denote $P_{\Gamma_1}(z)$ (correspondingly, $N_{\Gamma_1}(z)$) the sum of characteristic functions of domains Δ_n such that $a_1 \in \Delta_n$ and $s_n = +1$ (correspondingly, $s_n = -1$.) The functions $P_{\Gamma_1}(z)$ and $N_{\Gamma_1}(z)$ are defined in $\mathbb{C} \setminus a_1$, real, positive and integer-valued. Thus, there is valid

Proposition 2 *If $\Gamma_1 \in \mathcal{T}(a_1)$, then near the point a_1*

$$K_{\Gamma_1}(z) = K_{I_1}(z) + P_{\Gamma_1}(z) - N_{\Gamma_1}(z) + O(1). \tag{28}$$

The arc Γ has high torsion at the point a_1 if the difference $P_{\Gamma_1}(z) - N_{\Gamma_1}(z)$ is not integrable in a neighborhood of this point.

Analogous result is valid for the arc Γ_2 and end-point a_2.

The arc γ_n and the subsegment I_n divide the ring $R_n = \{|x_{n+1}| < |z - a_1| < |x_n|\}$ onto two domains. We denote D_n the one that lies on the left from γ_n. The boundary ∂D_n consists of an arc γ_n, a segment I_n (oriented in the opposite I direction), and, maybe, one of circles C_n, C_{n+1} or both these circles. We put $\sigma_n = +1$ (correspondingly, $\sigma_n = -1$) if ∂D_n contains positively (correspondingly, negatively) directed circle C_n, and $\sigma_n = 0$ if ∂D_n does not contain this circle. Analogously, $\eta_n = +1$ (correspondingly, $\eta_n = -1$) if ∂D_n contains positively (correspondingly, negatively) directed circle C_{n+1}, and $\eta_n = 0$ if ∂D_n does not contain this circle.

Obviously, $\overline{z - a_1} = |x_n|^2(z - a_1)^{-1}$ for $z \in C_n$. Let us consider function $\xi^{(1)}_{k,l,n}(z)$ equaling to $|x_n|^{2l}(z - a_1)^{k-l}$ in the disk $|z - a_1| \leq x_n$ and 0 outside of this disk. Then sum

$$\Xi^{(1)}_n(z) = \sum_{0 \leq k+l \leq m} c_{k,l}\xi^{(1)}_{k,l,n}(z)$$

has jump $u(t)$ on positively directed circle C_n, and sum

$$\Psi_1(z) = \sum_{n=1}^{\infty} \sigma_n \Xi^{(1)}_n(z) \tag{29}$$

has jump $u(t)$ on system of circles $\cup_{n=1}^{\infty}\sigma_n C_n$, where factor σ_n means that the circle C_n is directed positively for $\sigma_n = +1$, negatively for $\sigma_n = -1$, and this term is missed for $\sigma_n = 0$.

We introduce an analogous function Ψ_2 for end point a_2 and an arc Γ_2, and put $\Psi = \Psi_1 + \Psi_2$.

Let us consider one more jump problem

$$\Omega^+(t) - \Omega^-(t) = u(t), \quad t \in \Lambda, \tag{30}$$

where Λ is a union of all boundaries of domains D_n, and Ω^+, Ω^- stand for limit values of function Ω from these domains and from the opposite side respectively. Clearly, this jump problem has solution

$$\Omega_1(z) = \sum_{n=1}^{+\infty}\left(u(z)\chi_n(z) - \frac{1}{2\pi i}\iint_{D_n}\frac{\partial u}{\partial \overline{\tau}}\frac{d\tau\, d\overline{\tau}}{\tau - z}\right), \tag{31}$$

where χ_n is the characteristic function of domain D_n. The domains D_n does not overlap. Hence, the solution Ω is bounded.

On the other hand,

$$\Omega_1(z) = \Phi_1(z) - \Phi_{I_1}(z) + \Psi_1(z),$$

where $\Phi_1(z)$ is a desired solution of jump problem (26) on arc Γ_1,

$$\Phi_{I_1}(z) = \frac{1}{2\pi i}\int\limits_I \frac{ut)\,dt}{t-z} \tag{32}$$

is a customary integral of Cauchy type over segment I_1, and $\Psi_1(z)$ is function (29). Hence,

$$\Phi_1(z) = \Omega_1(z) + \Phi_{I_1}(z) - \Psi_1(z).$$

Clearly, the boundary values $\Phi_1^{\pm}(t)$ satisfy the Hölder condition with any exponent lesser 1 in a neighborhood of any point $t \in \Gamma_1 \setminus \{a_1, a0\}$.

