# Globally modified Navier-Stokes equations coupled with the heat equation:existence result and time discrete approximation

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**Abstract**

We provide in this article an investigation of the globally modified Navier-Stokes problem coupled with the heat equation. After deriving the variational formulation of this problem, we prove the existence and the uniqueness of the solution using the method of Faedo-Galerkin and some compactness results. Next, we propose a time discretization of these equations based on Euler's implicit scheme. We prove the existence of solution with the aid of Brouwer's fixed point and study the stability of discrete in time solution by using the energy approach.

## Contents



# 1 Governing equations and its mathematical setting

## 1.1 Formulation of the problem

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with regular boundary $\Gamma = \partial\Omega$. We define the function $F_N : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$F_N(r) = \min\{1, N/r\}, \; r \in \mathbb{R}^+, \tag{1.1}$$

for $N \in \mathbb{R}^+$ and taking $\widetilde{T} > 0$, we consider the following globally modified Navier-Stokes equations coupled with the heat equations (GMNSEHE)

$$\begin{cases} \boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + F_N(\|\boldsymbol{u}\|_{\mathbf{V}})(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega\times(0,\widetilde{T})\,, \\ \text{div } \boldsymbol{u} = 0 \ \text{ in } \Omega\times(0,\widetilde{T}), \\ T_t - \alpha\Delta T + F_N(\|(\boldsymbol{u},T)\|_{\mathbb{V}})(\boldsymbol{u}\cdot\nabla)T = g \text{ in } \Omega\times(0,\widetilde{T})\,, \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \ \ T(\boldsymbol{x},0) = T_0(\boldsymbol{x}) \text{ in } \Omega\,, \\ \boldsymbol{u} = 0, \ \ \text{and} \ \ T = T_b \quad \text{on} \ \ \Gamma\times(0,\widetilde{T})\,, \end{cases} \tag{1.2}$$

where $\|\boldsymbol{u}\|_{\mathbf{V}}$ and $\|(\boldsymbol{u},T)\|_{\mathbb{V}}$ are defined as in (1.6) and (1.7) below. As usual, $\boldsymbol{u}, T$ and $p$ represent respectively the fluid velocity, the temperature and the pressure. $\nu$ is the constant viscosity of the fluid, and $\alpha$ represent the thermal conductivity. $\boldsymbol{f}$ is the external force acting on the fluid while $g$ is the radiant heating. $\boldsymbol{u}_0$ is the initial velocity and $T_0$ is the initial heat.

The GMNSEHE (1.2) is inspired from the globally modified Navier-Stokes equations (GMNSE) studied in [4]. As clearly demonstrated in [4], $F_N(\|\boldsymbol{u}\|_{\mathbf{V}})$ prevents the rapid grow of velocity gradient and helps to obtain uniqueness of weak solution in 3d, property which is lacking for Navier Stokes in 3D. Hence Mathematically, the globally modified Navier Stokes has an advantage for now over the Navier Stokes equations. In this work, we show that the factors $F_N(\|\boldsymbol{u}\|)$ and $F_N(\|(\boldsymbol{u},T)\|_{\mathbb{V}})$ help us to control the values of $\|\boldsymbol{u}\|_{\mathbf{V}}$ and $\|(\boldsymbol{u},T)\|_{\mathbb{V}}$ and subsequently permit us the establish uniqueness of weak solution of (1.2). As we are aware of, this remarkable property is unreachable for (1.2) without the weighted terms (see [2, 22]). It is worth noting that many challenges in the mathematical and numerical analysis of the full 3D Navier-Stokes equation are still lacking at present. Since the uniqueness theorem for the global weak solutions (or the global existence of strong solutions) of the initial-value problem of the 3D Navier-Stokes system is not yet proved, the known theory of global attractors of infinite-dimensional dynamical systems is not applicable to the 3D Navier-Stokes system. Thus, the use of "regularized approximation equations" to study the classical 3D Navier-Stokes systems has become an effective tool both from the numerical and the theoretical point of views. just like it has been noted in [23], many works make use of the LANS-$\alpha$ model to approximate many problems related to turbulence flows.
In [4], the authors proposed a three-dimensional system of a globally modified Navier-Stokes equations (GMNSE). They studied the existence and uniqueness of strong solutions and established the existence of global $\mathbb{V}$-attractors. Also, using a limiting argument they obtained the existence of bounded entire weak solutions of the three dimensional Navier-Stokes equations (NSE) with time independent forcing. As noted in [4], the GMNSE prevents large gradients dominating the dynamic and leading to explosion. Several articles are devoted to the mathematical analysis of the modified problems involving Navier-Stokes equations, see for instance [5, 6, 9, 11, 12, 15, 16, 17, 18, 19], as well as the review paper [7] in which the authors present some recent developments on the GMNSE The globally modified Navier-Stokes equations are useful in obtaining new results about the 3D NSE. Indeed there were used in [12] to show that the attainability set of weak solutions of the 3D NSE is weakly compact and weakly connected. We refer the reader to [8, 18, 19] for other modifications on the nonlinear terms in some mathematical models.
Motivated by the above works, we consider in the present article the globally modified of the model (1.2). More precisely, we propose a time semi-discretization of the time-dependent globally modified Navier-Stokes problem coupled with the heat equation (GMNSHE). The outline of the paper is as follows:

- We recall in section 2 the variational formulation of the problem. We also present the mathematical tools for its resolution.
- In section 3, we establish the existence of strong solutions of the GMNSHE in three-dimensions

and their continuous dependence on $N$ and on the initial value in the space $\mathbf{V}$. In addition, we show that the weak solution of the GMNSHE is unique in the class of weak solutions. We also investigate the relationship between the Galerkin approximations of the GMNSHE and the NSHE for a fixed finite dimension.

- Section 4 is devoted to the time semi-discretization of GMNSHE. We present a numerical scheme to approximate the unique solution obtained in section 3 and study its stability.

## 1.2 Mathematical setting

Let us now recall from [20] the functional spaces of the model (1.2) and its abstract formulation. Unless otherwise specified, the domain of interest $\Omega$ is bounded connected, and have a boundary $\partial\Omega = \Gamma$ that is at least $C^{0,1}$, i.e Lipschitz-continuous. Let $k = (k_1, k_2, k_3)$ be a triplet of non-negative integers and set $|k| = k_1 + k_2 + k_3$, we define the partial derivative $\partial^k$ of order $|k|$ by

$$\partial^k \phi = \frac{\partial^{|k|}\phi}{\partial^{k_1}x_1\partial^{k_2}x_2\partial^{k_3}x_3}\,.$$

The usual definitions of $L^p$ spaces and $H^m$ spaces applies with the scalar product of $L^2$ being denoted by $(\cdot,\cdot)$. These definitions are extended directly to vector-valued functions, with the notation

$$\mathbb{L}^2(\Omega) := (L^2(\Omega))^3,\ \mathbb{H}^m(\Omega) := (H^m(\Omega))^3,\ \mathbb{H}^m_0(\Omega) := (H^m_0(\Omega))^3, \mathbb{L}^2_0(\Omega) := (L^2_0(\Omega))^3$$

where $L^2_0(\Omega) = \left\{ q \in L^2(\Omega);\ \int_\Omega q(x)dx = 0 \right\}$. It is noted that for a vector $\boldsymbol{w}$ we set

$$\|\boldsymbol{w}\|^r_{\mathbb{L}^r(\Omega)} = \int_\Omega |\boldsymbol{w}(\boldsymbol{x})|^r dx$$

where $|\cdot|$ denotes the Euclidean norm $|\boldsymbol{w}|^2 = \boldsymbol{w}\cdot\boldsymbol{w}$. We shall frequently use Sobolev imbedding: for a real number $p \in \mathbb{R}$, $1 \le p \le 6$, the space $\mathbb{H}^1(\Omega)$ is imbedded into $\mathbb{L}^p(\Omega)$. In particular, there exists a constant $c_p$ (that depends only on $p$, $\Omega$ and $d = 3$) such that

$$\text{for all}\ \ \boldsymbol{v} \in \mathbb{H}^1_0,\ \ \|\boldsymbol{v}\|_{\mathbb{L}^p(\Omega)} \le c_p\|\nabla\boldsymbol{v}\|\,. \tag{1.3}$$

When $p = 2$, this is Poincare's inequality and $c_2$ is Poincare's constant. In the case of the maximum norm, the following imbedding holds

$$\text{for all}\ r > d = 3,\ \ \mathbb{W}^{1,r}(\Omega) \subset \mathbb{L}^\infty(\Omega)$$

in particular, for each $r > d = 3$, there exists $c_{\infty,r}$ such that

$$\text{for all}\ \ \boldsymbol{v} \in \mathbb{H}^1_0(\Omega) \cap \mathbb{W}^{1,r},\ \ \|\boldsymbol{v}\|_{\mathbb{L}^\infty(\Omega)} \le c_{\infty,r}\|\nabla\boldsymbol{v}\|_{\mathbb{L}^r(\Omega)}\,. \tag{1.4}$$

Owing to Poincare's inequality, the semi-norm $|\cdot|$ is a norm on $\mathbb{H}^1_0(\Omega)$, equivalent to the full norm. As it is directly related gradient operator, we take this semi-norm as norm on $\mathbb{H}^1_0(\Omega)$, and we use it to define the dual norm on its dual space $\mathbb{H}^{-1}(\Omega)$:

$$\text{for all}\ \ \boldsymbol{f} \in \mathbb{H}^{-1}(\Omega),\quad \|\boldsymbol{f}\|_{\mathbb{H}^{-1}(\Omega)} = \sup_{\boldsymbol{v}\in H^1_0(\Omega)} \frac{\langle \boldsymbol{f}, \boldsymbol{v}\rangle}{\|\nabla\boldsymbol{v}\|}$$

where $\langle\cdot\rangle$ is the duality pairing between $\mathbb{H}^{-1}(\Omega)$ and $\mathbb{H}^1_0(\Omega)$ . We also introduce the following spaces

$$
\begin{aligned}
&\mathcal{V} = \left\{\boldsymbol{u} \in (\mathcal{C}^\infty_c(\Omega))^3 : \operatorname{div} \boldsymbol{u} = 0\right\} \ ,\\
&\mathbf{V} = \text{the closure of } \mathcal{V} \text{ in } \mathbb{H}^1_0(\Omega)\, ,\\
&\mathbf{H} = \left\{\boldsymbol{u} \in \mathbb{L}^2(\mathcal{M}) : \operatorname{div} \boldsymbol{u} = 0 \text{ and } \boldsymbol{u} = 0 \text{ on } \Gamma\right\} \ ,\\
&\mathbb{H} = \mathbf{H} \times L^2(\Omega)\, ,\\
&\mathbb{V} = \mathbf{V} \times H^1(\Omega)\, ,\\
&H_1 = \left\{T \in L^2(\Omega) :\ T = T_b \text{ on } \Gamma\right\} \ ,\\
&V_1 = \left\{T \in H^1(\Omega) :\ T = T_b \text{ on } \Gamma\right\} \ .
\end{aligned}
$$

We have (see[20])

$$
\mathbb{V} \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{V}' \tag{1.5}
$$

where the first injection is compact. We endow $\mathbf{H}$ with the inner product of $\mathbb{L}^2(\Omega)$ and the norm of $\mathbb{L}^2(\Omega)$ denoted respectively by $(\cdot,\cdot)_{\mathbf{H}}$ and $|\cdot|_{\mathbf{H}}$.
We equip $\mathbf{V}$ thanks to Poincaré's inequality with the following inner product

$$
((\boldsymbol{u},\boldsymbol{v}))_{\mathbf{V}} = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{\mathbf{H}}\, .
$$

and the norm

$$
\|\boldsymbol{u}\|_{\mathbf{V}} = (\nabla \boldsymbol{u}, \nabla \boldsymbol{u})_{\mathbf{H}}\, . \tag{1.6}
$$

Hereafter, we set

$$
((\boldsymbol{u},T),(\boldsymbol{v},S))_{\mathbb{V}} = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{\mathbf{H}} + (\nabla T, \nabla S) \text{ and } \|(\boldsymbol{u},\boldsymbol{v})\|^2_{\mathbb{V}} = \|\boldsymbol{u}\|^2_{\mathbf{V}} + \|\boldsymbol{v}\|^2\ , \tag{1.7}
$$

where $\|\cdot\|$ denotes the norm in $H^1(\Omega)$ and $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$. The dual spaces of $V$ and $H^m_0(\Omega)$ are denoted by $V'$ and $H^{-m}(\Omega)$ respectively and their norms by $\|\cdot\|_{V'}$ and $\|\cdot\|_{-m}$ respectively. We will also use the following operators $\mathcal{A}$ and $\mathcal{A}_1$ defined from $\mathbf{V}$ to $\mathbf{V}'$ and $V_1$ to $V_1'$ respectively by

$$
\begin{aligned}
&\langle \mathcal{A}\boldsymbol{u}, \boldsymbol{v}\rangle = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{\mathbf{H}} \text{ for all } \boldsymbol{u},\boldsymbol{v} \in \mathbf{V}\, ,\\
&\langle \mathcal{A}_1 T, S\rangle = (\nabla T, \nabla S) \text{ for all } \ T,S \in V_1\, .
\end{aligned}
$$

From the regularity theory for the Stokes equation [20, 21], it is known that

$$
\begin{aligned}
&D(\mathcal{A}) = \mathbb{H}^2(\Omega) \cap \mathbf{V}\, ,\\
&D(\mathcal{A}_1) = H^2(\Omega) \cap V_1\, ,
\end{aligned}
$$

and the following holds true

$$
\begin{aligned}
&D(\mathcal{A}) \subset \mathbf{V} \subset \mathbf{H}\, ,\\
&D(\mathcal{A}_1) \subset V_1 \subset H_1\, ,
\end{aligned} \tag{1.8}
$$

each injection being continuous and compact; hence

$$
|\boldsymbol{u}|_{\mathbf{H}} \le \frac{1}{\sqrt{\lambda}}\, \|\boldsymbol{u}\|_{\mathbf{V}} \text{ for all } \boldsymbol{u} \in \mathbf{V},\ |T| \le \frac{1}{\sqrt{\lambda^1}}\, \|T\| \text{ for all } T \in H^1_0(\Omega) \tag{1.9}
$$

where $\lambda$, $\lambda^1$ are respectively the first eigenvalues of the compact operators $\mathcal{A}^{-1}$ from $\mathbf{H}$ into itself and $\mathcal{A}_1^{-1}$ from $H_1$ into itself. $|.|$ and $\|.\|$ represent respectively the norm in $L^2(\Omega)$ and $H^1(\Omega)$. In addition, the following Agmon type inequality holds (See [21], page 30):

$$\|\boldsymbol{u}\|_{\mathbf{L}^\infty(\Omega)} \le C\,|\nabla \boldsymbol{u}|_{\mathbf{H}}^{1/2}\,|\mathcal{A}\boldsymbol{u}|_{\mathbf{H}}^{1/2}\,. \tag{1.10}$$

Also, of importance in this part are the bilinear forms $\mathcal{B}, \mathcal{B}_N$ from $\mathbf{V}\times\mathbf{V}$ to $\mathbf{V}'$ defined by

$$\begin{aligned}
\langle \mathcal{B}(\boldsymbol{u}_1, \boldsymbol{u}_2), \boldsymbol{u}_3\rangle_{\mathbf{V}',\mathbf{V}} &= b(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)\,,\\
\langle \mathcal{B}_N(\boldsymbol{u}_1, \boldsymbol{u}_2), \boldsymbol{u}_3\rangle_{\mathbf{V}',\mathbf{V}} &= b_N(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)\,,
\end{aligned}$$

for all $u_i \in \mathbf{V}(i = 1, 2, 3)$, where $b(\cdot,\cdot,\cdot)$ is a continuous trilinear form defined on $\mathbb{H}^1(\Omega)\times\mathbb{H}^1(\Omega)\times\mathbb{H}^1(\Omega)$ by

$$\begin{aligned}
b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &= \sum_{i,j=1}^{3}\int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx\,,\\
b_N(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &= F_N(\|\boldsymbol{u}\|_{\mathbf{V}})b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})\,.
\end{aligned}$$

Similarly, we also introduce the bilinear form $\mathcal{B}_1, \mathcal{B}_{N,1}$ from $\mathbf{V}\times V_1$ to $V_1'$ defined by

$$\begin{aligned}
(\mathcal{B}_1(\boldsymbol{u}, T_1), T_2) &= b_1(\boldsymbol{u}, T_1, T_2)\,,\\
(\mathcal{B}_{N,1}(\boldsymbol{u}, T_1), T_2) &= b_{N,1}(\boldsymbol{u}, T_1, T_2)\,,
\end{aligned}$$

for all $\boldsymbol{u} \in \mathbf{V},\ T_1, T_2 \in V_1$, where $b_{N,1}$ is a continuous operator defined on $\mathbf{V}\times H^1(\Omega)\times H^1(\Omega)$ by

$$\begin{aligned}
b_1(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &= \sum_{i=1}^{3}\int_\Omega u_i \frac{\partial v}{\partial x_i} w dx\,,\\
b_{N,1}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &= F_N(\|\boldsymbol{u}\|_{\mathbb{V}})b_1(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})\,.
\end{aligned}$$

We will also use the bilinear forms $a_0(\cdot,\cdot)$ and $a_1(\cdot,\cdot)$ given by:

$$\begin{aligned}
&a_0(\boldsymbol{u}, \boldsymbol{v}) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{\mathbf{H}},\ \boldsymbol{u}, \boldsymbol{v} \in \mathbf{V}\,,\\
&a_1(T, S) = (\nabla T, \nabla S),\ T, S \in H^1(\Omega)\,,\\
&a_2(\boldsymbol{u}, q) = -(q, \operatorname{div}\boldsymbol{u})\quad \boldsymbol{u} \in \mathbf{H}^1(\Omega),\ \ q \in L^2(\Omega)\,.
\end{aligned}$$

From (1.10) and (1.9), we deduce most of the properties of the forms $b(\cdot,\cdot,\cdot)$ and $b_N(\cdot,\cdot,\cdot)$, given in the following lemmas where $C_b$ is a positive constant (depending on the domain) which can vary from one line to another.

**Lemma 1.1** *[4, 14]*

1. $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v})$, *and* $b_N(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) = 0\ \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$,
2. $|b_N(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le C_b\,|\boldsymbol{u}|_{\boldsymbol{H}}^{1/4}\,\|\boldsymbol{u}\|_{\boldsymbol{V}}^{3/4}\,|\boldsymbol{v}|_{\boldsymbol{H}}^{1/4}\,\|\boldsymbol{v}\|_{\boldsymbol{V}}^{3/4}\,\|\boldsymbol{w}\|_{\boldsymbol{V}},\ \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$,
3. $|b_N(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le NC_b\,\|\boldsymbol{u}\|_{\boldsymbol{V}}\,\|\boldsymbol{w}\|_{\boldsymbol{V}},\ \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$,
4. $|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le C_b\,\|\boldsymbol{u}\|_{\boldsymbol{V}}^{1/2}\,|\mathcal{A}\boldsymbol{u}|_{\boldsymbol{H}}^{1/2}\,\|\boldsymbol{v}\|_{\boldsymbol{V}}\,|\boldsymbol{w}|_{\boldsymbol{H}}\,, \forall \boldsymbol{u} \in D(\mathcal{A}),\ \boldsymbol{v} \in \boldsymbol{V},\ \boldsymbol{w} \in \boldsymbol{H}$.
5. $|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \le C_b\,\|\boldsymbol{u}\|_{\boldsymbol{V}}\,\|\boldsymbol{v}\|_{\boldsymbol{V}}\,|\boldsymbol{w}|_{\boldsymbol{H}}^{1/2}\,\|\boldsymbol{w}\|_{\boldsymbol{V}}^{1/2},\ \ \forall \boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w} \in \boldsymbol{V}$.

**Lemma 1.2** *[4, 15] For all* $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$ *with* $\boldsymbol{v} \neq 0$,

1. $|F_N(\|\boldsymbol{u}\|_{\boldsymbol{V}}) - F_N(\|\boldsymbol{v}\|_{\boldsymbol{V}})| \le \frac{\|\boldsymbol{u}-\boldsymbol{v}\|_{V}}{\|\boldsymbol{v}\|_{V}}$.

*2.* $|F_N(\|\boldsymbol{u}\|_{\boldsymbol{V}}) - F_N(\|\boldsymbol{v}\|_{\boldsymbol{V}})| \leq \frac{\|\boldsymbol{u}-\boldsymbol{v}\|_V}{N} F_N(\|\boldsymbol{u}\|_{\boldsymbol{V}}) F_N(\|\boldsymbol{v}\|_{\boldsymbol{V}})$ .