Analogous considerations enable us to find function $\Phi_2(z)$ with the same properties on the arc Γ. The sum

$$\Phi := \Phi_1 + \Phi_2 = \Omega_1(z) + \Phi_{I_1}(z) - \Psi_1(z) + \Omega_2(z) + \Phi_{I_2}(z) - \Psi_2(z) \tag{33}$$

satisfies the boundary condition (26) for $t \in \Gamma \setminus \{a_{1,2}\}$. Thus, there is valid

Theorem 5 *If u is a polynomial* (25)*, then there exists a holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ such that*

– it vanishes at the infinity;

– at any point $t \in \Gamma \setminus \{a_{1,2}\}$ it has boundary values from both sides related by equation (26)*;*

– its boundary values $\Phi^{\pm}(t)$ satisfy the Hölder condition with any exponent less than 1 *in a neighborhood of any point $t \in \Gamma_1 \setminus \{a_1, a_0\}$.*

One of that functions is given by formula (33)*.*

Clearly, the determined by formula (33) solution Φ at end points of Γ has asymptotic

$$\Phi(z) = -\Psi_j(z) + u(a_j)K_{I_j}(z) + O(1), \quad z \to a_j, \quad j = 1, 2. \tag{34}$$

This result can be improved. Indeed, the polynomial (25) is representable as the following sum:

$$\sum_{k=0}^{m} c_{k,0} z^k + \sum_{1\le k+l\le m}{}' c_{k,l} z^k \overline{z}^l$$

where prime means that any term contain $\overline{z}$. The first of these polynomials $h(z)$ is holomorphic, and by virtue of the previous subsection there exists a function $H(z)$ with jump h and asymptotic $H(z) = h(a_j)K_\Gamma(z) + O(1)$ at the end points. For the second polynomial $v(z) = u(z) - h(z)$ we apply the considerations from the proof of preceding theorem. As a result, we obtain

Corollary 6 *For any polynomial* (25) *we can find a holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi^*(z)$ satisfying all properties of function $\Phi(z)$ from theorem 5, and such that*

$$\Phi^*(z) = -\Psi_j^*(z) + u(a_j)K_{I_j}(z) + h(a_j)K_\Gamma(z) + O(1), \quad z \to a_j, \quad j = 1, 2,$$

where Ψ_j^* is an analog of function Ψ_j for polynomial v.

2.4 Taylor-differentiable jumps

In this subsection we consider on arc Γ functions representable as sums

$$u(z)=\sum_{j=1}^{2}\Pi_j(z)+w(z)v(z), \tag{35}$$

where

$$\Pi_j(z)=\sum_{0\le k+l\le m_j} c^{(j)}_{k,l}(z-a_j)^k\overline{(z-a_j)}^l, \quad j=1,2,$$

are the Taylor polynomials of orders m_j for function $u(z)$ at points a_j, $j=1,2$, weight $w(z)$ has zeros of orders greater or equal than m_j at end points a_j, $j=1,2$, and function v is continuous on Γ.

We refer a function u to class $H_V^{\mathbf{m}}(\Gamma)$, $\mathbf{m}=(m_1,m_2)$ if it is representable in the form (35) with $v\in H_V(\Gamma)$.

Let order of an arc $\Gamma\in\mathcal{T}$ at its end a_j does not exceed q_j, $j=1,2$. Denote class of all that arcs as $\mathcal{T}^{\mathbf{q}}$, where $\mathbf{q}=(q_1,q_2)$.

The results of three previous subsections imply

Theorem 6 *Let an arc Γ belong to class $\mathcal{T}^{\mathbf{q}}$, $u\in H_V^{\mathbf{m}}(\Gamma)$, and $m_j\ge q_j$, $j=1,2$. If conditions* (11) *and* (12) *are fulfilled, then there exists a holomorphic in $\overline{\mathbb{C}}\setminus\Gamma$ function $\Phi(z)$ such that*

– it vanishes at the infinity;

– at any point $t\in\Gamma\setminus\{a_{1,2}\}$ it has boundary values from both sides related by equality (26)*;*

– its boundary values $\Phi^{\pm}(t)$ satisfy the Hölder condition with any exponent less than (15) *in a neighborhood of any point $t\in\Gamma_1\setminus\{a_1,a_0\}$;*

– at the ends of Γ this function has asymptotic (34)*, where $\Psi_j(z)$ is an analog of sum* (29) *for polynomial $\Pi_j(z)$, $j=1,2$.*

One of functions with these properties is the sum of functions built in the previous subsections for polynomials Π_j, $j=1,2$, and for product wv. Its terms corresponding polynomials Π_j can be improved as it is done in Corollary 6.

2.5 Riemann problem on arc with high torsion.

We consider here the jump problem and homogeneous Riemann problem.

The results of preceding subsection enable us to obtain a function Φ with given jump $g\in H_V^{\mathbf{m}}(\Gamma)$ on open arc $\Gamma\setminus\{a_{1,2}\}$. It remains to study its behavior at end points. Let $g(a_1)=0$. According to formulas (34) and (29), function Φ from Theorem 6 has asymptotic

$$\Phi(z)=-\Psi_1(z)+O(1)=-\sum_{n=1}^{\infty}\sigma_n\sum_{0\le k+l\le m}c^{(1)}_{k,l}\xi^{(1)}_{k,l,n}(z)+O(1)$$

at point a_1, where $c_{k,l}^{(1)}$ are the Taylor coefficients of g at this point. We rewrite the definition of functions $\xi_{k,l,n}^{(1)}(z)$ as

$$\xi_{k,l,n}^{(1)}(z) = |x_n|^{2l}(z-a_1)^{k-l}\chi_n^{(1)}(z),$$

where $\chi_n^{(1)}(z)$ is the characteristic function of disk $\{z : |z-a_1| \le x_n\}$. Then

$$\Phi(z) = -\sum_{n=1}^{\infty} \sigma_n \sum_{0\le k+l\le m} c_{k,l}^{(1)}|x_n|^{2l}(z-a_1)^{k-l}\chi_n^{(1)}(z) + O(1).$$