**Lemma 1.3** *[4] For all $M$, $N$, $p$, $r \in \mathbb{R}^+$,*

$$|F_M(p) - F_N(r)| \leq \frac{|M-N|}{r} + \frac{|p-r|}{r} .$$

**Remark 1.1** *Similar properties are satisfied by $b_1(\cdot,\cdot,\cdot)$ and $b_{N1}(\cdot,\cdot,\cdot)$ .*

In the following, we shall use, if necessary, the notation $\phi(t)$ for the function

$$\boldsymbol{x} \to \phi(\boldsymbol{x}, t) .$$

As usual for handling time dependent problems, it is convenient to consider functions defined on a time interval $(a, b)$ with values in a functional space, say $\boldsymbol{Y}$ (see [3]). More precisely, we let $\|\cdot\|_{\boldsymbol{Y}}$ be the norm on $\boldsymbol{Y}$ and for any number $r$ with $1 \leq r \leq \infty$, we define

$$L^r(a, b; \boldsymbol{Y}) = \left\{ w \text{ measurable in } \ (a, b) \ ; \quad \int_a^b \|w(t)\|_{\boldsymbol{Y}}^r dt < \infty \right\}$$

equipped with the norm

$$\|w\|_{L^r(a,b;\boldsymbol{Y})}^r = \int_a^b \|w(t)\|_{\boldsymbol{Y}}^r dt$$

with the usual modification if $r = \infty$. It is Banach space if $\boldsymbol{Y}$ is a Banach space, and when $r = 2$, it is a Hilbert space if $\boldsymbol{Y}$ is also a Hilbert space. Of particular interest here will be the space $L^2(0, \widetilde{T}; \mathbb{H})$, $L^2(0, \widetilde{T}; \mathbb{H}_0^1(\Omega))$, etc...
The analysis of (1.2) will required the following

**Lemma 1.4** *Let $\widetilde{T} > 0$ and let $\kappa$ be a non-negative function in $L^1(0, \widetilde{T})$. Let $c > 0$ be a constant and $\psi \in \mathcal{C}^0(0, \widetilde{T})$ a function that satisfies*

$$\textit{for all } t \in [0, \widetilde{T}] \quad , \quad 0 \leq \psi(t) \leq c + \int_0^t \kappa(s)\psi(s) ds \ ,$$

*then $\psi$ satisfies the bound*

$$\textit{for all } t \in [0, \widetilde{T}] \ , \ \psi(t) \leq c \exp\left( \int_0^t \kappa(s) ds \right) .$$

We will require the following compactness in time result to pass to the limit [3, 13]

**Theorem 1.1** *Let $E, F, G$ be three Banach spaces with continuous imbedding $E \subset F \subset G$, such that the imbedding of $E$ into $F$ being compact. Then for any number $q \in [1, \infty]$, the space*

$$\{v \in L^q(0, \widetilde{T}; E), \ \partial_t v \in L^1(0, \widetilde{T}; G)\}$$

*is compactly imbedded into $L^q(0, \widetilde{T}; F)$.*

In the next paragraph, we propose a weak formulation of our problem.

### 1.3 Variational formulation

We propose here a weak formulation of Problem (1.2) given by the following definition.

**Definition 1.1** *Suppose that* $(\boldsymbol{u}_0, T_0) \in \mathbb{H}$ *and* $\boldsymbol{f} \in L^2(0,\widetilde{T};\mathbb{H}^{-1}(\Omega)), g \in L^2(0,\widetilde{T};H^{-1}(\Omega))$. *A weak solution to (1.2) is any pair* $(\boldsymbol{u},T) \in L^2(0,\widetilde{T};\boldsymbol{V}) \times L^2(0,\widetilde{T};H_1)$ *such that*

$$\begin{cases} \dfrac{d\boldsymbol{u}}{dt} + \nu\mathcal{A}\boldsymbol{u} + \mathcal{B}_N(\boldsymbol{u},\boldsymbol{u}) = \boldsymbol{f} \text{ in } \mathcal{D}(0,\widetilde{T};\boldsymbol{V}'), \\ \dfrac{dT}{dt} + \alpha\mathcal{A}_1 T + \mathcal{B}_{N1}(\boldsymbol{u},T) = g \text{ in } \mathcal{D}(0,\widetilde{T};V_1'), \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \ \ T(\boldsymbol{x},0) = T_0(\boldsymbol{x}) \ \text{ in } \Omega, \\ \boldsymbol{u} = 0 \ \text{ and } \ T = T_b \ \text{ on } \ \Gamma \times (0,\widetilde{T}). \end{cases} \tag{1.11}$$

*or equivalently for all* $(\boldsymbol{v},S) \in \mathbb{H}_0^1(\Omega) \times H_0^1(\Omega)$,

$$\begin{cases} \left\langle \dfrac{d\boldsymbol{u}(t)}{dt}, \boldsymbol{v} \right\rangle + \nu a_0(\boldsymbol{u}(t),\boldsymbol{v}) + b_N(\boldsymbol{u}(t),\boldsymbol{u}(t),\boldsymbol{v}) = \langle \boldsymbol{f}(t), \boldsymbol{v}\rangle, \\ \left\langle \dfrac{dT(t)}{dt}, S \right\rangle + \alpha a_1(T(t),S) + b_{N1}(\boldsymbol{u}(t),T(t),S) = \langle g(t), S\rangle, \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(x), \ \ T(\boldsymbol{x},0) = T_0(\boldsymbol{x}) \text{ in } \Omega, \\ \boldsymbol{u} = 0, \ \text{ and } \ T = T_b \ \text{ on } \ \Gamma \times (0,\widetilde{T}). \end{cases} \tag{1.12}$$

**Remark 1.2** *The previous definition provides also the variational formulation of problem (1.2) which is, due to the density of* $\mathcal{D}(\Omega)$ *in* $L^2(\Omega)$ *and* $H_0^1(\Omega)$ *equivalent to it.*

**Remark 1.3** *If the couple* $(\boldsymbol{u},T)$ *belong to* $L^2(0,\widetilde{T};\boldsymbol{V}) \times L^2(0,\widetilde{T};V_1)$ *and satisfies* $(1.11)_1 - (1.11)_2$, *then* $\left(\dfrac{d\boldsymbol{u}}{dt}, \dfrac{dT}{dt}\right) \in L^2(0,\widetilde{T};\mathbb{V}')$, *and we deduce from [20] that* $(\boldsymbol{u},T) \in \mathcal{C}([0,\widetilde{T};\mathbb{H})$. *In fact,* $\mathcal{B}_N(\boldsymbol{u},\boldsymbol{u}) \in L^2(0,\widetilde{T};\boldsymbol{V}')$, $\nu\mathcal{A}\boldsymbol{u} \in L^2(0,\widetilde{T};\boldsymbol{H}) \subset L^2(0,\widetilde{T};\boldsymbol{V}')$, $\boldsymbol{f} \in L^2(0,\widetilde{T};\mathbb{H}^{-1}) \subset L^2(0,\widetilde{T};\boldsymbol{V}')$ *and* $\mathcal{B}_{N1}(\boldsymbol{u},T) \in L^2(0,\widetilde{T};V_1')$, $\alpha\mathcal{A}_1 T \in L^2(0,\widetilde{T};H_1) \subset L^2(0,\widetilde{T};V_1')$, $g \in L^2(0,\widetilde{T};H_1^{-1}) \subset L^2(0,\widetilde{T};V_1')$.

Before stating the existence result, it is clear that we need to take care of the boundary condition involving the temperature. For that purpose, invoking to the trace's result, we let $\mathcal{R}$ be a continuous operator from $H^{1/2}(\Omega)$ in to $H^1(\Omega)$. Since $T_b \in L^2(0,\widetilde{T};H^{1/2}(\Gamma))$ we denote by $\bar{T}_b$ the function defined for *a.e.* $0 \le t \le \widetilde{T}$ by

$$\bar{T}_b(t) = \mathcal{R}T_b(t).$$

This function belongs to $L^2(0,\tilde{T};H^1(\Omega))$ and satisfies

$$\begin{aligned} \|\bar{T}_b\|_{L^2(0,\widetilde{T};H^1(\Omega))} &\le c_\Lambda \|T_b\|_{L^2(0,\widetilde{T};H^{1/2}(\Gamma))}, \\ \|\bar{T}_b\|_{L^2(0,\widetilde{T};L^4(\Omega))} &\le \epsilon \|T_b\|_{L^2(0,\widetilde{T};H^{1/2}(\Gamma))}. \end{aligned} \tag{1.13}$$

where $\epsilon > 0$ is any reel number and $c_\Lambda$ is a positive constant depending only on $\Omega$ and $\mathcal{R}$. When setting $T^* = T - \bar{T}_b$, the new variational formulation of problem (1.2) is as follows:
we seek for $(u,T^*) \in L^2(0,\widetilde{T};\mathbf{V}) \times L^2(0,\widetilde{T};H_0^1(\Omega))$ such that

$$
\begin{cases}
\dfrac{d\,\boldsymbol{u}}{dt} + \nu \mathcal{A}\boldsymbol{u} + \mathcal{B}_N(\boldsymbol{u},\boldsymbol{u}) = \boldsymbol{f}\,, \text{ in } \mathcal{D}(0,\widetilde{T};\mathbf{V}')\,,\\
\dfrac{d\,T^*}{dt} + \alpha\mathcal{A}_1 T^* + F_N(\|(\boldsymbol{u},T^*+\bar{T}_b)\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{u},T^*) = g - \dfrac{d\,\bar{T}_b}{dt}\\
-\alpha\mathcal{A}_1\bar{T}_b - F_N(\|(\boldsymbol{u},T^*+\bar{T}_b)\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{u},\bar{T}_b) \text{ in } \mathcal{D}(0,\widetilde{T};V_1')\,,\\
\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}),\;\; T^*(\boldsymbol{x},0) = T_0(\boldsymbol{x}) - \bar{T}_b(\boldsymbol{x},0) \text{ in } \Omega\,,\\
\boldsymbol{u} = 0, \quad \text{and} \quad T^* = 0 \quad \text{on} \quad \Gamma\times(0,\widetilde{T})\,.
\end{cases}
\tag{1.14}
$$

or equivalently for all $(\boldsymbol{v},S)\in\mathbf{V}\times H_0^1(\Omega)$,

$$
\begin{cases}
\left\langle \dfrac{d\,\boldsymbol{u}(t)}{dt},\boldsymbol{v}\right\rangle + \nu a_0(\boldsymbol{u}(t),\boldsymbol{v}) + b_N(\boldsymbol{u}(t),\boldsymbol{u}(t),\boldsymbol{v}) = \langle \boldsymbol{f}(t),\boldsymbol{v}\rangle\,,\\
\left\langle \dfrac{d\,T^*(t)}{dt},S\right\rangle + \alpha a_1(T^*(t),S) + F_N(\|(\boldsymbol{u},T^*+\bar{T}_b)\|_{\mathbb{V}}) b_1(\boldsymbol{u}(t),T^*(t),S)\\
= \langle g(t),S\rangle - \left(\dfrac{d\,\bar{T}_b}{dt},S\right) - \alpha a_1(\widetilde{T}_b,S) - F_N(\|(\boldsymbol{u},T^*+\bar{T}_b)\|_{\mathbb{V}}) b_1(\boldsymbol{u}(t),\bar{T}_b(t),S),\\
\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}),\;\; T^*(x,0) = T_0(\boldsymbol{x}) - \bar{T}_b(\boldsymbol{x},0) \text{ in } \Omega,\\
\boldsymbol{u} = 0, \quad \text{and} \quad T^* = 0 \quad \text{on} \quad \Gamma\times(0,\widetilde{T})\,.
\end{cases}
\tag{1.15}
$$

# 2 Existence theory and qualitative properties of the solution

## 2.1 Existence and uniqueness

Here, we prove that problem (1.12) has a unique weak solution which is, under some conditions a strong one. In this section, we construct solutions by combining; Galerkin's scheme, a priori estimates and compactness results. The method of proof is classical (see [13]), but it is worth mentioning that the nonlinearity involved here are particular. We give details proofs so as to render our work self contained. As mentioned earlier, we would like to see how the added terms can control de velocity gradient and help us to obtain uniqueness in 3d. We begin this journey by showing that the weak solutions are properly defined (see theorem 2.1), next we show that the solution is uniquely defined (see theorem 2.2).

**Theorem 2.1** *Suppose that $\boldsymbol{f}\in L^2(0,\widetilde{T};\mathbb{H}^{-1}(\Omega))$, $g\in L^2(0,\widetilde{T};H^{-1}(\Omega))$, $T_b\in H^1(0,\widetilde{T};H^{1/2}(\Gamma))$, the initial temperature on the boundary $T_b^0$ belongs to $H^{1/2}(\Gamma)$ and $(\boldsymbol{u}_0,T_0,)\in\mathbb{H}$ be given. There exists a weak solution $y=(\boldsymbol{u},T)$ of (1.12), which is in fact a strong solution if $(\boldsymbol{u}_0,T_0)\in\mathbb{V}$, in the sense that*

$$
y\in\mathcal{C}(0,\widetilde{T};\mathbb{V})\cap L^2(0,\widetilde{T};D(\mathcal{A})\times D(\mathcal{A}_1)). \tag{2.1}
$$

**proof.** It is done on several steps:

<u>*Step*1</u>: *Faedo Galerkin Approximation.*
Let $\overline{\{(\phi_i,\psi_i), i=1,2,...\}}\subset\mathbb{V}$ be an orthonormal basis of $\mathbb{H}$, where $\{\phi_i, i=1,2,....\}$, $\{\psi_i, i=1,2,....\}$ are eigenvectors of $\mathcal{A}$ and $\mathcal{A}_1$ respectively. We set $V_n\times W_n=\text{span}\,\{(\phi_1,\psi_1),...,(\phi_n,\psi_n)\}$ and denote by $P_n=(P_n^1,P_n^2)$, the orthogonal projector from $\mathbb{H}$ onto $V_n\times W_n$ for the scalar product $(\cdot,\cdot)_{\mathbb{H}}$ defined before. Note that $P_n$ is also the orthogonal projector from $D(\mathcal{A}\times\mathcal{A}_1),\mathbb{V},\mathbb{V}'$ onto $V_n\times W_n$. In $V_n\times W_n$, a smooth Galerkin's approximation of problem (1.14) is as follows:

we look for $(\boldsymbol{u}_n, T_n^*) = \left(\sum_{i=1}^{n} \boldsymbol{u}_{ni}\phi_i, \sum_{i=1}^{n} T_{ni}\psi_i\right) \in L^2(0,\widetilde{T}; V_n) \times L^2(0,\widetilde{T}; W_n)$ such that

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{u}_n + \nu\mathcal{A}\boldsymbol{u}_n + \mathcal{B}_N(\boldsymbol{u}_n,\boldsymbol{u}_n) = P_n^1\boldsymbol{f} \ \text{ in } \mathcal{D}(0,\widetilde{T}; V_n')\,, \\ \dfrac{d}{dt}T_n^* + \alpha\mathcal{A}_1T_n^* + F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{u}_n, T_n^*) = P_n^2 g - \\ \dfrac{d}{dt}\bar{T}_b - \alpha\mathcal{A}_1\bar{T}_b - F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{u}_n,\bar{T}_b) \text{ in } \mathcal{D}(0,\widetilde{T}; W_n')\,, \\ \boldsymbol{u}_n(x,0) = P_n^1 u_0(x), \ \ T_n^*(x,0) = P_n^2T_0(x) - P_n^2\bar{T}_b(x,0) \text{ in } \Omega\,, \\ \boldsymbol{u}_n = 0, \ \ \text{and} \ \ T_n^* = 0 \quad \text{on} \ \ \Gamma\times(0,\widetilde{T})\,. \end{cases} \tag{2.2}$$

or equivalently for all $(\boldsymbol{v}, S) \in V_n \times W_n$,

$$\begin{cases} \left\langle \dfrac{d}{dt}\boldsymbol{u}_n(t), \boldsymbol{v}\right\rangle = \left\langle P_n^1\boldsymbol{f}(t), \boldsymbol{v}\right\rangle - \nu(\mathcal{A}\boldsymbol{u}_n(t), \boldsymbol{v}) - b_N(\boldsymbol{u}_n(t), \boldsymbol{u}_n(t), \boldsymbol{v})\,, \\ \left\langle \dfrac{d}{dt}T_n^*(t), S\right\rangle = \left\langle P_n^2 g(t), S\right\rangle - \alpha(\mathcal{A}_1T_n^*(t), S) - F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}})b_1(\boldsymbol{u}_n(t), T_n^*(t), S) - \\ \left(\dfrac{d}{dt}\bar{T}_b, S\right) - \alpha(\mathcal{A}_1\bar{T}_b(t), S) - F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}})b_1(\boldsymbol{u}_n(t), \bar{T}_b(t), S)\,, \\ \boldsymbol{u}_n(x,0) = P_n^1u_0(x), \ \ T_n^*(x,0) = P_n^2T_0(x) - P_n^2\bar{T}_b(x,0) \quad \text{in } \Omega\,, \\ \boldsymbol{u}_n = 0, \ \ \text{and} \ \ T_n^* = 0 \quad \text{on} \ \ \Gamma\times(0,\widetilde{T})\,, \end{cases} \tag{2.3}$$

where $\boldsymbol{u}_{ni}(t)$, $T_{ni}(t)$ are $\mathcal{C}^1$ functions, $\left\langle P_n^1\boldsymbol{f}(t), \boldsymbol{u}\right\rangle = \langle \boldsymbol{f}(t), \boldsymbol{u}_n\rangle$ and $\left\langle P_n^2 g(t), T\right\rangle = \langle g(t), T_n\rangle$ for $(\boldsymbol{u}, T) \in \mathbb{V}$. (2.3) is a Cauchy problem and the mapping

$$(\boldsymbol{v}, s) \longrightarrow \begin{pmatrix} P_n^1\boldsymbol{f} + \nu\mathcal{A}\boldsymbol{v} - \mathcal{B}_N(\boldsymbol{v},\boldsymbol{v}) \\ P_n^2 g - \dfrac{d}{dt}\overline{T}_b - \alpha\mathcal{A}_1\overline{T}_b - F_N(\left\|(\boldsymbol{v}, s+\bar{T}_b)\right\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{v},\overline{T}_b) - \\ F_N(\left\|(\boldsymbol{v}, s+\bar{T}_b)\right\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{v}, s) - \alpha\mathcal{A}_1 s \end{pmatrix}$$

is locally Lipschitz-continuous on $\mathbb{H}^1(\Omega) \times H^1(\Omega)$ (see **Appendix**).
It follows from the Cauchy-Lipschitz theorem that problem (2.2) has a unique solution $(\boldsymbol{u}_n, T_n^*) \in \mathcal{C}(0,\widetilde{T}_n; V_n) \times \mathcal{C}(0,\widetilde{T}_n; W_n)$ for some $\widetilde{T}_n \leq \widetilde{T}$ and the problem is to show that $\widetilde{T}_n$ is in fact independent of time. The following a priori estimates (see lemma 2.1 and lemma 2.2) on $\boldsymbol{u}_n$ and $T_n^*$, will be enough to conclude that $\widetilde{T}_n = \widetilde{T}$.

We next want to construct the limit of $(\boldsymbol{u}_n, T_n^*)$ given via the equations (2.3), and we hope that the limit will solve (1.15). For that purpose, we next derive some a priori estimates and next we use compactness results to pass to the limit in (2.3).

<u>*Step*2</u>: *A priori estimates and passage to the limit*

**Lemma 2.1** *The functions $\boldsymbol{u}_n$ and $T_n^*$ are uniformly bounded on $L^2(0,\widetilde{T}; \boldsymbol{V}) \cap L^\infty(0,\widetilde{T}; \boldsymbol{H})$ and $L^2(0,\widetilde{T}; H_0^1(\Omega)) \cap L^\infty(0,\widetilde{T}; L^2(\Omega))$ respectively.*

**proof** Taking $v = \boldsymbol{u}_n(t)$ in $(2.3)_1$ and $S = T_n(t)$ in $(2.3)_2$ and using Lemma 1.1, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|\boldsymbol{u}_n(t)\right|^2_{\mathbf{H}} + \nu\left|\nabla \boldsymbol{u}_n(t)\right|^2_{\mathbf{H}} \leq c_1\left\|f(t)\right\|_{\mathbf{V}'}\left\|\boldsymbol{u}_n(t)\right\|_{\mathbf{V}}\,,\\
&\frac{1}{2}\frac{d}{dt}\left|T_n^*(t)\right|^2 + \alpha\left|\nabla T_n^*(t)\right|^2 \leq c_2\left\|g(t)\right\|_{-1}\left\|T_n^*(t)\right\| + c_3\left\|\frac{d}{dt}\bar{T}_b(t)\right\|\left\|T_n^*(t)\right\| +\\
&\alpha c_4\left|\nabla \bar{T}_b(t)\right|\left|\nabla T_n^*(t)\right| + \frac{N}{\left\|(\boldsymbol{u}_n(t),\bar{T}_b(t))\right\|_{\mathbb{V}}}c_5\left\|\boldsymbol{u}_n(t)\right\|_{\mathbf{V}}\left\|\bar{T}_b(t)\right\|\left\|T_n^*(t)\right\|\,.
\end{aligned} \tag{2.4}$$

Which leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|\boldsymbol{u}_n(t)\right|^2_{\mathbf{H}} + \nu\left\|\boldsymbol{u}_n(t)\right\|^2_{\mathbf{V}} \leq \frac{c_1^2}{2\nu}\left\|f(t)\right\|^2_{\mathbf{V}'} + \frac{\nu}{2}\left\|\boldsymbol{u}_n(t)\right\|^2_{\mathbf{V}}\,,\\
&\frac{1}{2}\frac{d}{dt}\left|T_n^*(t)\right|^2 + \alpha\left\|T_n^*(t)\right\|^2 \leq \frac{c_2^2}{\alpha}\left\|g(t)\right\|^2_{-1} + \frac{\alpha}{8}\left\|T_n^*(t)\right\|^2 + \frac{2c_3^2}{\alpha}\left\|\frac{d}{dt}\bar{T}_b(t)\right\|^2 + \frac{\alpha}{8}\left\|T_n^*(t)\right\|^2\\
&2c_4^2\left\|\bar{T}_b(t)\right\|^2 + \frac{\alpha}{8}\left\|T_n^*(t)\right\|^2 + \frac{2N^2c_5^2}{\alpha}\left\|\bar{T}_b(t)\right\|^2 + \frac{\alpha}{8}\left\|T_n^*(t)\right\|^2\,.
\end{aligned} \tag{2.5}$$

Hence

$$\begin{aligned}
&\left|\boldsymbol{u}_n(t)\right|^2_{\mathbf{H}} + \nu\int_0^{\widetilde{T}}\left\|\boldsymbol{u}_n(t)\right\|^2_{\mathbf{V}}dt \leq \frac{c_1^2}{\nu}\int_0^{\widetilde{T}}\left\|f(t)\right\|^2_{\mathbf{V}'}dt + \left|\boldsymbol{u}_0\right|^2_{\mathbf{H}}\,,\\
&\left|T_n^*(t)\right|^2 + \alpha\int_0^{\widetilde{T}}\left\|T_n^*(t)\right\|^2 dt \leq \frac{2c_2^2}{\alpha}\int_0^{\widetilde{T}}\left\|g(t)\right\|^2_{-1}dt + \frac{4c_3^2}{\alpha}\int_0^{\widetilde{T}}\left\|\frac{d}{dt}\bar{T}_b(t)\right\|^2 dt +\\
&\left(4c_4^2 + \frac{4N^2c_5^2}{\alpha}\right)\int_0^{\widetilde{T}}\left\|\bar{T}_b(t)\right\|^2 dt + \left|T_0^*\right|^2\,.
\end{aligned} \tag{2.6}$$

□

From (2.6), we infer that $\left\|\boldsymbol{u}_n\right\|_{L^\infty(0,\widetilde{T};\mathbf{H})}$, $\left\|\boldsymbol{u}_n\right\|_{L^2(0,\widetilde{T};\mathbf{V})}$, $\left\|T_n\right\|_{L^\infty(0,\widetilde{T};L^2(\Omega))}$, $\left\|T_n\right\|_{L^2(0,\widetilde{T};H_0^1(\Omega))}$ are uniformly bounded independently of $n$. Then, considering also (1.5), we use Theorem 1.1 to extract a subsequence of $(\boldsymbol{u}_n, T_n^*)$ denoted again by $(\boldsymbol{u}_n, T_n^*)$ satisfying

$$(\boldsymbol{u}_n, T_n^*) \to (\boldsymbol{u}, T^*)\begin{cases}\text{weak-star in } L^\infty(0,T;\mathbb{H})\,,\\ \text{weakly in } L^2(0,T;\mathbf{V}\times H_0^1(\Omega))\,,\\ \text{strongly in } L^2(0,T;\mathbb{H})\,,\\ \text{a.e., in } (0,T)\times\Omega\,,\end{cases} \tag{2.7}$$

with $(\boldsymbol{u}, T^*) \in L^\infty(0,T;\mathbb{H}) \cap L^2(0,T;\mathbf{V}\times H_0^1(\Omega))$.

With the weak convergence (2.7), we can pass to the limit in the linear terms in (2.3), meaning that as $n \to \infty$,

$$\begin{aligned}
&\left\langle \frac{d}{dt}\boldsymbol{u}_n(t), \boldsymbol{v}\right\rangle \to \left\langle \frac{d}{dt}\boldsymbol{u}(t), \boldsymbol{v}\right\rangle\,,\\
&(\mathcal{A}\boldsymbol{u}_n(t), \boldsymbol{v}) \to (\mathcal{A}\boldsymbol{u}(t), \boldsymbol{v})\,,\\
&\left\langle \frac{d}{dt}T_n^*(t), S\right\rangle \to \left\langle \frac{d}{dt}T^*(t), S\right\rangle\,,\\
&(\mathcal{A}_1 T_n^*(t), S) \to (\mathcal{A}_1 T^*(t), S)\,.
\end{aligned} \tag{2.8}$$

Now, it remains to deal with terms involving nonlinearities and projections in (2.3). Starting with nonlinear terms, it is worth noticing that the weak convergence in $L^2(0,T;\mathbf{V}\times H_0^1(\Omega))$ is not enough to ensure that

$$
\begin{aligned}
&F_N(\|\boldsymbol{u}_n\|_{\mathbf{V}}) \to F_N(\|\boldsymbol{u}\|_{\mathbf{V}}) \text{ as } n\to\infty\,,\\
&F_N(\|(\boldsymbol{u}_n,T_n^*)\|_{\mathbb{V}}) \to F_N(\|(\boldsymbol{u},T^*)\|_{\mathbb{V}}) \text{ as } n\to\infty\,.
\end{aligned}
$$

Hence we need to derive stronger a priori estimates and this is the goal of our next result.

**Lemma 2.2** *The functions $\boldsymbol{u}_n$ and $T_n^*$ are uniformly bounded on $L^\infty(0,\widetilde{T};\boldsymbol{V})\cap L^2(0,\widetilde{T};\mathcal{D}(\mathcal{A}))$ and $L^\infty(0,\widetilde{T};H_0^1(\Omega))\cap L^2(0,\widetilde{T};\mathcal{D}(\mathcal{A}_1))$ respectively.*

**proof.** Taking the inner product of $(2.2)_1$ by $\mathcal{A}\boldsymbol{u}_n(t)$ and the inner product of $(2.2)_2$ by $\mathcal{A}_1T_n^*(t)$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_n(t)\|_{\mathbf{V}}^2+\nu\,|\mathcal{A}\boldsymbol{u}_n(t)|_{\mathbf{H}}^2 \le \langle f(t),\mathcal{A}\boldsymbol{u}_n(t)\rangle - F_N(\|\boldsymbol{u}_n\|_{\mathbf{V}})b(\boldsymbol{u}_n,\boldsymbol{u}_n,\mathcal{A}\boldsymbol{u}_n),\\
&\frac{1}{2}\frac{d}{dt}\|T_n^*(t)\|^2+\alpha\,|\mathcal{A}_1T_n^*(t)|^2 \le \langle g(t),\mathcal{A}_1T_n^*(t)\rangle_\Omega - (\frac{d}{dt}\bar{T}_b(t),\mathcal{A}_1T_n^*(t)) -\\
&\alpha(\nabla\bar{T}_b(t),\mathcal{A}_1T_n^*(t)) - F_N(\left\|(u_n(t),\bar{T}_b(t)+T_n^*(t))\right\|_{\mathbb{V}})b_1(u_n(t),\bar{T}_b(t),\mathcal{A}_1T_n^*(t))\\
&-F_N(\left\|(\boldsymbol{u}_n(t),\bar{T}_b(t)+T_n^*(t))\right\|_{\mathbb{V}})b_1(\boldsymbol{u}_n(t),T_n^*(t),\mathcal{A}_1T_n^*(t))\,.
\end{aligned}
\tag{2.9}
$$

Using Lemma 1.1 and Young's inequality, we have the following estimates:

$$
|-F_N(\|\boldsymbol{u}_n\|_{\mathbf{V}})b(\boldsymbol{u}_n,\boldsymbol{u}_n,\mathcal{A}\boldsymbol{u}_n)| \le \frac{3(c_bN)^4}{4\nu}\|\boldsymbol{u}_n\|_{\mathbf{V}}^2+\frac{\nu}{4}|\mathcal{A}\boldsymbol{u}_n(t)|_{\mathbf{H}}^2 \tag{2.10}
$$

$$
|\langle f(t),\mathcal{A}\boldsymbol{u}_n(t)\rangle_\Omega| \le \frac{c^2}{\nu}\|f(t)\|_{\mathbf{V}'}^2+\frac{\nu}{4}|\mathcal{A}\boldsymbol{u}_n(t)|_{\mathbf{H}}^2 \tag{2.11}
$$

$$
|\langle g(t),\mathcal{A}_1T_n^*(t)\rangle_\Omega| \le \frac{4c_1^2}{\alpha}\|g(t)\|_{-1}^2+\frac{\alpha}{16}|\mathcal{A}_1T_n^*(t)|^2 \tag{2.12}
$$

$$
\left|-(\frac{d}{dt}\bar{T}_b(t),\mathcal{A}_1T_n^*(t))\right| \le \frac{4c_2^2}{\alpha}\left\|d_t\bar{T}_b(t)\right\|^2+\frac{\alpha}{16}|\mathcal{A}_1T_n^*(t)|^2 \tag{2.13}
$$

$$
\left|-\alpha(\nabla\bar{T}_b(t),\mathcal{A}_1T_n^*(t))\right| \le \frac{4c_3^2}{\alpha}\left|\nabla\bar{T}_b(t)\right|^2+\frac{\alpha}{16}|\mathcal{A}_1T_n^*(t)|^2 \tag{2.14}
$$

$$
\left|-F_N(\left\|(\boldsymbol{u}_n(t),\bar{T}_b(t)+T_n^*(t))\right\|_{\mathbb{V}})b_1(\boldsymbol{u}_n(t),\bar{T}_b(t),\mathcal{A}_1T_n^*(t))\right| \le \frac{3(c_bN)^4}{\alpha}\|\boldsymbol{u}_n(t)\|_{\mathbf{V}}^2+\frac{\alpha}{16}|\mathcal{A}_1T_n^*(t)|^2 \tag{2.15}
$$

$$
\left|-F_N(\left\|(\boldsymbol{u}_n(t),\bar{T}_b(t)+T_n^*(t))\right\|_{\mathbb{V}})b_1(\boldsymbol{u}_n(t),T_n^*(t),\mathcal{A}_1T_n^*(t))\right| \le \frac{3(c_bN)^4}{4\alpha}\|\boldsymbol{u}_n(t)\|_{\mathbf{V}}^2+\frac{\alpha}{4}|\mathcal{A}_1T_n^*(t)|^2\,. \tag{2.16}
$$

Then using (2.10)- (2.16) in (2.9), we have

$$
\begin{aligned}
&\frac{d}{dt}\|\boldsymbol{u}_n(t)\|_{\mathbf{V}}^2+\nu\,|\mathcal{A}\boldsymbol{u}_n(t)|_{\mathbf{H}}^2 \le \frac{2c^2}{\nu}\|f(t)\|_{\mathbf{V}'}^2+\frac{3(c_bN)^4}{2\nu}\|\boldsymbol{u}_n(t)\|_{\mathbf{V}}^2\,,\\
&\text{and}\\
&\frac{d}{dt}\|T_n^*(t)\|^2+\alpha\,|\mathcal{A}_1T_n^*(t)|^2 \le \frac{8c_1^2}{\alpha}\|g(t)\|_{-1}^2+\frac{8c_2^2}{\alpha}\left\|d_t\bar{T}_b(t)\right\|^2+\\
&\frac{8c_3^2}{\alpha}\left\|\bar{T}_b(t)\right\|^2+\frac{15(Nc_b)^4}{4\alpha}\|\boldsymbol{u}_n(t)\|_{\mathbf{V}}^2\,.
\end{aligned}
\tag{2.17}
$$

Hence

$$\|\boldsymbol{u}_n(t)\|_{\mathbf{V}}^2 + \nu \int_0^{\widetilde{T}} |\mathcal{A}\underline{\mathrm{u}}_n(t)|_{\mathbf{H}}^2 \, dt \leq \frac{2c^2}{\nu} \int_0^{\widetilde{T}} \|f(t)\|_{\mathbf{V}'}^2 \, dt + \frac{3(c_b N)^4}{2\nu} \int_0^{\widetilde{T}} \|\boldsymbol{u}_n(t)\|_{\mathbf{V}}^2 \, dt + \|\boldsymbol{u}_0\|_{\mathbf{H}}^2 \ ,$$

and

$$\|T_n^*(t)\|^2 + \alpha \int_0^{\widetilde{T}} |\mathcal{A}_1 T_n^*(t)|^2 \, dt \leq \frac{8c_1^2}{\alpha} \int_0^{\widetilde{T}} \|g(t)\|_{-1}^2 \, dt + \frac{8c_2^2}{\alpha} \int_0^{\widetilde{T}} \left\| \frac{d}{dt}\bar{T}_b(t) \right\|^2 dt +$$
$$\frac{8c_3^2}{\alpha} \int_0^{\widetilde{T}} \left\|\bar{T}_b(t)\right\|^2 dt + \frac{15(Nc_b)^4}{4\alpha} \int_0^{\widetilde{T}} \|\boldsymbol{u}_n(t)\|_{\mathbf{V}}^2 \, dt + \|T_0^*\|^2 \ . \tag{2.18}$$

and the lemma is proved. □

From (2.2), we have

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{u}_n(t) = -\nu\mathcal{A}\boldsymbol{u}_n(t) - \mathcal{B}_N(\boldsymbol{u}_n(t), \boldsymbol{u}_n(t)) + P_n^1 f(t) \, , \\ \dfrac{d}{dt}T_n^*(t) = -\alpha\mathcal{A}_1 T_n^*(t) - F_N(\|(\boldsymbol{u}_n, T_n^* + \bar{T}_b)\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{u}_n(t), T_n^*(t)) + P_n^2 g(t) - \\ \dfrac{d}{dt}\bar{T}_b - \alpha\mathcal{A}_1\bar{T}_b(t) - F_N(\|(\boldsymbol{u}_n, T_n^* + \bar{T}_b)\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{u}_n(t), \bar{T}_b(t)) \, , \end{cases} \tag{2.19}$$

It follows from (2.19) and Lemma 1.1 that $\left(\dfrac{d}{dt}\boldsymbol{u}_n, \dfrac{d}{dt}T_n^*\right)$ is also bounded in $L^2(0, \widetilde{T}, \mathbb{H})$. Using Lemma 2.2, (1.8), and the compactness result (Theorem 1.1), there exists an element $(\boldsymbol{u}, T^*) \in L^2(0, \widetilde{T}; \mathbf{V} \times H_0^1(\Omega)) \cap L^2(0, \tilde{T}; \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}_1))$ and a subsequence of $(\boldsymbol{u}_n, T_n^*)$ denoted again by $(\boldsymbol{u}_n, T_n^*)$ satisfying

$$(\boldsymbol{u}_n, T_n^*) \to (\boldsymbol{u}, T^*) \begin{cases} \text{weak-star in } L^\infty(0, \tilde{T}; \mathbf{V} \times H_0^1(\Omega)), \\ \text{weakly in } L^2(0, \widetilde{T}; \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}_1)), \\ \text{strongly in } L^2(0, \widetilde{T}; \mathbf{V} \times H_0^1(\Omega)), \\ \text{a.e., in } (0, T) \times \Omega, \end{cases} \tag{2.20}$$

and

$$\left(\frac{d}{dt}\boldsymbol{u}_n, \frac{d}{dt}T_n^*\right) \to \left(\frac{d}{dt}\boldsymbol{u}, \frac{d}{dt}T^*\right) \text{ weakly in } L^2(0, \widetilde{T}; \mathbf{H} \times H_0^1(\Omega)) \, . \tag{2.21}$$

From (2.20), we infer that

$$\begin{aligned} &F_N(\|\boldsymbol{u}_n\|_{\mathbf{V}}) \to F_N(\|\boldsymbol{u}\|_{\mathbf{V}}) \text{ as } n \to \infty \, , \\ &F_N(\|(\boldsymbol{u}_n, T_n^* + \bar{T}_b)\|_{\mathbb{V}}) \to F_N(\|(\boldsymbol{u}, T^* + \bar{T}_b)\|_{\mathbb{V}}) \text{ as } n \to \infty \, . \end{aligned} \tag{2.22}$$

Our next task is to compute the limit when $n$ goes to infinity of $F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}})b_1(\boldsymbol{u}_n, T_n^*, S)$ and $F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}})b_1(\boldsymbol{u}_n, \bar{T}_b, S)$ for $(\boldsymbol{v}, S) \in V_n \times W_n$. Firstly,

$$
\begin{aligned}
F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}})b_1(\boldsymbol{u}_n, T_n^*, S) &= F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}}) \int_\Omega (\boldsymbol{u}_n(t)\cdot\nabla)T_n^*(t) S dx \\
&= F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}}) \sum_{i,j=1}^{3} \int_\Omega \boldsymbol{u}_n^i(t)\partial_i T_n^{*j}(t) S^j dx \\
&= \sum_{i,j=1}^{3} \int_\Omega F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}}) \boldsymbol{u}_n^{mi}(t)\partial_i T_n^{*j}(t) S^j dx \\
&= -\sum_{i,j=1}^{3} \int_\Omega F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}}) \boldsymbol{u}_n^{i}(t) T_n^{*j}(t) \partial_i S^j dx\,.
\end{aligned}
$$

But

$$
\begin{aligned}
\left\|\boldsymbol{u}_n^i(t)T_n^{*j}(t)\right\|_{\mathbb{L}^{3/2}(\Omega)} &= \left(\int_\Omega \left|\boldsymbol{u}_n^i(t)T_n^{*j}(t)\right|^{3/2} dx\right)^{2/3} \\
&\le \left(\int_\Omega \left|\boldsymbol{u}_n^i(t)\right|^{3/2\times 4/3} dx\right)^{2/3\times 3/4} \left(\int_\Omega \left|T_n^{*j}(t)\right|^{3/2\times 4} dx\right)^{2/3\times 1/4} \\
&= \left(\int_\Omega \left|\boldsymbol{u}_n^i(t)\right|^{2} dx\right)^{1/2} \left(\int_\Omega \left|T_n^{*j}(t)\right|^{6} dx\right)^{1/6} \\
&= \left|\boldsymbol{u}_n^i(t)\right|_{\mathbf{H}} \left\|T_n^{*j}(t)\right\|_{\mathbf{L}^6(\Omega)} \\
&\le C\left|\boldsymbol{u}_n(t)\right|_{\mathbf{H}} \|T_n^*(t)\|\,.
\end{aligned}
$$

Hence, $\boldsymbol{u}_n^i T_n^{*j}$ is bounded in $L^2(0,T;\mathbb{L}^{3/2}(\Omega))$. Next, we note that

$$
\begin{aligned}
&0 < F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}}) \le 1\,, \text{and} \\
&F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}})\boldsymbol{u}_n^i T_n^{*j} \text{ is still bounded in } L^2(0,T;\mathbb{L}^{3/2}(\Omega)) \subset L^{3/2}(0,T;\mathbb{L}^{3/2}(\Omega))\,.
\end{aligned}
$$

Thus there exists $\chi_{ij} \in L^{3/2}(0,T;\mathbb{L}^{3/2}(\Omega))$ such that

$$
F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}})\boldsymbol{u}_n^i T_n^{*j} \to \chi_{ij} \textit{ in } L^{3/2}(0,T;\mathbb{L}^{3/2}(\Omega)) - \text{weak}\,. \tag{2.23}
$$

In addition, relation (2.22) implies

$$
F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}})\boldsymbol{u}_n^i T_n^{*j} \to F_N(\|(\boldsymbol{u}, T^*+\bar{T}_b)\|_{\mathbb{V}})\boldsymbol{u}^i T^{*j} \textit{ a.e. in } (0,T)\times\Omega\,. \tag{2.24}
$$

Then we apply Lemma 1.3 in [13] to conclude from (2.23) and (2.24) that

$$
F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}})\boldsymbol{u}_n^i T_n^{*j} \to F_N(\|(\boldsymbol{u}, T^*+\bar{T}_b)\|_{\mathbb{V}})\boldsymbol{u}^i T^{*j} \text{ in } L^{3/2}(0,T;\mathbb{L}^{3/2}(\Omega)) \text{ weak}\,,
$$

which implies the following convergence result

$$
-\sum_{i,j=1}^{3} \int_\Omega F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}})\boldsymbol{u}_n^i(t) T_n^{*j}(t)\partial_i S_j dx \to -\sum_{i,j=1}^{3} \int_\Omega F_N(\|(\boldsymbol{u}, T^*+\bar{T}_b)\|_{\mathbb{V}})\boldsymbol{u}^i(t) T^{*j}(t)\partial_i S_j dx\,.
$$

Hence,

$$
F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}})b_1(\boldsymbol{u}_n, T_n^*, S) \to F_N(\|(\boldsymbol{u}_n, T_n^*+\bar{T}_b)\|_{\mathbb{V}})b_1(\boldsymbol{u}, T^*, S), \quad S \in W_n\,. \tag{2.25}
$$

Secondly,

$$\begin{aligned}F_N(\|(\boldsymbol{u}_n,T_n^*+\bar{T}_b)\|_{\mathbb{V}})b_1(\boldsymbol{u}_n,\bar{T}_b,S)=&F_N(\|(\boldsymbol{u}_n,T_n^*+\bar{T}_b)\|_{\mathbb{V}})\int_\Omega(\boldsymbol{u}_n(t).\nabla)\bar{T}_b(t)Sdx\\ =&F_N(\|(\boldsymbol{u}_n,T_n^*+\bar{T}_b)\|_{\mathbb{V}})\sum_{i,j=1}^3\int_\Omega \boldsymbol{u}_n^i(t)\partial_i\bar{T}_b^j(t)S^jdx\\ =&\sum_{i,j=1}^3\int_\Omega F_N(\|(\boldsymbol{u}_n,T_n^*+\bar{T}_b)\|_{\mathbb{V}})\boldsymbol{u}_n^{mi}(t)\partial_i\bar{T}_b^j(t)S^jdx\\ =&-\sum_{i,j=1}^3\int_\Omega F_N(\|(\boldsymbol{u}_n,T_n^*+\bar{T}_b)\|_{\mathbb{V}})\boldsymbol{u}_n^i(t)\bar{T}_b^j(t)\partial_iS^jdx\,.\end{aligned}$$