Easy calculations show that $\left|\sum\limits_{n=1}^{\infty} \chi_n^{(1)}(z)\right| \le C|z-a_1|^{-q_1}$, where C is a positive constant. Therefore, the function Φ is bounded at the end point a_1 if $c_{k,l}^{(1)} = 0$ for $k-l-q_1 < 0$. The asymptotic at the point a_2 is analogous.

We obtain the following result concerning the jump problem on arcs of high torsion.

Theorem 7 *Let* $\Gamma \in \mathcal{T}^{\mathbf{q}}$, $g \in H_V^{\mathbf{m}}(\Gamma)$, $m_j \ge q_j$ *for* $j = 1,2$, *and the Taylor coefficients of jump* g *at end-points of* Γ *satisfy the conditions* $c_{k,l}^{(j)} = 0$ *for* $k-l-q_j < 0$. *If there are valid inequations* (11), (12), *then the jump problem has a solution in the class of bounded functions. Additionally, under condition* (16) *this solution is unique in the class* $HC_V(\Gamma)$. *Here* $h(t)$ *stands for local Hausdorff dimension of an arc* Γ.

Let us pass to homogeneous Riemann problem (17) on arcs of high torsion in the class of bounded functions. We assume that $\Gamma \in \mathcal{T}^{\mathbf{q}}$ and $G(t) = \exp f(t)$, $f \in H_V^{\mathbf{m}}(\Gamma)$, $m_j \ge q_j$ for $j = 1,2$. By means of Theorem 6 we build the function $\mathcal{F}(z)$ with jump f on open arc $\Gamma \setminus \{a_{1,2}\}$ (here we does not require its boundedness) and put $X(z) := \exp \mathcal{F}(z)$. In just same way as for arcs of moderate torsion we conclude that any bounded solution of the problem (17) in class $HC_V(\Gamma)$ is representable as $\Phi(z) = X(z)F(z)$, where function $F(z)$ is holomorphic in $\overline{\mathbb{C}} \setminus \{a_{1,2}\}$. Then $\Phi(z) = X(z)(F_1((z-a_1)^{-1}) + F_2((z-a_1)^{-1})$, where F_1 and F_2 are entire functions. The estimations of this subsection show that order of F_j does not exceed q_j, and

$$|F_1((z-a_1)^{-1})| \le C \exp \operatorname{Re} \sum_{n=1}^{\infty} \sigma_n \sum_{0\le k+l\le m} c_{k,l}^{(1)}|x_n|^{2l}(z-a_1)^{k-l}\chi_n^{(1)}(z),$$

where $c_{k,l}^{(j)}$ are the Taylor coefficients of function f at point a_j. Let us consider rational functions

$$\mathcal{R}_j(z) := \sum_{0\le k+l\le m} c_{k,l}^{(j)}(z-a_1)^{k-l}, \quad j = 1,2,$$

and sets

$$\mathbb{Z}_j := \{z : \operatorname{Re} \mathcal{R}_j(z) = 0\}, \quad j = 1,2.$$

If $\mathcal{R}_1(z)$ has either zero or pole of order $p_1 > 0$ at point a_1, then set $\mathbb{Z}_1$ in a sufficiently small neighborhood of a_1 is a union of p_1 smooth arcs dividing this neighborhood into $2p_1$ curvilinear sectors with angles π/p_1 at this point. Function $F_1((z-a_1)^{-1})$ is bounded on this set. Therefore, by virtue of the Phragmen – Lindelöf principle (see, for instance, [36]) for $p_1 > q_1$ the function F_1 is bounded. Then this function is constant. But the product $CX(z)$ is bounded in a neighborhood of a_1 for $C \neq 0$ if and only if the Taylor polynomial of function f at this point is identical zero. Thus, there is valid

Theorem 8 *Let $\Gamma \in \mathcal{T}^{\mathbf{q}}$, $f \in H_V^{\mathbf{m}}(\Gamma)$, $m_j \geq q_j$ for $j = 1, 2$, $G(t) = \exp f(t)$, and Taylor coefficients of jump g at end-points of Γ are not identical zeros. If these polynomials generate rational functions $\mathcal{R}_j$ with zeros or poles of orders $p_1 > q_1$ at points $a_{1,2}$, and conditions* (11), (12), (16) *are fulfilled, then the unique bounded solution of problem 17 in the class $HC_V(\Gamma)$ is identical zero.*

Clearly, this result concerns only one special case of the Riemann boundary-value problem on arc with high torsion, and basically this problem is open.

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