Using (2.7) and (2.22) one obtains the following

$$\int_\Omega F_N(\|(\boldsymbol{u}_n,T_n^*+\bar{T}_b)\|_{\mathbb{V}})\boldsymbol{u}_n^i(t)\bar{T}_b^j(t)\partial_iS^jdx\to\int_\Omega F_N(\|(\boldsymbol{u},T^*+\bar{T}_b)\|_{\mathbb{V}})\boldsymbol{u}^i(t)\bar{T}_b^j(t)\partial_iS^jdx\,,$$

thus

$$F_N(\|(\boldsymbol{u}_n,T_n^*+\bar{T}_b)\|_{\mathbb{V}})b_1(\boldsymbol{u}_n,\bar{T}_b,S)\to F_N(\|(\boldsymbol{u}_n,T_n^*+\bar{T}_b)\|_{\mathbb{V}})b_1(\boldsymbol{u},\bar{T}_b,S),\quad S\in W_n\,. \tag{2.26}$$

For the initial data, we have

$$P_n(\boldsymbol{u}_0,T_0^*)\to(\boldsymbol{u}_0,T_0^*)\ \text{ in }\ \mathbb{H}. \tag{2.27}$$

Indeed, $P_n(\boldsymbol{u}_0,T_0^*)=(\boldsymbol{u}_n(0),T_n^*(0))$. Since $(\boldsymbol{u}_n,T_n^*)\in\mathcal{C}([0,T];\mathbb{H})$ and $(\boldsymbol{u}_n,T_n^*)\to(\boldsymbol{u},T^*)$ strongly in $\mathbb{H}$; then (2.27) follows.
In addition, since $\langle P_n^1\boldsymbol{f}(t),\boldsymbol{u}\rangle=\langle\boldsymbol{f}(t),\boldsymbol{u}_n\rangle$ and $\boldsymbol{u}_n\to\boldsymbol{u}$ strongly in $\mathbf{H}$, then

$$\langle P_n^1\boldsymbol{f}(t),\boldsymbol{u}\rangle\to\langle\boldsymbol{f}(t),\boldsymbol{u}\rangle\,.$$

Similarly, we prove that

$$\langle P_n^2g(t),T\rangle\to\langle g(t),T\rangle\,.$$

<u>Step 3</u>: *recovering the pressure*

The method is standard and proceed as follows. First, we integrate the first equation of (1.15) respecting to $t$, we define the functional for all $L$ by: for all $\boldsymbol{v}\in\mathbb{H}^1(\Omega)$,

$$L(\boldsymbol{v})=\int_0^t\left((\boldsymbol{f}(s),\boldsymbol{v})-\nu a_0(\boldsymbol{u}(s),\boldsymbol{v})-b_N(\boldsymbol{u}(s),\boldsymbol{u}(s),v)\right)ds-(\boldsymbol{u}(s),\boldsymbol{v})+(\boldsymbol{u}_0,\boldsymbol{v})$$

which is a continuous linear functional on $\mathbb{H}^1(\Omega)$ and vanish on $\mathbf{V}$. Hence from [10], there exists a unique function $P(t)\in L_0^2(\Omega)$ such that for all $v\in\mathbb{H}_0^1(\Omega)$, for all $t\in(0,\tilde{T})$

$$\begin{aligned}L(\boldsymbol{v})&=-(\operatorname{div}\boldsymbol{v},P(t))_{\mathbf{H}}\\ |P(t)|&\le\sup_{\boldsymbol{v}\in\mathbb{H}_0^1(\Omega)}\frac{L(\boldsymbol{v})}{\|\boldsymbol{v}\|_{\mathbf{V}}}\end{aligned} \tag{2.28}$$

By defining $p(t)=\dfrac{d}{dt}P(t)$, we conclude that $(\boldsymbol{u},p,T=T^*+\overline{T}_b)$ is the weak solution of problem (1.2).
Hence we have constructed the weak solutions of problem (1.2). □

We now show that problem (1.2) has a unique solution.

**Theorem 2.2** *Suppose that* $\boldsymbol{f} \in L^2(0,\widetilde{T};\mathbb{H}^{-1}(\Omega))$, $g \in L^2(0,\widetilde{T};H^{-1}(\Omega))$, $T_b \in H^1(0,\widetilde{T};H^{1/2}(\Gamma))$, *the initial temperature on the boundary* $T_b^0$ *belongs to* $H^{1/2}(\Gamma)$ *and* $(\boldsymbol{u}_0, T_0,) \in \mathbb{H}$ *be given. The weak solution of problem (1.2) given by theorem 2.1 is unique.*

**proof** Let $(\boldsymbol{u}_1, \boldsymbol{T}_1)$ and $(\boldsymbol{u}_2, \boldsymbol{T}_2)$ two weak solutions of (1.11), then we have when setting $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $T = T_1 - T_2$,

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{u}(t) + \nu\mathcal{A}\boldsymbol{u}(t) \leq -B_N(\boldsymbol{u}_1(t),\boldsymbol{u}_1(t)) + B_N(\boldsymbol{u}_2(t),\boldsymbol{u}_2(t)) \\ \dfrac{d}{dt}T(t) + \alpha\mathcal{A}_1 T(t) \leq -B_{N,1}(\boldsymbol{u}_1(t),T_1(t)) + B_{N,1}(\boldsymbol{u}_2(t),T_2(t)) \\ (\boldsymbol{u}(0),T(0)) = (\boldsymbol{0},0)\,. \end{cases} \tag{2.29}$$

or for all $(\boldsymbol{v}, S) \in \mathbb{H}_0^1(\Omega) \times H_0^1(\Omega)$,

$$\begin{cases} \left\langle \dfrac{d}{dt}\boldsymbol{u}(t),\boldsymbol{v} \right\rangle + \nu(\mathcal{A}\boldsymbol{u}(t),\boldsymbol{v}) \leq -b_N(\boldsymbol{u}_1(t),\boldsymbol{u}_1(t),\boldsymbol{v}) + b_N(\boldsymbol{u}_2(t),\boldsymbol{u}_2(t),\boldsymbol{v}) \\ \left\langle \dfrac{d}{dt}T(t),S \right\rangle + \alpha(\mathcal{A}_1 T(t),S) \leq -b_{N,1}(\boldsymbol{u}_1(t),T_1(t),S) + b_{N,1}(\boldsymbol{u}_2(t),T_2(t),S) \\ (\boldsymbol{u}(0),T(0)) = (\boldsymbol{0},0). \end{cases} \tag{2.30}$$

Taking $\boldsymbol{v} = \boldsymbol{u}(t)$ in $(2.30)_1$ and $S = T(t)$ in $(2.30)_2$, we have

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\left|\boldsymbol{u}(t)\right|_{\mathbf{H}}^2 + \nu\left\|\boldsymbol{u}(t)\right\|_{\mathbf{V}}^2 \leq -b_N(\boldsymbol{u}_1(t),\boldsymbol{u}_1(t),\boldsymbol{u}(t)) + b_N(\boldsymbol{u}_2(t),\boldsymbol{u}_2(t),\boldsymbol{u}(t)) \\ \dfrac{1}{2}\dfrac{d}{dt}\left|T(t)\right|^2 + \alpha\left\|T(t)\right\|^2 \leq -b_{N,1}(\boldsymbol{u}_1(t),T_1(t),T(t)) + b_{N,1}(\boldsymbol{u}_2(t),T_2(t),T(t)) \\ (\boldsymbol{u}(0),T(0)) = (\boldsymbol{0},0)\,. \end{cases} \tag{2.31}$$

Now, we estimate each term of the right hand side of (2.30). First,

$$\begin{aligned} &-b_N(\boldsymbol{u}_1(t),\boldsymbol{u}_1(t),\boldsymbol{u}(t)) + b_N(\boldsymbol{u}_2(t),\boldsymbol{u}_2(t),\boldsymbol{u}(t)) \\ &= -F_N(\left\|\boldsymbol{u}_1(t)\right\|_{\mathbf{V}})b(\boldsymbol{u}(t),\boldsymbol{u}_1(t),\boldsymbol{u}(t)) - (F_N(\left\|\boldsymbol{u}_1(t)\right\|_{\mathbf{V}}) - F_N(\left\|\boldsymbol{u}_2(t)\right\|_{\mathbf{V}}))\,b(\boldsymbol{u}_2(t),\boldsymbol{u}_1(t),\boldsymbol{u}(t))\,. \end{aligned}$$

But using standard inequalities

$$\begin{aligned} \left|-F_N(\left\|\boldsymbol{u}_1(t)\right\|_{\mathbf{V}})b(\boldsymbol{u}(t),\boldsymbol{u}_1(t),\boldsymbol{u}(t))\right| &\leq c_b\frac{N}{\left\|\boldsymbol{u}_1(t)\right\|_{\mathbf{V}}}\left\|\boldsymbol{u}(t)\right\|_{\mathbf{V}}\left\|\boldsymbol{u}_1(t)\right\|_{\mathbf{V}}\left|\boldsymbol{u}(t)\right|_{\mathbf{H}}^{1/2}\left\|\boldsymbol{u}(t)\right\|_{\mathbf{V}}^{1/2} \\ &= Nc_b\left\|\boldsymbol{u}(t)\right\|_{\mathbf{V}}^{3/2}\left|\boldsymbol{u}(t)\right|_{\mathbf{H}}^{1/2} \\ &\leq \frac{\nu}{4}\left\|\boldsymbol{u}(t)\right\|_{\mathbf{V}}^2 + \frac{3(Nc_b)^4}{4\nu}\left|\boldsymbol{u}(t)\right|_{\mathbf{H}}^2\,, \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} &\left|-(F_N(\left\|\boldsymbol{u}_1(t)\right\|_{\mathbf{V}}) - F_N(\left\|\boldsymbol{u}_2(t)\right\|_{\mathbf{V}}))\,b(\boldsymbol{u}_2(t),\boldsymbol{u}_1(t),\boldsymbol{u}(t))\right| \\ &\leq \frac{\left\|\boldsymbol{u}_2(t)-\boldsymbol{u}_1(t)\right\|_{\mathbf{V}}}{N}\left|b(\boldsymbol{u}_2(t),\boldsymbol{u}_1(t),\boldsymbol{u}(t))\right|F_N(\left\|\boldsymbol{u}_1(t)\right\|_{\mathbf{V}})F_N(\left\|\boldsymbol{u}_2(t)\right\|_{\mathbf{V}}) \\ &\leq c_bF_N(\left\|\boldsymbol{u}_1(t)\right\|_{\mathbf{V}})F_N(\left\|\boldsymbol{u}_2(t)\right\|_{\mathbf{V}})\frac{\left\|\boldsymbol{u}(t)\right\|_{\mathbf{V}}}{N}\left\|\boldsymbol{u}_2(t)\right\|_{\mathbf{V}}\left\|u_1(t)\right\|_{\mathbf{V}}\left|\boldsymbol{u}(t)\right|_{\mathbf{H}}^{1/2}\left\|\boldsymbol{u}(t)\right\|_{\mathbf{V}}^{1/2} \\ &= Nc_b\left\|\boldsymbol{u}(t)\right\|_{\mathbf{V}}^{3/2}\left|\boldsymbol{u}(t)\right|_{\mathbf{H}}^{1/2} \\ &\leq \frac{\nu}{4}\left\|\boldsymbol{u}(t)\right\|_{\mathbf{V}}^2 + \frac{3(Nc_b)^4}{4\nu}\left|\boldsymbol{u}(t)\right|_{\mathbf{H}}^2\,. \end{aligned} \tag{2.33}$$

Similarly

$$
\begin{aligned}
&- b_{N1}(\boldsymbol{u}_1(t), T_1(t), T(t)) + b_{N1}(\boldsymbol{u}_2(t), T_2(t), T(t)) \\
&= -F_N(\|(\boldsymbol{u}_1(t), T_1(t))\|_{\mathbb{V}}) b_1(\boldsymbol{u}(t), T_1(t), T(t)) \\
&- (F_N(\|(\boldsymbol{u}_1(t), T_1(t))\|_{\mathbb{V}}) - F_N(\|(\boldsymbol{u}_2(t), T_2(t))\|_{\mathbb{V}}))\, b_1(\boldsymbol{u}_2(t), T_1(t), T(t))\,.
\end{aligned}
$$

Again, with usual inequalities one has

$$
\begin{aligned}
&|-F_N(\|(\boldsymbol{u}_1(t), T_1(t)\|_{\mathbb{V}}) b_1(\boldsymbol{u}(t), T_1(t), T(t))| \\
&\leq c_b \frac{N}{\|(\boldsymbol{u}_1(t), T_1(t))\|_{\mathbb{V}}} \|\boldsymbol{u}(t)\|_{\mathbf{V}} \|T_1(t)\| \, |T(t)|^{1/2} \|T(t)\|^{1/2} \\
&\leq N c_b \|\boldsymbol{u}(t)\|_{\mathbf{V}} |T(t)|^{1/2} \|T(t)\|^{1/2} \\
&\leq \frac{\alpha}{4} \|\boldsymbol{u}(t)\|^2_{\mathbf{V}} + \frac{\alpha}{4} \|T(t)\|^2 + \frac{(Nc_b)^4}{\alpha^3} |T(t)|^2\,,
\end{aligned}
\tag{2.34}
$$

and

$$
\begin{aligned}
&|- (F_N(\|(\boldsymbol{u}_1(t), T_1(t))\|_{\mathbb{V}}) - F_N(\|(\boldsymbol{u}_2(t), T_2(t))\|_{\mathbb{V}}))\, b_1(\boldsymbol{u}_2(t), T_1(t), T(t))| \\
\leq& \frac{\|(\boldsymbol{u}(t), T(t))\|_{\mathbb{V}}}{N} |b_1(\boldsymbol{u}_2(t), T_1(t), T(t))| \, F_N(\|(\boldsymbol{u}_1(t), T_1(t))\|_{\mathbb{V}}) F_N(\|(\boldsymbol{u}_2(t), T_2(t))\|) \\
\leq& c_b F_N(\|(\boldsymbol{u}_1(t), T_1(t))\|_{\mathbb{V}}) F_N(\|(\boldsymbol{u}_2(t), T_2(t))\|_{\mathbb{V}}) \frac{\|(\boldsymbol{u}(t), T(t))\|_{\mathbb{V}}}{N} \|\boldsymbol{u}_2(t)\|_{\mathbf{V}} \|T_1(t)\| \, |T(t)|^{1/2} \times \\
&\|T(t)\|^{1/2} \\
=& N c_b \|(\boldsymbol{u}(t), T(t))\|^{3/2}_{\mathbb{V}} |T(t)|^{1/2} \\
\leq& \frac{\alpha}{4} \|(\boldsymbol{u}(t), T(t))\|^2_{\mathbb{V}} + \frac{3(Nc_b)^4}{4\alpha} |T(t)|^2. \\
=& \frac{\alpha}{4} \|\boldsymbol{u}(t)\|^2_{\mathbf{V}} + \frac{\alpha}{4} \|T(t)\|^2 + \frac{3(Nc_b)^4}{4\alpha} |T(t)|^2\,.
\end{aligned}
\tag{2.35}
$$

Using (2.32) - (2.35) in (2.29), we obtain

$$
\begin{cases}
\dfrac{d}{dt} |\boldsymbol{u}(t)|^2_{\mathbf{H}} + \nu \|\boldsymbol{u}(t)\|^2_{\mathbf{V}} \leq \dfrac{3(Nc_b)^4}{2\nu} |\boldsymbol{u}(t)|^2_{\mathbf{H}} \\
\frac{d}{dt} |T(t)|^2 + \alpha \|T(t)\|^2 \leq \alpha \|\boldsymbol{u}(t)\|^2_{\mathbf{V}} + \left( \frac{2(Nc_b)^4}{\alpha^3} + \frac{3(Nc_b)^4}{2\alpha} \right) |T(t)|^2 \\
(\boldsymbol{u}(0), T(0)) = (0, 0)\,.
\end{cases}
\tag{2.36}
$$

Dropping momentarily the term $\nu \|\boldsymbol{u}(t)\|^2_{\mathbf{V}}$ in $(2.36)_1$ and using lemma 1.4, we have

$$
|\boldsymbol{u}(t)|^2_{\mathbf{H}} \leq |\boldsymbol{u}(0)|^2_{\mathbf{H}} \, e^{\frac{3(Nc_b)^4}{2\nu} t},
$$

consequently, $\boldsymbol{u}_1(t) = \boldsymbol{u}_2(t)$ since $\boldsymbol{u}(0) = 0$.
Using this in $(2.36)_2$ and Lemma 1.4 again, we have

$$
|T(t)|^2 \leq |T(0)|^2_{\mathbf{V}} \, e^{\left( \frac{2(Nc_b)^4}{\alpha^3} + \frac{3(Nc_b)^4}{2\alpha} \right) t},
$$

hence, $T_1(t) = T_2(t)$ since $T(0) = 0$ and the theorem is proved. □

## 2.2 Continuous dependence on initial values and the parameter N

The goal of the paragraph is to show that the solution $(\boldsymbol{u}(t), T(t), p(t))$ of (1.2) depends continuously on the parameter $N$ as well as on the initial value $(\boldsymbol{u}_0, T_0)$. This result was already obtained when the temperature is zero in [4]. The non trivial task here is to re-adapt their proof by taking into account the coupling between the velocity and temperature. More precisely, we prove the following result

**Theorem 2.3** *Let $\boldsymbol{f} \in L^2(0,\widetilde{T};\mathbb{L}^2(\Omega))$, $g \in L^2(0,\widetilde{T};L^2(\Omega))$, $N_i > 0$, $(\boldsymbol{u}_{0i}, T_{0i}) \in \mathbb{V}$, $i = 1;2$ be given. Assume $y = (\boldsymbol{u}_i, T_i)$ be the solutions of (1.2) corresponding to the parameter $N_i$ and the initial values $y_{0i} = (\boldsymbol{u}_{0i}, T_{0i})$, $i = 1;2$. Then*

$$(\boldsymbol{u}_1, T_1) \to (\boldsymbol{u}_2, T_2) \ \ in \ \ C(0,\widetilde{T};\mathbb{V}) \cap \mathcal{D}(0,\widetilde{T};\mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}_1))$$

*when $N_1 \to N_2$ and $(\boldsymbol{u}_{02}, T_{01}) \to (\boldsymbol{u}_{02}, T_{02})$. More precisely, the following estimates hold true.*

$$\|(\boldsymbol{u}(t),T(t))\|_{\mathbb{V}}^2 \le \left\{ \|(\boldsymbol{u}(0),T(0))\|_{\mathbb{V}}^2 + \frac{12c_b^2}{\alpha_3}|N_1-N_2|^2 \int_0^{\widetilde{T}} (|\mathcal{A}\boldsymbol{u}_2(s)|_{\boldsymbol{H}}^2 + |\mathcal{A}_1 T_2(s)|^2) ds \right\} \times$$
$$exp\left( \frac{6(Nc_b)^4}{\alpha_3}\widetilde{T} + \alpha_4 \int_0^{\widetilde{T}} |\mathcal{A}\boldsymbol{u}_2(t)|_{\boldsymbol{H}}^2 dt \right). \tag{2.37}$$

*and*

$$\alpha_3 \int_0^{\widetilde{T}} (|\mathcal{A}\boldsymbol{u}(t)|_{\boldsymbol{H}}^2 + |\mathcal{A}_1 T(t)|^2 dt \le$$
$$\left\{ \|(\boldsymbol{u}(0),T(0))\|_{\mathbb{V}}^2 + \frac{12c_b^2}{\alpha_3}|N_1-N_2|^2 \int_0^{\widetilde{T}} (|\mathcal{A}\boldsymbol{u}_2(s)|_{\boldsymbol{H}}^2 + |\mathcal{A}_1 T_2(s)|^2) ds \right\} \times$$
$$\left[ 1 + \left( \frac{6(Nc_b)^4}{\alpha_3}\widetilde{T} + \alpha_4 \int_0^{\widetilde{T}} |\mathcal{A}\boldsymbol{u}_2(t)|_{\boldsymbol{H}}^2 dt \right) \times exp\left[ \frac{6(Nc_b)^4}{\alpha_3}\widetilde{T} + \alpha_4 \int_0^{\widetilde{T}} |\mathcal{A}\boldsymbol{u}_2(t)|_{\boldsymbol{H}}^2 dt \right] \right] \tag{2.38}$$

*$\alpha_3$, $\alpha_4$ will be defined later.*

**proof** Setting $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $T = T_1 - T_2$, we have for almost every $t \in (0,\widetilde{T})$

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{u}(t) + \nu\mathcal{A}\boldsymbol{u}(t) & \le -B_{N_1}(\boldsymbol{u}_1(t),\boldsymbol{u}_1(t)) + B_{N_2}(\boldsymbol{u}_2(t),\boldsymbol{u}_2(t)) \\ \dfrac{d}{dt}T(t) + \alpha\mathcal{A}_1 T(t) & \le -B_{N_1,1}(\boldsymbol{u}_1(t),T_1(t)) + B_{N_2,1}(\boldsymbol{u}_2(t),T_2(t)). \end{cases} \tag{2.39}$$

Taking the inner product of $(2.39)_1$ with $\mathcal{A}\boldsymbol{u}(t)$ and of $(2.39)_2$ with $\mathcal{A}_1 T(t)$, we have

$$\begin{cases} \dfrac{1}{2}\dfrac{dt}{dt}\|\boldsymbol{u}(t)\|_{\mathbf{V}}^2 + \nu|\mathcal{A}\boldsymbol{u}(t)|_{\mathbf{H}}^2 & \le -b_{N_1}(\boldsymbol{u}_1(t),\boldsymbol{u}_1(t),\mathcal{A}\boldsymbol{u}(t)) + b_{N_2}(\boldsymbol{u}_2(t),\boldsymbol{u}_2(t),\mathcal{A}\boldsymbol{u}(t)) \\ \dfrac{1}{2}\dfrac{d}{dt}\|T(t)\|^2 + \alpha|\mathcal{A}_1 T(t)|^2 & \le -b_{N_1,1}(\boldsymbol{u}_1(t),T_1(t),\mathcal{A}_1 T(t)) + b_{N_2,1}(\boldsymbol{u}_2(t),T_2(t),\mathcal{A}_1 T(t)). \end{cases} \tag{2.40}$$

We now need to treat the right hand side of (2.40). First from the linearity one has

$$\begin{aligned} &-b_{N_1}(\boldsymbol{u}_1(t),\boldsymbol{u}_1(t),\mathcal{A}\boldsymbol{u}(t)) + b_{N_1}(\boldsymbol{u}_2(t),\boldsymbol{u}_2(t),\mathcal{A}\boldsymbol{u}(t)) \\ &= -F_{N_1}(\|\boldsymbol{u}_1(t)\|_{\mathbf{V}})b(\boldsymbol{u}(t),\boldsymbol{u}_1(t),\mathcal{A}\boldsymbol{u}(t)) - F_{N_2}(\|\boldsymbol{u}_2(t)\|_{\mathbf{V}})b(\boldsymbol{u}_2(t),\boldsymbol{u}(t),\mathcal{A}\boldsymbol{u}(t)) \\ &- (F_{N_1}(\|\boldsymbol{u}_1(t)\|_{\mathbf{V}}) - F_{N_2}(\|\boldsymbol{u}_2(t)\|_{\mathbf{V}}))\, b(\boldsymbol{u}_2(t),\boldsymbol{u}_1(t),\mathcal{A}\boldsymbol{u}(t)). \end{aligned} \tag{2.41}$$

The right hand side of (2.41) is treated using standard inequalities as follows;

$$\begin{aligned}|-F_{N_1}(\|\boldsymbol{u}_1(t)\|_{\mathbf{V}})b(\boldsymbol{u}(t),\boldsymbol{u}_1(t),\mathcal{A}\boldsymbol{u}(t))| &\leq c_b\tfrac{N_1}{\|\boldsymbol{u}_1(t)\|_{\mathbf{V}}}\|\boldsymbol{u}(t)\|_{\mathbf{V}}^{1/2}\|\boldsymbol{u}_1(t)\|_{\mathbf{V}}|\mathcal{A}\boldsymbol{u}(t)|_{\mathbf{H}}^{3/2}\\ &= N_1c_b\|\boldsymbol{u}(t)\|_{\mathbf{V}}^{1/2}|\mathcal{A}\boldsymbol{u}(t)|_{\mathbf{H}}^{3/2}\\ &\leq \tfrac{\nu}{8}|\mathcal{A}\boldsymbol{u}(t)|_{\mathbf{H}}^2+\tfrac{3(N_1c_b)^4}{2\nu}\|\boldsymbol{u}(t)\|_{\mathbf{V}}^2\,.\end{aligned}\tag{2.42}$$

$$\begin{aligned}&|-(F_{N_1}(\|\boldsymbol{u}_1(t)\|_{\mathbf{V}})-F_{N_2}(\|\boldsymbol{u}_2(t)\|_{\mathbf{V}}))\,b(\boldsymbol{u}_2(t),\boldsymbol{u}_1(t),\mathcal{A}\boldsymbol{u}(t))|\\ &\leq\left(\frac{|N_1-N_2|}{\|\boldsymbol{u}_1(t)\|_{\mathbf{V}}}+\frac{\|\boldsymbol{u}(t)\|_{\mathbf{V}}}{\|\boldsymbol{u}_1(t)\|_{\mathbf{V}}}\right)c_b|\mathcal{A}\boldsymbol{u}_2(t)|_{\mathbf{H}}\|\boldsymbol{u}_1(t)\|_{\mathbf{V}}|\mathcal{A}\boldsymbol{u}(t)|_{\mathbf{H}}\\ &\leq\frac{2}{\nu_1}(|N_1-N_2|+\|\boldsymbol{u}(t)\|_{\mathbf{V}})^2c_b^2|\mathcal{A}\boldsymbol{u}_2(t)|_{\mathbf{H}}^2+\frac{\nu_1}{8}|\mathcal{A}\boldsymbol{u}(t)|_{\mathbf{H}}^2\\ &\leq\frac{2}{\nu}\left(|N_1-N_2|^2+\|\boldsymbol{u}(t)\|_{\mathbf{V}}^2\right)c_b^2|\mathcal{A}\boldsymbol{u}_2(t)|_{\mathbf{H}}^2+\frac{\nu}{8}|\mathcal{A}\boldsymbol{u}(t)|_{\mathbf{H}}^2\,,\end{aligned}\tag{2.43}$$

and

$$\begin{aligned}|-F_{N_2}(\|\boldsymbol{u}_2(t)\|_{\mathbf{V}})b(\boldsymbol{u}_2(t),\boldsymbol{u}(t),\mathcal{A}\boldsymbol{u}(t))| &\leq|b(\boldsymbol{u}_2(t),\boldsymbol{u}(t),\mathcal{A}\boldsymbol{u}(t))|\\ &\leq c_b|\mathcal{A}\boldsymbol{u}_2(t)|_{\mathbf{H}}\|\boldsymbol{u}(t)\|_{\mathbf{V}}|\mathcal{A}\boldsymbol{u}(t)|_{\mathbf{H}}\\ &\leq\tfrac{\nu}{4}|\mathcal{A}\boldsymbol{u}(t)|_{\mathbf{H}}^2+\tfrac{c_b^2}{\nu}|\mathcal{A}\boldsymbol{u}_2(t)|_{\mathbf{H}}^2\|\boldsymbol{u}(t)\|_{\mathbf{V}}^2\,.\end{aligned}\tag{2.44}$$

Secondly, exploiting the same linearity one has

$$\begin{aligned}&-b_{N_1,1}(\boldsymbol{u}_1(t),T_1(t),T(t))+b_{N_2,1}(\boldsymbol{u}_2(t),T_2(t),\mathcal{A}_1T(t))\\ =&-F_{N_1}(\|(\boldsymbol{u}_1(t),T_1(t))\|_{\mathbb{V}})b_1(\boldsymbol{u}(t),T_1(t),\mathcal{A}_1T(t))-F_{N_2}(\|(\boldsymbol{u}_2(t),T_2(t))\|_{\mathbb{V}})b_1(\boldsymbol{u}_2(t),T(t),\mathcal{A}_1T(t))\\ &-(F_{N_1}(\|(\boldsymbol{u}_1(t),T_1(t))\|_{\mathbb{V}})-F_{N_2}(\|(\boldsymbol{u}_2(t),T_2(t))\|_{\mathbb{V}}))\,b_1(\boldsymbol{u}_2(t),T_1(t),\mathcal{A}_1T(t))\,.\end{aligned}\tag{2.45}$$

Again, we treat the right hand side of (2.45) using standard inequalities as follows;

$$\begin{aligned}|-F_{N_1}(\|(\boldsymbol{u}_1(t),T_1(t)\|_{\mathbb{V}})b_1(\boldsymbol{u}(t),T_1(t),\mathcal{A}_1T(t))| &\leq c_b\frac{N_1}{\|(\boldsymbol{u}_1(t),T_1(t))\|_{\mathbb{V}}}\|\boldsymbol{u}(t)\|_{\mathbf{V}}^{1/2}\|T_1(t)\||\mathcal{A}_1T(t)|^{3/2}\\ &=N_1c_b\|\boldsymbol{u}(t)\|_{\mathbf{V}}^{1/2}|\mathcal{A}_1T(t)|^{3/2}\\ &\leq\frac{\alpha}{8}|\mathcal{A}_1T(t)|^2+\frac{3(N_1c_b)^4}{2\alpha}\|\boldsymbol{u}(t)\|_{\mathbf{V}}^2\,.\end{aligned}\tag{2.46}$$

$$\begin{aligned}&|-(F_{N_1}(\|(\boldsymbol{u}_1(t),T_1(t))\|_{\mathbb{V}})-F_{N_2}(\|(\boldsymbol{u}_2(t),T_2(t))\|_{\mathbb{V}}))\,b_1(\boldsymbol{u}_2(t),T_1(t),\mathcal{A}_1T(t))|\\ &\leq\left(\frac{|N_1-N_2|}{\|(\boldsymbol{u}_1(t),T_1(t))\|_{\mathbb{V}}}+\frac{\|y(t)\|_{\mathbb{V}}}{\|(\boldsymbol{u}_1(t),T_1(t)\|_{\mathbb{V}}}\right)c_b|\mathcal{A}\boldsymbol{u}_2(t)|_{\mathbf{H}}\|T_1(t)\||\mathcal{A}_1T(t)|\\ &\leq\frac{2}{\alpha}(|N_1-N_2|+\|y(t)\|_{\mathbb{V}})^2c_b^2|\mathcal{A}\boldsymbol{u}_2(t)|_{\mathbf{H}}^2+\frac{\alpha}{8}|\mathcal{A}_1T(t)|^2\\ &\leq\frac{4}{\alpha}\left(|N_1-N_2|^2+\|y(t)\|_{\mathbb{V}}^2\right)c_b^2|\mathcal{A}\boldsymbol{u}_2(t)|_{\mathbf{H}}^2+\frac{\alpha}{8}|\mathcal{A}_1T(t)|^2\,,\end{aligned}\tag{2.47}$$

and

$$\begin{aligned}|-F_{N_2}(\|(\boldsymbol{u}_2(t),T_2(t))\|_{\mathbb{V}})b_1(\boldsymbol{u}_2(t),T(t),\mathcal{A}_1T(t))| &\leq|b_1(\boldsymbol{u}_2(t),T(t),\mathcal{A}_1T(t))|\\ &=c_b|\mathcal{A}\boldsymbol{u}_2(t)|_{\mathbf{H}}\|T(t)\||\mathcal{A}_1T(t)|\\ &\leq\frac{\alpha}{4}|\mathcal{A}_1T(t)|^2+\frac{c_b^2}{\alpha}|\mathcal{A}\boldsymbol{u}_2(t)|_{\mathbf{H}}^2\|T(t)\|^2\,.\end{aligned}\tag{2.48}$$

Using (2.42) - (2.48) in (2.40), we obtain

$$
\left\{
\begin{array}{ll}
\dfrac{d}{dt}\left\|\boldsymbol{u}(t)\right\|_{\mathbf{V}}^{2}+\nu\left|\mathcal{A}\boldsymbol{u}(t)\right|_{\mathbf{H}}^{2} & \leq\left(\frac{3(N_1c_b)^4}{\nu}+\frac{6c_b^2}{\nu}\left|\mathcal{A}\boldsymbol{u}_2(t)\right|_{\mathbf{H}}^{2}\right)\left\|\boldsymbol{u}(t)\right\|_{\mathbf{V}}^{2}+ \\
 & \frac{4c_b^2}{\nu}\left|N_1-N_2\right|^2\left|\mathcal{A}\boldsymbol{u}_2(t)\right|_{\mathbf{H}}^{2} \\
\dfrac{d}{dt}\left\|T(t)\right\|^{2}+\alpha\left|\mathcal{A}_1T(t)\right|^{2} & \leq\left\{\frac{3(N_1c_b)^4}{\alpha}+\frac{8c_b^2}{\alpha}\left|\mathcal{A}\boldsymbol{u}_2(t)\right|_{\mathbf{H}}^{2}\right\}\left\|\boldsymbol{u}(t)\right\|_{\mathbf{V}}^{2}+ \\
 & \frac{8c_b^2}{\alpha}\left|N_1-N_2\right|^2\left|\mathcal{A}\boldsymbol{u}_2(t)\right|_{\mathbf{H}}^{2}+\frac{10c_b^2}{\alpha}\left|\mathcal{A}\boldsymbol{u}_2(t)\right|_{\mathbf{H}}^{2}\left\|T(t)\right\|^{2}.
\end{array}
\right.
\tag{2.49}
$$

Adding these two inequalities, we obtain

$$
\frac{d}{dt}\left\|(\boldsymbol{u}(t),T(t))\right\|_{\mathbb{V}}^{2}+\alpha_3(\left|\mathcal{A}\boldsymbol{u}(t)\right|_{\mathbf{H}}^{2}+\left|\mathcal{A}_1T(t)\right|^{2}+\frac{12c_b^2}{\alpha_3}\left|N_1-N_2\right|^2\left(\left|\mathcal{A}\boldsymbol{u}_2(t)\right|_{\mathbf{H}}^{2}+\left|\mathcal{A}_1T(t)\right|^{2}\right)
$$

$$
\leq\left(\frac{6(Nc_b)^4}{\alpha_3}+\alpha_4\left|\mathcal{A}\boldsymbol{u}_2(t)\right|_{\mathbf{H}}^{2}\right)\left\|(\boldsymbol{u}(t),T(t))\right\|_{\mathbb{V}}^{2} \tag{2.50}
$$

where $\alpha_3=\min\left(\nu_1,\alpha\right);\alpha_4=\frac{14c_b^2}{\alpha_3}+\frac{10c_b^2}{\alpha}$.
Dropping momentarily the term $\alpha_3(\left|\mathcal{A}\boldsymbol{u}(t)\right|_{\mathbf{H}}^{2}+\left|\mathcal{A}_1T(t)\right|^{2})$ in (2.50) and using Lemma 1.4, we have

$$
\left\|(\boldsymbol{u}(t),T(t))\right\|_{\mathbb{V}}^{2}\leq\left\{\left\|(\boldsymbol{u}(0),T(0))\right\|_{\mathbb{V}}^{2}+\frac{12c_b^2}{\alpha_3}\left|N_1-N_2\right|^2\int_0^{\widetilde{T}}(\left|\mathcal{A}\boldsymbol{u}_2(s)\right|_{\mathbf{H}}^{2}+\left|\mathcal{A}_1T_2(s)\right|^{2})ds\right\}\times
$$

$$
\exp\left(\frac{6(Nc_b)^4}{\alpha_3}\widetilde{T}+\alpha_4\int_0^{\widetilde{T}}\left|\mathcal{A}\boldsymbol{u}_2(t)\right|_{\mathbf{H}}^{2}dt\right). \tag{2.51}
$$

Using (2.51) in (2.50), we get

$$
\alpha_3\int_0^{\widetilde{T}}(\left|\mathcal{A}\boldsymbol{u}(t)\right|_{\mathbf{H}}^{2}+\left|\mathcal{A}_1T(t)\right|^{2}dt\leq
$$

$$
\left\{\left\|(\boldsymbol{u}(0),T(0))\right\|_{\mathbb{V}}^{2}+\frac{12c_b^2}{\alpha_3}\left|N_1-N_2\right|^2\int_0^{\widetilde{T}}(\left|\mathcal{A}\boldsymbol{u}_2(s)\right|_{\mathbf{H}}^{2}+\left|\mathcal{A}_1T_2(s)\right|^{2})ds\right\}\times
$$

$$
\left[1+\left(\frac{6(Nc_b)^4}{\alpha_3}\widetilde{T}+\alpha_4\int_0^{\widetilde{T}}\left|\mathcal{A}\boldsymbol{u}_2(t)\right|_{\mathbf{H}}^{2}dt\right)\times\exp\left[\frac{6(Nc_b)^4}{\alpha_3}\widetilde{T}+\alpha_4\int_0^{\widetilde{T}}\left|\mathcal{A}\boldsymbol{u}_2(t)\right|_{\mathbf{H}}^{2}dt\right]\right] \tag{2.52}
$$

the proof of Theorem (2.3) follows. □

## 2.3 Comparison of Galerkin solutions of the GMNSHE and NSHE

We first note that NSHE stands for Navier-Stokes equation coupled with the heat equation. In this paragraph, we prove that the Galerkin's approximations of the GMNSHE (2.54) below are the same as the Galerkin's approximations for the NSHE (associated with the same initial value $(\boldsymbol{u}_0,T_0)$ over the time interval $[0,\widetilde{T}]$) for some value of $N$. The following inequalities (see [4]) will also be used:

$$
\begin{array}{l}
|\mathcal{A}\boldsymbol{u}_n|_{\mathbf{H}}\leq\lambda_n|\boldsymbol{u}_n|_{\mathbf{H}},\ \|\boldsymbol{u}_n\|_{\mathbf{V}}\leq(\lambda_n)^{1/2}|\boldsymbol{u}_n|_{\mathbf{H}}, \\
|\mathcal{A}_1T_n|\leq\lambda_n^1|T_n|,\ \|T_n\|\leq(\lambda_n^1)^{1/2}|T_n|, \\
\lambda_1|\boldsymbol{u}_n|_{\mathbf{H}}^2\leq\|\boldsymbol{u}_n\|_{\mathbf{V}}^2,\ \lambda_1^1|T_n|^2\leq\|T_n\|^2,
\end{array}
\tag{2.53}
$$

where $\lambda_j$ and $\lambda_j^1$ are the corresponding eigenvalues of the operators $\mathcal{A}$ and $\mathcal{A}_1$. our next result establishes a link between GMNSHE and NSHE. We claim that

**Theorem 2.4** *We assume that $\boldsymbol{f} \in L^\infty(0,\widetilde{T};\mathbb{L}^2(\Omega))$, $g \in L^\infty(0,\widetilde{T};L^2(\Omega))$ for all $\widetilde{T} > 0$, and we consider the Galerkin's approximations of the GMNSHE and NSHE of fixed dimension n for the same initial value $(\boldsymbol{u}_0, T_0)$ over the time interval $[0,\widetilde{T}]$. Then there exists a subsequence $(\boldsymbol{u}_n^{(N_j)}, T_n^{(N_j)})_j$ of the sequence $(\boldsymbol{u}_n^N, T_n^N)_N$ which converges uniformly in $\mathcal{C}(0,\widetilde{T};\mathbb{R}^3) \times \mathcal{C}(0,\widetilde{T};\mathbb{R}^3)$ to a function $(\boldsymbol{u}_n^\infty, T_n^\infty)$ in $\mathcal{C}(0,\widetilde{T};\mathbb{R}^3) \times \mathcal{C}(0,\widetilde{T};\mathbb{R}^3)$ which is the corresponding solution of the n-dimensional Galerkin's approximations for the NSHE if N satisfies*

$$N \geq \max\left\{(\lambda_n)^{1/2}\mathcal{K}_4^{1/2}; \left(\lambda_n \mathcal{K}_4 + 2\left(\lambda_n^1 \mathcal{K}_5 + c_\Lambda^2 \|T_b\|^2_{H^{1/2}(\Gamma)}\right)\right)^{1/2}\right\}$$

*where $\mathcal{K}_4$ and $\mathcal{K}_5$ are defined below.*

**Proof.** We set $|f|_{\mathbf{H}\infty} = \|f\|_{L^\infty(0,\widetilde{T};\mathbb{L}^2(\Omega))}$ and $|g|_\infty = |g|_{L^\infty(0,\widetilde{T};L^2(\Omega))}$. The Galerkin's approximations of the GMNSHE with the parameter $N$ are given by

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{u}_n^N(t) + \nu\mathcal{A}\boldsymbol{u}_n^N(t) + \mathcal{B}_N(\boldsymbol{u}_n^N(t), \boldsymbol{u}_n^N(t)) = \boldsymbol{f}(t), \\ \dfrac{d}{dt}T_n^{*N}(t) + \alpha\mathcal{A}_1 T_n^{*N}(t) + F_N(\|(\boldsymbol{u}_n^N, T_n^{*N} + \bar{T}_b)\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{u}_n^N(t), T_n^{*N}(t)) = g(t) - \\ \dfrac{d}{dt}\bar{T}_b - \alpha\mathcal{A}_1\bar{T}_b(t) - F_N(\|(\boldsymbol{u}_n^N, T_n^{*N} + \bar{T}_b)\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{u}_n^N(t), \bar{T}_b(t)), \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \;\; T^*(\boldsymbol{x},0) = T_0(x) - \bar{T}_b(\boldsymbol{x},0)\,. \end{cases} \tag{2.54}$$

We deduce from (2.5) and (2.53) that

$$\begin{cases} \dfrac{d}{dt}\left|\boldsymbol{u}_n^N(t)\right|^2_{\mathbf{H}} + \nu\lambda_1\left|\boldsymbol{u}_n^N(t)\right|^2_{\mathbf{H}} \leq \dfrac{c_1^2}{\nu}\left|\boldsymbol{f}\right|^2_{\mathbf{H}\infty}, \\ \dfrac{d}{dt}\left|T_n^*(t)\right|^2 + \alpha\lambda_1^1\left|T_n^{*N}(t)\right|^2 \leq \dfrac{2c_2^2}{\alpha}\left|g(t)\right|^2_\infty + \dfrac{4c_3^2}{\alpha}\left\|\dfrac{d}{dt}\bar{T}_b(t)\right\|^2 + \\ \left(4c_4^2 + \frac{4\lambda_n c_5^2}{\alpha}\left|\boldsymbol{u}_n^N(t)\right|^2_{\mathbf{H}}\right)\left\|\bar{T}_b(t)\right\|^2. \end{cases} \tag{2.55}$$

Hence, the energy inequalities of the ODE (2.54) read

$$\begin{cases} \left|\boldsymbol{u}_n^N(t)\right|^2_{\mathbf{H}} \leq \left|\boldsymbol{u}_0\right|^2_{\mathbf{H}} + \frac{c_1^2}{\lambda_1\nu^2}\left|\boldsymbol{f}\right|^2_{\mathbf{H}\infty}, \\ \left|T_n^{*N}(t)\right|^2 \leq \left|T_0^*\right|^2 + \frac{2c_2^2}{\alpha}\widetilde{T}\left|g\right|^2_\infty + \frac{4c_3^2}{\alpha}\displaystyle\int_0^{\widetilde{T}}\left\|\dfrac{d}{dt}\bar{T}_b(t)\right\|^2 dt + \\ \left(4c_4^2 + \frac{4\lambda_n c_5^2}{\alpha}\mathcal{K}_4\right)\displaystyle\int_0^{\widetilde{T}}\left\|\bar{T}_b(t)\right\|^2 dt\,. \end{cases} \tag{2.56}$$

where $\mathcal{K}_4 = \left|\boldsymbol{u}_0\right|^2_{\mathbf{H}} + \frac{c_1^2}{\lambda_1\nu^2}\left|\boldsymbol{f}\right|^2_{\mathbf{H}\infty}$. In addition, from (2.54), (2.55) and (1.1), we have

$$\begin{aligned} \left|\frac{d}{dt}\boldsymbol{u}_n^N(t)\right|_{\mathbf{H}} &\leq \nu\left|\mathcal{A}\boldsymbol{u}_n^N\right|_{\mathbf{H}} + \left|B_N(\boldsymbol{u}_n^N, \boldsymbol{u}_n^N)\right|_{\mathbf{H}} + \left|\boldsymbol{f}\right|_{\mathbf{H}} \\ &\leq \left|\nu\mathcal{A}\boldsymbol{u}_n^N\right|_{\mathbf{H}} + \left|B(\boldsymbol{u}_n^N, \boldsymbol{u}_n^N)\right|_{\mathbf{H}} + \left|\boldsymbol{f}\right|_{\mathbf{H}\infty} \\ &\leq \nu\lambda_n\left|\boldsymbol{u}_n^N\right|_{\mathbf{H}} + c_b\|\boldsymbol{u}_n^N\|_{\mathbf{V}}^{3/2}\left|\mathcal{A}\boldsymbol{u}_n^N\right|_{\mathbf{H}}^{1/2} + \left|\boldsymbol{f}\right|_{\mathbf{H}\infty} \\ &\leq \nu\lambda_n\mathcal{K}_4^{1/2} + c_b\lambda_n^{5/4}\mathcal{K}_4 + \left|\boldsymbol{f}\right|_{\mathbf{H}\infty}\,. \end{aligned} \tag{2.57}$$

On the other hand, also from (2.54), (2.55) and (1.1),

$$
\begin{aligned}
&\left|\frac{d}{dt}T_n^{*N}(t)\right| \\
\leq&\alpha\left|\mathcal{A}_1 T_n^{*N}(t)\right| + F_N(\|(\boldsymbol{u}_n^N, T_n^{*N}+\bar{T}_b)\|_{\mathbb{V}})\left|\mathcal{B}_1(\boldsymbol{u}_n^N(t), T_n^{*N}(t))\right| + \left|\frac{d}{dt}\bar{T}_b\right| + \alpha\left|\mathcal{A}_1\bar{T}_b(t)\right| \\
&+ F_N(\|(\boldsymbol{u}_n^N, T_n^{*N}+\bar{T}_b)\|_{\mathbb{V}})\left|\mathcal{B}_1(\boldsymbol{u}_n^N(t), \bar{T}_b(t))\right| + |g(t)| \\
\leq&\alpha\left|\mathcal{A}_1 T_n^{*N}(t)\right| + \left|\mathcal{B}_1(\boldsymbol{u}_n^N(t), T_n^{*N}(t))\right| + \left|\frac{d}{dt}\bar{T}_b\right| + \alpha\left|\mathcal{A}_1\bar{T}_b(t)\right| + \left|\mathcal{B}_1(\boldsymbol{u}_n^N(t), \bar{T}_b(t))\right| + |g(t)| \\
\leq&\alpha\left|\mathcal{A}_1 T_n^{*N}(t)\right| + c_b\left|\mathcal{A}\boldsymbol{u}_n^N\right|_{\mathbf{H}}\left\|T_n^{*N}\right\| + \left|\frac{d}{dt}\bar{T}_b\right| + \alpha\left|\mathcal{A}_1\bar{T}_b(t)\right| + c_b\left|\mathcal{A}\boldsymbol{u}_n^N\right|_{\mathbf{H}}\left\|\bar{T}_b\right\| + |g(t)|_\infty \\
\leq&\alpha\lambda_n^1\left|T_n^{*N}(t)\right| + c_b\lambda_n\left|\boldsymbol{u}_n^N\right|_{\mathbf{H}}(\lambda_n^1)^{1/2}\left|T_n^{*N}\right| + \left|\frac{d}{dt}\bar{T}_b\right| + \alpha\left|\mathcal{A}_1\bar{T}_b(t)\right| + c_b\lambda_n\left|\boldsymbol{u}_n^N\right|_{\mathbf{H}}\left\|\bar{T}_b\right\| + |g(t)|_\infty \\
\leq&\alpha\lambda_n^1\mathcal{K}_5 + c_b\lambda_n(\lambda_n^1)^{1/2}\mathcal{K}_4\mathcal{K}_5 + \left|\frac{d}{dt}\bar{T}_b\right| + \alpha\left|\mathcal{A}_1\bar{T}_b(t)\right| + c_b\lambda_n\mathcal{K}_4\left\|\bar{T}_b\right\| + |g(t)|_\infty\ ,
\end{aligned}
\tag{2.58}
$$

where $\mathcal{K}_5 = |T_0^*|^2 + \frac{2c_2^2}{\alpha}\widetilde{T}\,|g|_\infty^2 + \frac{4c_3^2}{\alpha}\int_0^{\widetilde{T}}\left\|\frac{d}{dt}\bar{T}_b(t)\right\|^2 dt + \left(4c_4^2 + \frac{4\lambda_n c_5^2}{\alpha}\mathcal{K}_4\right)\int_0^{\widetilde{T}}\left\|\bar{T}_b(t)\right\|^2 dt.$
Using these estimates (providing the uniformly boundaries in both $N$ and $n$ of $(\boldsymbol{u}_n^N, T_n^{*N})_N$ and $\left(\frac{d}{dt}\boldsymbol{u}_n^N, \frac{d}{dt}T_n^{*N}\right)$, it follows from the Ascoli theorem that there exists a subsequence $(\boldsymbol{u}_n^{N_j}, T_n^{*N_j})_j$ of $(\boldsymbol{u}_n^N, T_n^{*N})_N$ which converges uniformly to a function $(\boldsymbol{u}_n^\infty, T_n^{*\infty})$ in $\mathcal{C}(0,\widetilde{T};\mathbb{R}^3)\times\mathcal{C}(0,\widetilde{T};\mathbb{R}^3)$. Setting $T_n^\infty = T_n^{*\infty} + \bar{T}_b$, $(\boldsymbol{u}_n^\infty, T_n^\infty)$ is the corresponding solution of the $n-$dimensional Galerkin ODE for NSHE. This follows from the uniqueness of solutions of the Galerkin ODE for a given initial value and the fact that

$$
1 \geq F_N\left(\|\boldsymbol{u}_n^N\|_{\mathbf{V}}\right) = \min\left(1, \frac{N}{\|\boldsymbol{u}_n^N\|_{\mathbf{V}}}\right) \geq \min\left(1, \frac{N}{(\lambda_n)^{1/2}\mathcal{K}_4^{1/2}}\right)\ , \tag{2.59}
$$

$$
\begin{aligned}
1 \geq F_N\left(\|\left(\boldsymbol{u}_n^N, T_n^{*N}+\bar{T}_b\right)\|_{\mathbb{V}}\right) &= \min\left(1, \frac{N}{\|\left(\boldsymbol{u}_n^N, T_n^{*N}+\bar{T}_b\right)\|_{\mathbb{V}}}\right) \\
&\geq \min\left(1, \frac{N}{\left(\lambda_n\mathcal{K}_4 + 2\left(\lambda_n^1\mathcal{K}_5 + c_\Lambda^2\|T_b\|_{H^{1/2}(\Gamma)}^2\right)\right)^{1/2}}\right),
\end{aligned}
\tag{2.60}
$$

so,

$$
\begin{gathered}
F_N(\|\boldsymbol{u}_n^N\|_{\mathbf{V}}) = 1 \text{ and } F_N(\|(\boldsymbol{u}_n^N, T_n^{*N}+\bar{T}_b)\|_{\mathbb{V}}) = 1 \text{ for} \\
N \geq \max\left\{(\lambda_n)^{1/2}\mathcal{K}_4^{1/2};\left(\lambda_n\mathcal{K}_4 + 2\left(\lambda_n^1\mathcal{K}_5 + c_\Lambda^2\|T_b\|_{H^{1/2}(\Gamma)}^2\right)\right)^{1/2}\right\}.
\end{gathered}
$$

□

# 3 Time discretization of problem (1.2)

In this section our goals are as follows; formulate the time discrete scheme and analyse it. By analysing, we mean: existence, uniqueness and stability.

### 3.1 Numerical scheme

We propose in this paragraph the time semi-discretization of problem (1.2) based on a backward Euler's scheme. As in [9], we divide the interval $[0,\widetilde{T}]$ in to $M$ intervals of equal length. Let $k=\frac{\widetilde{T}}{M}$ the time step. We associate with $k$ and the functions $f$, $g$ and $T_b$ the elements

$$\boldsymbol{f}^m=\frac{1}{k}\int_{(m-1)k}^{mk}\boldsymbol{f}(t)dt,\ g^m=\frac{1}{k}\int_{(m-1)k}^{mk}g(t)dt,\ T_b^m=\frac{1}{k}\int_{(m-1)k}^{mk}T_b(t)dt\,,$$

with $m=1,2,...M$. Most of time, we will use $(\boldsymbol{u},T)$ instead of $(\boldsymbol{u}(t),T(t))$.
For any data $(\boldsymbol{f},g,T_b)\in\mathcal{C}(0,\widetilde{T};\mathbb{H}^{-1}(\Omega))\times\mathcal{C}(0,\widetilde{T};H^{-1}(\Omega))\times\mathcal{C}(0,\widetilde{T};H^{1/2}(\Gamma))$, $(\boldsymbol{u}_0,T_0)\in\mathbf{V}\times H^1(\Omega)$. We consider the following scheme: for all $m=1,2,...,M$, *a.e.* $t\in(0,\widetilde{T})$ find $(\boldsymbol{u}^m,T^m)\in\mathbf{V}\times H^(\Omega)$ such that

$$\begin{cases}\boldsymbol{u}^0=\boldsymbol{u}_0,\ T^0=T_0 \text{ on } \Omega\,,\\ T^m=T_b^m \text{ on } \Gamma\,,\\ \dfrac{\boldsymbol{u}^m-\boldsymbol{u}^{m-1}}{k}+\nu\mathcal{A}\boldsymbol{u}^m+\mathcal{B}_N(\boldsymbol{u}^m,\boldsymbol{u}^m)=\boldsymbol{f}^m\,,\\ \dfrac{T^m-T^{m-1}}{k}+\alpha\mathcal{A}_1T^m+\mathcal{B}_{N1}(\boldsymbol{u}^m,T^m)=g^m\,.\end{cases}\tag{3.1}$$

Following the analysis in the continuous case, it is suitable to lift the boundary data $T_b^m$. For this purpose, according to the analysis done before, we set $\bar{T}_b^m=\mathcal{R}T_b^m$ where

$$\left\|\bar{T}_b^m\right\|_{L^2(0,\widetilde{T};H^1(\Omega))}\le c_\Lambda\left\|T_b^m\right\|_{H^{1/2}(\Gamma)} \text{ and } \left\|\bar{T}_b^m\right\|_{L^2(0,\widetilde{T};L^4(\Omega))}\le\epsilon\left\|T_b^m\right\|_{L^2(0,\widetilde{T};H^{1/2}(\Gamma))}\,.\tag{3.2}$$

We set $T^{*m}=T^m-\bar{T}_b^m$, we seek for $(\boldsymbol{u}^m,T^{*m})\in\mathbf{V}\times H^1(\Omega)$ such that

$$\begin{cases}\boldsymbol{u}^0=\boldsymbol{u}_0,\ T^{*0}=T_0-\bar{T}_b^0 \text{ on } \Omega\,,\\ T^m=T_b^m,\ \boldsymbol{u}^m=0,\ T^{*m}=0 \text{ on } \Gamma\\ \boldsymbol{u}^m+k\nu\mathcal{A}\boldsymbol{u}^m+k\mathcal{B}_N(\boldsymbol{u}^m,\boldsymbol{u}^m)=\boldsymbol{u}^{m-1}+k\boldsymbol{f}^m\,,\\ T^{*m}+k\alpha\mathcal{A}_1T^{*m}+kF_N(\|(\boldsymbol{u}^m,T^{*m}+\bar{T}_b^m)\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{u}^m,T^{*m})=T^{m-1}\\ +\bar{T}_b^{m-1}+kg^m-k\alpha\mathcal{A}_1\bar{T}_b^m-kF_N(\|(\boldsymbol{u}^m,T^{*m}+\bar{T}_b^m)\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{u}^m,\bar{T}_b^m)\,.\end{cases}\tag{3.3}$$

or equivalently for all $(\boldsymbol{v},S)\in\mathbf{V}\times H_0^1(\Omega)$,

$$\begin{cases}\boldsymbol{u}^0=\boldsymbol{u}_0,\ T^{*0}=T_0-\bar{T}_b^0 \text{ on } \Omega\,,\\ T^m=T_b^m,\ \boldsymbol{u}^m=0,\ T^{*m}=0 \text{ on } \Gamma\\ (\boldsymbol{u}^m,\boldsymbol{v})+ka_0(\boldsymbol{u}^m,\boldsymbol{v})+kb_N(\boldsymbol{u}^m,\boldsymbol{u}^m,\boldsymbol{v})=(\boldsymbol{u}^{m-1},v)+k\left\langle\boldsymbol{f}^m,v\right\rangle_\Omega\,,\\ (T^{*m},S)+k\alpha a_1(T^{*m},S)+kF_N(\|(\boldsymbol{u}^m,T^{*m}+\bar{T}_b^m)\|_{\mathbb{V}})b_1(\boldsymbol{u}^m,T^{*m},S)=(T^{m-1},S)\\ +(\bar{T}_b^{m-1},S)+k\left\langle g^m,S\right\rangle_\Omega-k\alpha a_1(\bar{T}_b^m,S)-kF_N(\|(\boldsymbol{u}^m,T^{*m}+\bar{T}_b^m)\|_{\mathbb{V}})b_1(\boldsymbol{u}^m,\bar{T}_b^m,S)\,.\end{cases}\tag{3.4}$$

### 3.2 Existence of solutions

Our goal in this paragraph is to construct the weak solutions to (3.4) by using; Galerkin's scheme, Brouwer's fixe point, a priori estimates and compactness results.

**Theorem 3.1** *Assume that the data* $(\boldsymbol{f},g,T_b)$ *belongs to* $\mathcal{C}(0,\widetilde{T};\mathbb{H}^{-1}(\Omega)\times\mathcal{C}(0,\widetilde{T};H^{-1}(\Omega)\times\mathcal{C}(0,\widetilde{T};H^{1/2}(\Gamma))$, *that the initial temperature on the boundary* $T_b^0$ *belongs to* $H^{1/2}(\Gamma)$ *and* $(\boldsymbol{u}_0,T_0)\in\boldsymbol{V}\times H^1(\Omega)$, *then problem (3.3) has at least one solution* $(\boldsymbol{u}^m,T^{*m})\in\mathcal{D}(\mathcal{A})\times\mathcal{D}(\mathcal{A}_1)$.

**proof** Following [9], the existence of a solution $y_m = (\boldsymbol{u}^m, T^{*m})$ of problem (3.3) is proved by the Galerkin's method in several steps as follows.

<u>*Step* 1</u>:Existence of approximate solutions.

Let $p \geq 1$ be an integer, knowing $(\boldsymbol{u}^1, T^{*1}), ..., (\boldsymbol{u}^{m-1}, T^{*m-1})$, we define an approximate solution of problem (3.3) by

$$\begin{cases} \boldsymbol{u}_p^m = \sum\limits_{i=1}^{p} g_{ip}^m \boldsymbol{v}_i, \ T_p^{*m} = \sum\limits_{i=1}^{p} h_{ip}^m w_i, \ g_{ip}^m, h_{ip}^m \in \mathbb{R}\,, \\ \boldsymbol{u}_p^0 = (\boldsymbol{u}_0)|_{\langle \boldsymbol{v}_1,...,\boldsymbol{v}_p\rangle}, \ T_p^{*0} = \left(T_0 - \bar{T}_b^0\right)|_{\langle w_1,...,w_p\rangle}\,, \\ T_p^m = T_b^m|_{\langle w_1,...,w_p\rangle}, \ \boldsymbol{u}_p^m = 0, \ T_p^{*m} = 0 \ \text{ on } \Gamma \\ \boldsymbol{u}_p^m + k\nu \mathcal{A}\boldsymbol{u}_p^m + k\mathcal{B}_N(\boldsymbol{u}_p^m, \boldsymbol{u}_p^m) = \boldsymbol{u}^{m-1} + k\boldsymbol{f}^m\,, \\ T_p^{*m} + k\alpha\mathcal{A}_1 T_p^{*m} + kF_N(\|(\boldsymbol{u}_p^m, T_p^{*m} + \bar{T}_b^m)\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{u}_p^m, T_p^{*m}) = T^{m-1} \\ +\bar{T}_b^{m-1} + kg^m - k\alpha\mathcal{A}_1\bar{T}_b^m - kF_N(\|(\boldsymbol{u}_p^m, T_p^{*m} + \bar{T}_b^m)\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{u}_p^m, \bar{T}_b^m)\,. \end{cases} \tag{3.5}$$

where $(\boldsymbol{v}_i)_{1\leq i\leq p} \subset \mathcal{D}(\mathcal{A})$ and $(w_i)_{1\leq i\leq p} \subset \mathcal{D}(\mathcal{A}_1)$ are respectively the eigen-vectors of the operators $\mathcal{A}$ and $\mathcal{A}_1$; $Y|_W$ is the restriction of $Y$ on the space $W$. Let $Z_p = \langle v_1, ..., v_p\rangle \times \langle w_1, ..., w_p\rangle$ the space generated by the indicated vectors. To prove the existence of $(\boldsymbol{u}_p^m, T_p^{*m})$ defined via (3.5), we consider the operator $\varphi : Z_p \to Z_p'$ given as follows; for all $U = (\boldsymbol{u}, T)$, $V = (v, S) \in Z_p$,

$$\begin{aligned} \langle \varphi(U), V\rangle_{Z_p,Z_p'} =& (\boldsymbol{u}, \boldsymbol{v}) + (T, S) + k\nu a_0(\boldsymbol{u}, \boldsymbol{v}) + k\alpha a_1(T, S) + kb_N(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - (T^{m-1}, S) - (\bar{T}_b^{m-1}, S) \\ &+ kF_N(\|(\boldsymbol{u}, T + \bar{T}_b^m)\|_{\mathbb{V}})b_1(\boldsymbol{u}, T, S) - (\boldsymbol{u}^{m-1}, \boldsymbol{v}) - k\left\langle \boldsymbol{f}^m, v\right\rangle_\Omega \\ &- k\left\langle g^m, S\right\rangle_\Omega + k\alpha a_1(\bar{T}_b^m, S) + kF_N(\|(\boldsymbol{u}, T + \bar{T}_b^m)\|_{\mathbb{V}})b_1(\boldsymbol{u}, \bar{T}_b^m, S) \end{aligned} \tag{3.6}$$

we apply a consequence of Brouwer's fixed point theorem, see ([24], Lemma 41, page 23). So, our task is to show that $\varphi$ is continuous and $\langle \varphi(U), U\rangle_{Z_p,Z_p'}$ is positive outside a sphere.

**Continuity of** $\varphi$. Let $(U)_n = (\boldsymbol{u}_n, T_n^*)_n \subset \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}_1)$ a sequence such that $(\boldsymbol{u}_n, T_n^*) \to (\boldsymbol{u}, T^*) = U$, it is enough to prove that $\varphi(U_n) \to \varphi(U)$. Note that there is no need to specify wether it is weak or strong convergence since $Z_p$ is a finite dimensional space. Let $V = (\boldsymbol{v}, S) \in \mathbf{V} \times H_0^1(\Omega)$,

$$\begin{aligned} \langle \varphi(U_n), V\rangle_{Z_p,Z_p'} &= (\boldsymbol{u}_n, \boldsymbol{v}) + (T_n^*, S) + k\nu a_0(\boldsymbol{u}_n, \boldsymbol{v}) + k\alpha a_1(T_n^*, S) + kb_N(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{v}) + \\ kF_N(\|(\boldsymbol{u}_n, T_n^* + \bar{T}_b^m)\|_{\mathbb{V}})&b_1(\boldsymbol{u}_n, T_n^*, S) - (\boldsymbol{u}^{m-1}, \boldsymbol{v}) - (T^{m-1}, S) - (\bar{T}_b^{m-1}, S) - k\left\langle \boldsymbol{f}^m, \boldsymbol{v}\right\rangle - \\ &k\left\langle g^m, S\right\rangle_\Omega + k\alpha a_1(\bar{T}_b^m, S) + kF_N(\|(\boldsymbol{u}_n, T_n^* + \bar{T}_b^m)\|_{\mathbb{V}})b_1(\boldsymbol{u}_n, \bar{T}_b^m, S)\,. \end{aligned} \tag{3.7}$$

Taking the limit of (3.7) when $n \to +\infty$ and arguing as in the continuous case (see step 2 of the proof theorem 2.1), we can show that $\varphi(U_n) \to \varphi(U)$ and the continuity of $\varphi$ follows.

**Coercivity of $\varphi$.** Let $U = (\boldsymbol{u}, T)$, then

$$
\begin{aligned}
&\langle \varphi(U), U \rangle_{Z_p, Z_p'} \\
=&(\boldsymbol{u}, \boldsymbol{u}) + (T, T) + k\nu a_0(\boldsymbol{u}, \boldsymbol{u}) + k\alpha a_1(T, T) + k b_N(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}) \\
&+ kF_N(\|(\boldsymbol{u}, T + \bar{T}_b^m)\|_{\mathbb{V}}) b_1(\boldsymbol{u}, T, T) - (\boldsymbol{u}^{m-1}, \boldsymbol{u}) - (T^{m-1}, T) - (\bar{T}_b^{m-1}, T) \\
&- k \langle \boldsymbol{f}^m, \boldsymbol{u} \rangle_\Omega - k \langle g^m, T \rangle_\Omega + k\alpha a_1(\bar{T}_b^m, T) + kF_N(\|(\boldsymbol{u}, T + \bar{T}_b^m)\|_{\mathbb{V}}) b_1(\boldsymbol{u}, \bar{T}_b^m, T) \\
=& |\boldsymbol{u}|_{\mathbf{H}}^2 + |T|^2 + k\nu \|\boldsymbol{u}\|_{\mathbf{V}}^2 + k\alpha \|T\|^2 - (\boldsymbol{u}^{m-1}, \boldsymbol{u}) - (T^{m-1}, T) - (\bar{T}_b^{m-1}, T) \\
&- k \langle \boldsymbol{f}^m, \boldsymbol{u} \rangle_\Omega - k \langle g^m, T \rangle_\Omega + k\alpha a_1(\bar{T}_b^m, T) + kF_N(\|(\boldsymbol{u}, T + \bar{T}_b^m)\|_{\mathbb{V}}) b_1(\boldsymbol{u}, \bar{T}_b^m, T) \\
\geq& \min\{k\nu, k\alpha\} \left( \|\boldsymbol{u}\|_{\mathbf{V}}^2 + \|T\|^2 \right) - \left\|\boldsymbol{u}^{m-1}\right\|_{\mathbf{V}} \|\boldsymbol{u}\|_{\mathbf{V}} - \left\|T^{m-1}\right\| \|T\| - \left\|\bar{T}_b^{m-1}\right\| \|T\| \\
&- k \|\boldsymbol{f}^m\|_{\mathbf{V}'} \|\boldsymbol{u}\|_{\mathbf{V}} - k \|g^m\|_{-1} \|T\| - k\alpha \left\|\bar{T}_b^m\right\| \|T\| + k \left| F_N(\|(\boldsymbol{u}, T + \bar{T}_b^m)\|_{\mathbb{V}}) b_1(\boldsymbol{u}, \bar{T}_b^m, T) \right| \\
\geq& \min\{k\nu, k\alpha\} \left( \|\boldsymbol{u}\|_{\mathbf{V}}^2 + \|T\|^2 \right) - \left\|\boldsymbol{u}^{m-1}\right\|_{\mathbf{V}} \|\boldsymbol{u}\|_{\mathbf{V}} - \left\|T^{m-1}\right\| \|T\| - c_\Lambda \left\|T_b^{m-1}\right\|_\Gamma \|T\| \\
&- k \|\boldsymbol{f}^m\|_{\mathbf{V}'} \|\boldsymbol{u}\|_{\mathbf{V}} - k \|g^m\|_{-1} \|T\| - k\alpha c_\Lambda \|T_b^m\|_\Gamma \|T\| + \frac{k\epsilon c_\Lambda}{2} \|T_b^m\|_\Gamma \left( \|\boldsymbol{u}\|_{\mathbf{V}}^2 + \|T\|^2 \right) .
\end{aligned}
$$

We choose $\epsilon$ such that $\epsilon k c_\Lambda \|T_b^m\|_\Gamma \leq \min\{k\nu, k\alpha\}$; then

$$
\begin{aligned}
&\langle \varphi(U), U \rangle_{Z_p, Z_p'} \\
\geq& \min\{k\nu, k\alpha\} \left( \|\boldsymbol{u}\|_{\mathbf{V}}^2 + \|T\|^2 \right) - \left\|\boldsymbol{u}^{m-1}\right\|_{\mathbf{V}} \|\boldsymbol{u}\|_{\mathbf{V}} - \left\|T^{m-1}\right\| \|T\| - c_\Lambda \left\|T_b^{m-1}\right\|_\Gamma \|T\| \\
&- k \|\boldsymbol{f}^m\|_{\mathbf{V}'} \|\boldsymbol{u}\|_{\mathbf{V}} - k \|g^m\|_{-1} \|T\| - k\alpha c_\Lambda \|T_b^m\|_\Gamma \|T\| .
\end{aligned}
$$

Using now the fact that $a \leq (a^2 + b^2)^{1/2}$ for all $a, b \in \mathbb{R}, a \geq 0$, we have

$$
\begin{aligned}
&\langle \varphi(U), U \rangle_{Z_p, Z_p'} \\
\geq& \min\{k\nu, k\alpha\} \left( \|\boldsymbol{u}\|_{\mathbf{V}}^2 + \|T\|^2 \right) - \left( \|\boldsymbol{u}\|_{\mathbf{V}}^2 + \|T\|^2 \right)^{1/2} \times \\
&\left\{ \left\|\boldsymbol{u}^{m-1}\right\|_{\mathbf{V}} - \left\|T^{m-1}\right\| - c_\Lambda \left\|T_b^{m-1}\right\|_\Gamma - k \|\boldsymbol{f}^m\|_{\mathbf{V}'} - k \|g^m\|_{-1} - k\alpha c_\Lambda \|T_b^m\|_\Gamma \right\} \\
=& \left( \|\boldsymbol{u}\|_{\mathbf{V}}^2 + \|T\|^2 \right)^{1/2} \left\{ \min\{k\nu, k\alpha\} \left( \|\boldsymbol{u}\|_{\mathbf{V}}^2 + \|T\|^2 \right)^{1/2} \right\} \\
&- \left( \|\boldsymbol{u}\|_{\mathbf{V}}^2 + \|T\|^2 \right)^{1/2} \left\{ \left\|\boldsymbol{u}^{m-1}\right\|_{\mathbf{V}} + \left\|T^{m-1}\right\| + c_\Lambda \left\|T_b^{m-1}\right\|_\Gamma \right\} \\
&- \left\{ k \|\boldsymbol{f}^m\|_{\mathbf{V}'} + k \|g^m\|_{-1} + k\alpha c_\Lambda \|T_b^m\|_\Gamma \right\} \left( \|\boldsymbol{u}\|_{\mathbf{V}}^2 + \|T\|^2 \right)^{1/2} .
\end{aligned}
$$

So, $\langle \varphi(U), U \rangle_{Z_p, Z_p'}$ is nonnegative on the sphere of $\mathbf{V} \times H_0^1(\Omega)$ with radius

$$
\alpha \geq \frac{2}{\min\{k\nu, k\alpha\}} \left\{ \left\|\boldsymbol{u}^{m-1}\right\|_{\mathbf{V}} + \left\|T^{m-1}\right\| + c_\Lambda \left\|T_b^{m-1}\right\|_\Gamma + k \|\boldsymbol{f}^m\|_{\mathbf{V}'} + k \|g^m\|_{-1} + k\alpha c_\Lambda \|T_b^m\|_\Gamma \right\} .
$$

Then we deduce the existence of $(\boldsymbol{u}_p^m, T_p^{*p}) \in Z_p$, solution of (3.5).

Step 2: Some a priori estimates.

At this step, we recall that $k$ and $m$ are kept fixed, and we want to obtain a priori estimates on $(\boldsymbol{u}_p^m, T_p^{*m})$ independently of $p$ and then pass to the limit on (3.5) as $p$ goes to the infinity. Taking the inner product of $(3.5)_4$ with $2\boldsymbol{u}_p^m$ and Young's inequality, we have

$$\left|\boldsymbol{u}_p^m\right|_{\mathbf{H}}^2+\left|\boldsymbol{u}_p^m-\boldsymbol{u}_p^{m-1}\right|_{\mathbf{H}}^2+k\nu\left\|\boldsymbol{u}_p^m\right\|_{\mathbf{V}}^2\leq\left|\boldsymbol{u}_p^{m-1}\right|_{\mathbf{H}}^2+\frac{kc_1^2}{\nu}\left\|\boldsymbol{f}^m\right\|_{\mathbf{V}'}^2\,. \tag{3.8}$$

Similarly, taking the inner product of $(3.5)_5$ with $2T_p^{*m}$, we have

$$\left|T_p^{*m}\right|^2+\left|T_p^{*m}-T_p^{m-1}\right|^2+k\alpha\left\|T_p^m\right\|^2\leq\left|T_p^{m-1}\right|^2+\frac{4kc_2^2}{\alpha}\left\|g^m\right\|_{-1}^2+4k\alpha c_4^2\left\|\bar{T}_b^m\right\|^2+\frac{2N^2c_5^2}{\alpha}\mathcal{K}_6\,, \tag{3.9}$$

with $\mathcal{K}_6=\left|\boldsymbol{u}_p^{m-1}\right|_{\mathbf{H}}^2+\frac{kc_1^2}{\nu}\left\|\boldsymbol{f}^m\right\|_{\mathbf{V}'}^2$. Now, we take the inner product of $(3.5)_4$ with $\mathcal{A}\boldsymbol{u}_p^m$ and of $(3.5)_5$ with $\mathcal{A}_1T_p^{*m}$. One obtains

$$\begin{aligned}
k\nu\left|\mathcal{A}\boldsymbol{u}_p^m\right|^2\leq&k\left\langle\boldsymbol{f}^m,\mathcal{A}\boldsymbol{u}_p^m\right\rangle_\Omega-F_N(\left\|\boldsymbol{u}_p^m\right\|_{\mathbf{V}})b(\boldsymbol{u}_p^m,\boldsymbol{u}_p^m,\mathcal{A}\boldsymbol{u}_p^m)-(\boldsymbol{u}_p^m-\boldsymbol{u}_p^{m-1},\mathcal{A}\boldsymbol{u}_p^m)\,,\\
\alpha\left|\mathcal{A}_1T_p^{*m}\right|^2\leq&k\left\langle g^m,\mathcal{A}_1T_p^{*m}\right\rangle_\Omega-(T_p^{*m}-T_p^{m-1},\mathcal{A}_1T_p^{*m})-(\bar{T}_b^{m-1},\mathcal{A}_1T_p^{*m})\\
&-k\alpha(\mathcal{A}_1\bar{T}_b^m,\mathcal{A}_1T_p^{*m})-kF_N(\left\|(\boldsymbol{u}_p^m,\bar{T}_b^m+T_p^{*m})\right\|_{\mathbb{V}})b_1(\boldsymbol{u}_p^m,\bar{T}_b^m,\mathcal{A}_1T_p^{*m})\\
&-kF_N(\left\|(\boldsymbol{u}_p^m,\bar{T}_b^m+T_p^{*m})\right\|_{\mathbb{V}})b_1(\boldsymbol{u}_p^m,T_p^{*m},\mathcal{A}_1T_p^{*m})\,.
\end{aligned} \tag{3.10}$$

This leads to

$$\begin{aligned}
k\nu\left|\mathcal{A}\boldsymbol{u}_p^m\right|^2\leq&4kc^2\left\|\boldsymbol{f}^m\right\|_{\mathbf{V}'}^2+\frac{3kc_b^4N^4}{\nu}\left\|\boldsymbol{u}_p^m\right\|_{\mathbf{V}}^2+\frac{2}{k\nu}\left|\boldsymbol{u}_p^m-\boldsymbol{u}_p^{m-1}\right|_{\mathbf{H}}^2\,,\\
\alpha\left|\mathcal{A}_1T_p^{*m}\right|^2\leq&8kc_1^2\left\|g^m\right\|_{-1}^2+\frac{8c_2^2}{k\alpha}\left|T_p^{*m}-T_p^{m-1}\right|^2+\frac{9kc_b^4N^4}{\alpha}\mathcal{K}_6+8\alpha kc_3^2\left|\mathcal{A}_1\bar{T}_b^m\right|^2\\
&+\frac{4}{k\alpha}\left|\mathcal{A}_1\bar{T}_b^{m-1}\right|^2\,.
\end{aligned} \tag{3.11}$$

Since $k$ and $m$ are kept fixed, we conclude from (3.11) that $\left\{(\boldsymbol{u}_p^m,T_p^{*m})\right\}_p$ is bounded in $\mathcal{D}(\mathcal{A})\times\mathcal{D}(\mathcal{A}_1)$. As in the continuous case, we can extract a subsequence of $\left\{(\boldsymbol{u}_p^m,T_p^{*m})\right\}_p$ still noted $\left\{(\boldsymbol{u}_p^m,T_p^{*m})\right\}_p$ such that

$$(\boldsymbol{u}_p^m,T_p^{*m})\to(\boldsymbol{u}^m,T^{*m})\begin{cases}\text{weakly in } L^2(0,\widetilde{T};\mathcal{D}(\mathcal{A})\times\mathcal{D}(\mathcal{A}_1)),\\ \text{strongly in } L^2(0,\widetilde{T};\mathbf{V}\times H_0^1(\Omega))\,.\end{cases} \tag{3.12}$$

Arguing as in the continuous case, we can prove that $(\boldsymbol{u}^m,T^{*m})$ is the solution of problem (3.3).

□

## 3.3 Stability of the Numerical scheme

The objectives here are twofold. First, we follow [20] by computing some a priori estimates on $(u^m,T^m)$, solution of problem (3.1). We would like these estimates to be uniform with respect to $m$ and $k$. In fact, discretization in time of evolution equations can lead to unstable or conditionally stable schemes. Hence the importance of having uniform estimates with respect to approximation parameter. Next, we use the a priori estimates to deduce the unique solvability of (3.1).
We first claim that

**Lemma 3.1**

$$|\boldsymbol{u}^m|_{\boldsymbol{H}}^2\leq|\boldsymbol{u}_0|_{\boldsymbol{H}}^2+\frac{1}{\nu}\int_0^{\widetilde{T}}\|\boldsymbol{f}(t)\|_{\mathbf{V}'}^2\,dt\,. \tag{3.13}$$

$$k\sum_{m=1}^{M}\|\boldsymbol{u}^m\|_{\boldsymbol{V}}^2 \le \frac{1}{\nu}\left[|\boldsymbol{u}_0|_{\boldsymbol{H}}^2 + \frac{1}{\nu}\int_0^{\widetilde{T}}\|\boldsymbol{f}(t)\|_{\boldsymbol{V}'}^2\,dt\right]. \tag{3.14}$$

$$\sum_{m=1}^{M}\left|\boldsymbol{u}^m - \boldsymbol{u}^{m-1}\right|_{\boldsymbol{H}}^2 \le |\boldsymbol{u}_0|_{\boldsymbol{H}}^2 + \frac{1}{\nu}\int_0^{\widetilde{T}}\|\boldsymbol{f}(t)\|_{\boldsymbol{V}'}^2\,dt. \tag{3.15}$$

*The quantity* $k\sum_{m=1}^{M}\left\|\dfrac{\boldsymbol{u}^m-\boldsymbol{u}^{m-1}}{k}\right\|_{\boldsymbol{V}'}^2$ *is bounded independently of* $m$ *and* $k$ *.*
*Similarly,*

$$|T^m|^2 \le |T_0|^2 + \frac{1}{\alpha}\int_0^{\widetilde{T}}\|g(t)\|_{-1}^2\,dt. \tag{3.16}$$

$$k\sum_{m=1}^{M}\|T^m\|^2 \le \frac{1}{\alpha}\left[|T_0|^2 + \frac{1}{\alpha}\int_0^{\widetilde{T}}\|g(t)\|_{-1}^2\,dt\right]. \tag{3.17}$$

$$\sum_{m=1}^{M}\left|T^m - T^{m-1}\right|^2 \le |T_0|^2 + \frac{1}{\alpha}\int_0^{\widetilde{T}}\|g(t)\|_{-1}^2\,dt. \tag{3.18}$$

*The quantity* $k\sum_{m=1}^{M}\left\|\dfrac{T^m-T^{m-1}}{k}\right\|_{\boldsymbol{V}_1'}^2$ *is bounded independently of* $m$ *and* $k$*.*

**Proof** Taking the inner product of $(3.1)_3$ with $2u^m$ and using Young's inequality, we have

$$|\boldsymbol{u}^m|_{\mathbf{H}}^2 + \left|\boldsymbol{u}^{m-1}\right|_{\mathbf{H}}^2 + \left|\boldsymbol{u}^m - \boldsymbol{u}^{m-1}\right|_{\mathbf{H}}^2 + k\nu\,\|\boldsymbol{u}^m\|_{\mathbf{V}}^2 \le \frac{k}{\nu}\,\|\boldsymbol{f}^m\|_{\mathbf{V}'}^2$$

Summing this inequality over $m$, we obtain

$$|\boldsymbol{u}^m|_{\mathbf{H}}^2 + \sum_{i=1}^{m}\left|\boldsymbol{u}^i - \boldsymbol{u}^{i-1}\right|_{\mathbf{H}}^2 + k\nu\sum_{i=1}^{m}\left\|\boldsymbol{u}^i\right\|_{\mathbf{V}}^2 \le |\boldsymbol{u}_0|_{\mathbf{H}}^2 + \frac{k}{\nu}\sum_{i=1}^{m}\left\|\boldsymbol{f}^i\right\|_{\mathbf{V}'}^2. \tag{3.19}$$

We now would like to estimate the right hand side of (3.19).

$$\begin{aligned}
\left\|\boldsymbol{f}^i\right\|_{\mathbf{V}'}^2 &\le \frac{1}{k^2}\left[\int_{(i-1)k}^{ik}\|\boldsymbol{f}(t)\|_{\mathbf{V}'}\,dt\right]^2 \\
&\le \frac{1}{k^2}\left[\left(\int_{(i-1)k}^{ik}\|\boldsymbol{f}(t)\|_{\mathbf{V}'}^2\,dt\right)^{1/2}\left(\int_{(i-1)k}^{ik}dt\right)^{1/2}\right]^2 \\
&= \frac{1}{k}\int_{(i-1)k}^{ik}\|\boldsymbol{f}(t)\|_{\mathbf{V}'}^2\,dt.
\end{aligned}$$

Hence

$$\frac{k}{\nu}\sum_{i=1}^{m}\left\|\boldsymbol{f}^i\right\|_{\mathbf{V}'}^2 \le \sum_{i=1}^{M}\int_{(i-1)k}^{ik}\|\boldsymbol{f}(t)\|_{\mathbf{V}'}^2\,dt \le \int_0^{\widetilde{T}}\|\boldsymbol{f}(t)\|_{\mathbf{V}'}^2\,dt. \tag{3.20}$$

Then (3.13), (3.14) and (3.15) follow. In addition, taking the norm in $\mathbf{V}'$ of $(3.1)_3$, we obtain

$$\begin{aligned}
\left\|\frac{\boldsymbol{u}^m-\boldsymbol{u}^{m-1}}{k}\right\|_{\mathbf{V}'} &\le \|\boldsymbol{f}^m\|_{\mathbf{V}'} + \nu\,\|\mathcal{A}\boldsymbol{u}^m\|_{\mathbf{V}'} + \|\mathcal{B}_N(\boldsymbol{u}^m,\boldsymbol{u}^m)\|_{\mathbf{V}'} \\
&\le \|\boldsymbol{f}^m\|_{\mathbf{V}'} + (c_bN + \nu_2 c)\,\|\boldsymbol{u}^m\|_{\mathbf{V}}.
\end{aligned}$$

This leads to

$$\left\| \frac{\boldsymbol{u}^m - \boldsymbol{u}^{m-1}}{k} \right\|^2_{\mathbf{V}'} \leq 2 \left\| \boldsymbol{f}^m \right\|^2_{\mathbf{V}'} + C' \left\| \boldsymbol{u}^m \right\|^2_{\mathbf{V}}$$

where we have used the inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p),\ a \geq 0;\ b \geq 0;$ $1 \leq p < \infty,$ ([1], Lemma 2.24). So, we conclude that

$$\begin{aligned} k \sum_{m=1}^{M} \left\| \frac{\boldsymbol{u}^m - \boldsymbol{u}^{m-1}}{k} \right\|^2_{\mathbf{V}'} &\leq 2k \sum_{m=1}^{M} \left\| \boldsymbol{f}^m \right\|^2_{\mathbf{V}'} + 2C'k \sum_{m=1}^{M} \left\| \boldsymbol{u}^m \right\|^2_{\mathbf{V}} \\ &\leq 2 \int_0^{\widetilde{T}} \left\| \boldsymbol{f}(t) \right\|^2_{\mathbf{V}'} dt + \frac{2C'}{\nu} \left[ |\boldsymbol{u}_0|^2_{\mathbf{H}} + \frac{1}{\nu} \int_0^{\widetilde{T}} |||\boldsymbol{f}(t)|||^2_{-1} \, dt \right] \\ &= 2 \left[ 1 + \tfrac{C'}{\nu^2} \right] \int_0^{\widetilde{T}} \left\| \boldsymbol{f}(t) \right\|^2_{\mathbf{V}'} dt + \frac{2C'}{\nu} |\boldsymbol{u}_0|^2_{\mathbf{H}} \,. \end{aligned}$$

Similarly, we take the inner product of $(3.1)_4$ with $2T^m$ and use Young's inequality to obtain

$$|T^m|^2 + \left|T^{m-1}\right|^2 + \left|T^m - T^{m-1}\right|^2 + k\alpha \left\|T^m\right\|^2 \leq \frac{k}{\alpha} \left\|g^m\right\|^2_{-1} \,.$$

Summing this inequality over m we have

$$|T^m|^2 + \sum_{i=1}^{m} \left|T^i - T^{i-1}\right|^2 + k\alpha \sum_{i=1}^{m} \left\|T^i\right\|^2 \leq |T_0|^2 + \frac{k}{\alpha} \sum_{i=1}^{m} \left\|g^i\right\|^2_{-1} \,. \tag{3.21}$$

As before, the right hand side of (3.21) gives

$$\begin{aligned} \left\|g^i\right\|^2_{-1} &\leq \frac{1}{k^2} \left[ \int_{(i-1)k}^{ik} \left\|g(t)\right\|_{-1} dt \right]^2 \\ &\leq \frac{1}{k^2} \left[ \left( \int_{(i-1)k}^{ik} \left\|g(t)\right\|^2_{-1} dt \right)^{1/2} \left( \int_{(i-1)k}^{ik} dt \right)^{1/2} \right]^2 \\ &= \frac{1}{k} \int_{(i-1)k}^{ik} \left\|g(t)\right\|^2_{-1} dt \,. \end{aligned}$$

Hence

$$\frac{k}{\alpha} \sum_{i=1}^{m} \left\|g^i\right\|^2_{-1} \leq \sum_{i=1}^{M} \int_{(i-1)k}^{ik} \left\|g(t)\right\|^2_{-1} dt \leq \int_0^{\widetilde{T}} \left\|g(t)\right\|^2_{-1} dt \,. \tag{3.22}$$

Then (3.16), (3.17) and (3.18) follow. In addition, taking the norm in $H^{-1}(\Omega)$ of (3.1), we obtain

$$\begin{aligned} \left\| \frac{T^m - T^{m-1}}{k} \right\|_{V_1'} &\leq \left\|g^m\right\|_{-1} + \alpha \left\|\mathcal{A}_1 T^m\right\|_{V_1'} + \left\|\mathcal{B}_{N1}(\boldsymbol{u}^m, T^m)\right\|_{V_1'} \\ &\leq \left\|g^m\right\|_{-1} + (c_b N + \alpha c) \left\|T^m\right\| \,. \end{aligned}$$

From which we have

$$\left\| \frac{T^m - T^{m-1}}{k} \right\|^2_{V_1'} \leq 2 \left\|g^m\right\|^2_{-1} + 2C' \left\|T^m\right\|^2_{\mathbf{V}} \,.$$

Then

$$\begin{aligned}
k \sum_{m=1}^{M} \left\| \frac{T^m - T^{m-1}}{k} \right\|_{V_1'}^2 &\leq 2k \sum_{m=1}^{M} \|g^m\|_{-1}^2 + 2C'k \sum_{m=1}^{M} \|T^m\|_{\mathbf{V}}^2 \\
&\leq 2 \int_0^{\widetilde{T}} \|g(t)\|_{-1}^2 \, dt + \frac{2C'}{\alpha} \left[ |T_0|^2 + \frac{1}{\alpha} \int_0^{\widetilde{T}} \|g(t)\|_{-1}^2 \, dt \right] \\
&= 2 \left[ 1 + \tfrac{C'}{\alpha^2} \right] \int_0^{\widetilde{T}} \|g(t)\|_{-1}^2 \, dt + \frac{2C'}{\alpha} |T_0|^2 \, .
\end{aligned}$$

This ends the proof of Lemma 3.1 □

We need additional preparations to state the stability result. We recall from [20] the following definition.

**Definition 3.1** *An infinite set of functions $E$ is called $L^p(0,T;X)$ stable if and only if $E$ is a bounded subset of $L^p(0,T;X)$.*

Let us introduce the approximate functions

$$\begin{array}{rcll}
\boldsymbol{u}_k : & [0,\widetilde{T}] & \longrightarrow & \mathbf{V} \\
& t & \longmapsto & \boldsymbol{u}_k(t) = \boldsymbol{u}^m \text{ pour } t \in [(m-1)k, mk], \; m = 1, ..., N \, ,
\end{array}$$

and

$$\begin{array}{rcll}
T_k : & [0,\widetilde{T}] & \longrightarrow & V_1 \\
& t & \longmapsto & T_k(t) = T^m \text{ pour } t \in [(m-1)k, mk], \; m = 1, ..., N \, .
\end{array}$$

Then we have the following stability result.

**Theorem 3.2** *The functions $\boldsymbol{u}_k$ and $T_k$ are respectively $L^\infty(0,\widetilde{T};\boldsymbol{H}) \cap L^2(0,\widetilde{T};\, \boldsymbol{V})$ and $L^\infty(0,\widetilde{T};H_1) \cap L^2(0,\widetilde{T};V_1)$ stable.*

**Proof** Due to Lemma3.1, we have

$$\sup_{t \in [0,\widetilde{T}]} |\boldsymbol{u}_k|_{\mathbf{H}} \leq \left( |\boldsymbol{u}_0|_{\mathbf{H}}^2 + \frac{1}{\nu} \int_0^{\widetilde{T}} \|\boldsymbol{f}(t)\|_{\mathbf{V}'}^2 \, dt \right)^{1/2} ,$$

$$\int_0^{\widetilde{T}} \|\boldsymbol{u}_k(t)\|_{\mathbf{V}}^2 \, dt \leq \frac{1}{\nu} \left[ |\boldsymbol{u}_0|_{\mathbf{H}}^2 + \frac{1}{\nu} \int_0^{\widetilde{T}} \|\boldsymbol{f}(t)\|_{\mathbf{V}'}^2 \, dt \right] ,$$

$$\sup_{t \in [0,\widetilde{T}]} |T_k| \leq \left( |T_0|^2 + \frac{1}{\alpha} \int_0^{\widetilde{T}} \|g(t)\|_{-1}^2 \, dt \right)^{1/2} ,$$

$$\int_0^{\widetilde{T}} \|T_k(t)\|^2 \, dt \leq \frac{1}{\alpha} \left[ |T_0|^2 + \frac{1}{\alpha} \int_0^{\widetilde{T}} \|g(t)\|_{-1}^2 \, dt \right] .$$

Then, the theorem is proved. □

**Theorem 3.3** *Under the assumptions in theorem 3.1, and assuming that the discretization parameter $k$ is such that*

$$k < \min \left\{ \frac{\nu}{3(Nc_b)^4} \; , \; \frac{\alpha}{3(Nc_b)^4} \right\} , \tag{3.23}$$

*then is valid the problem (3.3) has only one solution $(\boldsymbol{u}^m, T^{*m}) \in \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}_1)$.*

**Proof.** It will be enough to prove that problem (3.1) has exactly one solution $(\boldsymbol{u}^m, T^m) \in \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}_1)$. Let $(\boldsymbol{u}_1^m, \boldsymbol{T}_1^m)$ and $(\boldsymbol{u}_2^m, \boldsymbol{T}_2^m)$ two weak solutions of (3.1), we set $\boldsymbol{u}^m = \boldsymbol{u}_1^m - \boldsymbol{u}_2^m$ and $T^m = T_1^m - T_2^m$, then $\boldsymbol{u}^m$ and $T^m$ satisfy

$$\begin{cases} \boldsymbol{u}^m - \boldsymbol{u}^{m-1} + k\nu\mathcal{A}\boldsymbol{u}^m = -kB_N(\boldsymbol{u}_1^m, \boldsymbol{u}_1^m) + kB_N(\boldsymbol{u}_2^m, \boldsymbol{u}_2^m) \\ T^m - T^{m-1} + k\alpha\mathcal{A}_1 T^m = -kB_{N,1}(\boldsymbol{u}_1^m, T_1^m) + kB_{N,1}(\boldsymbol{u}_2^m, T_2^m) \\ (\boldsymbol{u}^m(0), T^m(0)) = (\mathbf{0}, 0)\,. \end{cases} \tag{3.24}$$

Or for all $(\boldsymbol{v}, S) \in \mathbb{H}_0^1(\Omega) \times H_0^1(\Omega)$,

$$\begin{cases} \left\langle \boldsymbol{u}^m - \boldsymbol{u}^{m-1}, \boldsymbol{v}\right\rangle + k\nu(\mathcal{A}\boldsymbol{u}^m, \boldsymbol{v}) = -kb_N(\boldsymbol{u}_1^m, \boldsymbol{u}_1^m, \boldsymbol{v}) + kb_N(\boldsymbol{u}_2^m, \boldsymbol{u}_2^m, \boldsymbol{v}) \\ \left\langle\, T^m - T^{m-1}, S\right\rangle + k\alpha(\mathcal{A}_1 T^m, S) = -kb_{N,1}(\boldsymbol{u}_1^m, T_1^m, S) + kb_{N,1}(\boldsymbol{u}_2^m, T_2^m, S) \\ (\boldsymbol{u}^m(0), T^m(0)) = (\mathbf{0}, 0). \end{cases} \tag{3.25}$$

Taking $\boldsymbol{v} = \boldsymbol{u}^m$ in $(3.25)_1$ and $S = T^m$ in $(3.25)_2$, we have

$$\begin{cases} |\boldsymbol{u}^m|_{\mathbf{H}}^2 + k\nu\,\|\boldsymbol{u}^m\|_{\mathbf{V}}^2 \le -kb_N(\boldsymbol{u}_1^m, \boldsymbol{u}_1^m, \boldsymbol{u}^m) + kb_N(\boldsymbol{u}_2^m, \boldsymbol{u}_2^m, \boldsymbol{u}^m) + |\boldsymbol{u}^m|_{\mathbf{H}}\left|\boldsymbol{u}^{m-1}\right|_{\mathbf{H}} \\ |T^m|^2 + k\alpha\,\|T^m\|^2 \le -kb_{N,1}(\boldsymbol{u}_1^m, T_1^m, T^m) + kb_{N,1}(\boldsymbol{u}_2^m, T_2^m, T^m) + |T^m|\left|T^{m-1}\right| \\ (\boldsymbol{u}^m(0), T^m(0)) = (\mathbf{0}, 0)\,. \end{cases} \tag{3.26}$$

Reasoning as deriving (2.32)-(2.35), we infer from (3.26) that

$$\begin{cases} |\boldsymbol{u}^m|_{\mathbf{H}}^2 + k\nu\,\|\boldsymbol{u}^m\|_{\mathbf{V}}^2 \le \frac{3k(Nc_b)^4}{2\nu}\,|\boldsymbol{u}^m|_{\mathbf{H}}^2 + \frac{k\nu}{2}\,\|\boldsymbol{u}^m\|_{\mathbf{V}}^2 + \frac{1}{2}\,|\boldsymbol{u}^m|_{\mathbf{H}}^2 + \frac{1}{2}\left|\boldsymbol{u}^{m-1}\right|_{\mathbf{H}}^2. \\ |T^m|^2 + k\alpha\,\|T^m\|^2 \le \frac{k\alpha}{2}\,\|\boldsymbol{u}^m\|_{\mathbf{V}}^2 + \frac{3k(Nc_b)^4}{2\alpha}\,|T^m|^2 + \frac{k\alpha}{2}\,\|T^m\|^2 + \frac{1}{2}\,|T^m|^2 + \frac{1}{2}\left|T^{m-1}\right|^2. \\ (\boldsymbol{u}^m(0), T^m(0)) = (\mathbf{0}, 0)\,. \end{cases} \tag{3.27}$$

Then,

$$\begin{cases} |\boldsymbol{u}^m|_{\mathbf{H}}^2 + k\nu\,\|\boldsymbol{u}^m\|_{\mathbf{V}}^2 \le \frac{3k(Nc_b)^4}{\nu}\,|\boldsymbol{u}^m|_{\mathbf{H}}^2 + \left|\boldsymbol{u}^{m-1}\right|_{\mathbf{H}}^2. \\ |T^m|^2 + k\alpha\,\|T^m\|^2 \le k\alpha\,\|\boldsymbol{u}^m\|_{\mathbf{V}}^2 + \frac{3k(Nc_b)^4}{\alpha}\,|T^m|^2 + \left|T^{m-1}\right|^2. \\ (\boldsymbol{u}^m(0), T^m(0)) = (\mathbf{0}, 0) \end{cases} \tag{3.28}$$

At this stage, we continue the proof by induction on the space's dimension $m$. First, we mention that for $m = 0$, we just have $\boldsymbol{u}_0$ and $T_0$.
Now, let $m = 1$. Using it in (3.28) and taking into account the fact that $\boldsymbol{u}^0 = T^0 = 0$, one obtains

$$\begin{cases} \left|\boldsymbol{u}^1\right|_{\mathbf{H}}^2\left(1 - \frac{3k(Nc_b)^4}{\nu}\right) + k\nu\left\|\boldsymbol{u}^1\right\|_{\mathbf{V}}^2 \le 0\,. \\ \left|T^1\right|^2\left(1 - \frac{3k(Nc_b)^4}{\alpha}\right) + k\alpha\left\|T^1\right\|^2 \le k\alpha\left\|\boldsymbol{u}^1\right\|_{\mathbf{V}}^2. \end{cases} \tag{3.29}$$

Then, $\boldsymbol{u}_1^1 = \boldsymbol{u}_2^1$ and $T_1^1 = T_2^1$ since (3.23) holds.
On the other hand, we suppose that the solution of problem (3.1) is unique for $p = 1, 2, 3, ..., m$ and we want to prove that it remains unique for $p = m + 1$.
We recall that $(\boldsymbol{u}^{m+1}, T^{m+1})$ verifies

$$\begin{cases} \left|\boldsymbol{u}^{m+1}\right|_{\mathbf{H}}^2 + k\nu\left\|\boldsymbol{u}^{m+1}\right\|_{\mathbf{V}}^2 \le \frac{3k(Nc_b)^4}{\nu}\left|\boldsymbol{u}^{m+1}\right|_{\mathbf{H}}^2 + |\boldsymbol{u}^m|_{\mathbf{H}}^2. \\ \left|T^{m+1}\right|^2 + k\alpha\left\|T^{m+1}\right\|^2 \le k\alpha\left\|\boldsymbol{u}^{m+1}\right\|_{\mathbf{V}}^2 + \frac{3k(Nc_b)^4}{\alpha}\left|T^{m+1}\right|^2 + |T^m|^2. \\ (\boldsymbol{u}^{m+1}(0), T^{m+1}(0)) = (\mathbf{0}, 0)\,. \end{cases} \tag{3.30}$$

By the induction hypothesis $\boldsymbol{u}^m = T^m = 0$, we then infer from (3.30) that

$$\begin{cases} \left|\boldsymbol{u}^{m+1}\right|_{\mathbf{H}}^2 \left(1 - \frac{3k(Nc_b)^4}{\nu}\right) + k\nu \left\|\boldsymbol{u}^{m+1}\right\|_{\mathbf{V}}^2 \leq 0\,. \\ \left|T^{m+1}\right|^2 \left(1 - \frac{3k(Nc_b)^4}{\alpha}\right) + k\alpha \left\|T^{m+1}\right\|^2 \leq k\alpha \left\|\boldsymbol{u}^{m+1}\right\|_{\mathbf{V}}^2\,. \end{cases}$$

Using again (3.23), we obtain $\boldsymbol{u}_1^{m+1} = \boldsymbol{u}_2^{m+1}$ and $T_1^{m+1} = T_2^{m+1}$; this end the proof of Theorem 3.3 . $\square$

**Remark 3.1** *The condition (3.23) for uniqueness is restrictive, but we are all aware that for nonlinear problems, uniqueness in general is not guaranteed without restrictions. On the other hand even for Navier Stokes, there is a restriction on the discretization parameter in order ensure uniqueness (see [20]). Hence having (3.23) is not surprising.*

**Appendix**

**Theorem 3.4** *The application*

$$\begin{array}{lll} F:\boldsymbol{V}\times H_0^1(\Omega) & \rightarrow & \boldsymbol{V'}\times H^{-1}(\Omega) \\ (\boldsymbol{w},z) & \mapsto & (F_1(\boldsymbol{w},z),F_2(\boldsymbol{w},z)) \end{array}$$

*is locally lipschitz-continuous with*

$$F_1(\boldsymbol{w},z) = f - \nu\mathcal{A}\boldsymbol{w} - \mathcal{B}_N(\boldsymbol{w},\boldsymbol{w}) \ ,$$

$$F_2(\boldsymbol{w},z) = g - \frac{d}{dt}\overline{T}_b - \alpha\mathcal{A}_1\overline{T}_b - F_N(\left\|(\boldsymbol{w},z+\bar{T}_b)\right\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{w},\overline{T}_b) - F_N(\left\|(\boldsymbol{w},z+\bar{T}_b)\right\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{w},z) - \alpha\mathcal{A}_1 z.$$

**Proof** It is enough to show that $F_2$ is locally lipschitz-continuous in $\mathbf{V}\times H_0^1(\Omega)$.
Let $(\boldsymbol{w}_1,z_1),\ (\boldsymbol{w}_2,z_2)\in\mathbf{V}\times H_0^1(\Omega)$, we set $\boldsymbol{w}=\boldsymbol{w}_1-\boldsymbol{w}_2,\ z=(z_1-z_2)$, we look for a positive constants $C_2$ such that

$$\|F_2(\boldsymbol{w}_1,z_1)-F_2(\boldsymbol{w}_2,z_2)\|_{H^{-1}(\Omega)} \leq C_2\,\|(\boldsymbol{w},z)\|_{\mathbb{V}} \ .$$

We have

$$\begin{aligned} F_2(\boldsymbol{w}_1,z_1)-F_2(\boldsymbol{w}_2,z_2) &= -F_N(\left\|(\boldsymbol{w}_1,z_1+\bar{T}_b)\right\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{w}_1,z_1)+F_N(\left\|(\boldsymbol{w}_2,z_2+\bar{T}_b)\right\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{w}_2,z_2) \\ &\quad -F_N(\left\|(\boldsymbol{w}_1,z_1+\bar{T}_b)\right\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{w},\overline{T}_b)+F_N(\left\|(\boldsymbol{w}_2,z_2+\bar{T}_b)\right\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{w}_2,\bar{T}_b)- \\ &\quad \alpha\mathcal{A}_1 z_1+\alpha\mathcal{A}_1 z_2 \end{aligned} \tag{3.31}$$

Using (3.31) and arguing like proving the uniqueness result, we obtain

$$\begin{aligned} &\left|(-F_N(\left\|(\boldsymbol{w}_1,z_1+\bar{T}_b)\right\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{w}_1,z_1)+F_N(\left\|(\boldsymbol{w}_2,z_2+\bar{T}_b)\right\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{w}_2,z_2),z)\right| \\ &\leq Nc_b\,\|\boldsymbol{w}\|_{\mathbf{V}}\,\|z\|+Nc_b\,\|(\boldsymbol{w},z)\|_{\mathbb{V}}\,\|z\|\,; \end{aligned}$$

$$\begin{aligned} &\left|(-F_N(\left\|(\boldsymbol{w}_1,z_1+\bar{T}_b)\right\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{w},\overline{T}_b)+F_N(\left\|(\boldsymbol{w}_2,z_2+\bar{T}_b)\right\|_{\mathbb{V}})\mathcal{B}_1(\boldsymbol{w}_2,\bar{T}_b),z)\right| \\ &\leq Nc_b\,\|\boldsymbol{w}\|_{\mathbf{V}}\,\|z\|+Nc_b\,\|(\boldsymbol{w},z)\|_{\mathbb{V}}\,\|z\|\,; \end{aligned}$$

and $|(-\alpha\mathcal{A}_1 z_1+\alpha\mathcal{A}_1 z_2,z)|\leq\alpha\,\|z\|^2$. Then,
$|((F_2(\boldsymbol{w}_1,z_1)-F_2(\boldsymbol{w}_2,z_2)),z)|\leq(4Nc_b+\alpha)\,\|(\boldsymbol{w},z)\|_{\mathbb{V}}\,\|z\|$; consequently

$$\|F_2(\boldsymbol{w}_1,z_1)-F_2(\boldsymbol{w}_2,z_2)\|_{H^{-1}(\Omega)}\leq(4Nc_b+\alpha)\,\|(\boldsymbol{w},z)\|_{\mathbb{V}}$$

and we take $C_2=4Nc_b+\alpha$. □

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