The asymptotic numerical solution of highly oscillatory second-order differential equations

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Abstract

The paper is concerned with what is sometimes called "intrinsic oscillation", namely originating in the structure of the differential system itself, and as distinct to "extrinsic oscillation", whereby the oscillation is "pumped" into the system through an inhomogeneous term. This is an important distinction, because the two forms of oscillation are very different. In this paper, we address the highly oscillatory second-order initial value problems of the first type by extending the methods of the second. the asymptotic-numerical solvers for highly oscillatory second-order problems are developed, the error bounds are analyzed, and the accuracy is presented by numerical experiments.

Keywords: high oscillation; second-order ordinary differential equation; asymptotic expansions; extrinsic oscillation; modulated Fourier series.

1. Introduction

The asymptotic-solver for ordinary differential equations with highly oscillatory forcing terms was proved to be very accurate and affordable [2, 3, 4, 5, 6, 7]. In this paper, the asymptotic-numerical solvers are developed for highly oscillatory second-order initial value problem of the form

$$\begin{cases} \ddot{x}(t)+\omega^2 x(t) = g(t,x(t)), \quad t\in[0,T], \\ x(0)=x_0, \quad \dot{x}(0)=\dot{x}_0, \end{cases} \tag{1.1}$$

where $x(t):\mathbb{R}^+\to\mathbb{R}^d$, $\omega \gg 1$ and $g(t,x):\mathbb{R}^+\times\mathbb{R}^d\to\mathbb{R}^d$ are sufficiently differentiable. This is a model of highly oscillatory problems, which arise frequently in celestial mechanics, chemistry, biology, classical and quantum mechanics, and engineering. The integration of such systems has been a numerical challenge for a long time. The highly oscillatory nature of the solutions impose a very small step size on standard numerical methods for ODEs, however, this strategy is not always realistic, since as the step size decreases, the amount of computation will increase rapidly and the round-off error may accumulate enormously to a disaster. A lot of work has been made in efficient integrators for highly oscillatory problems. We are concerned with the special case of (1.1), where the components of $g(t,x):\mathbb{R}^+\times\mathbb{R}^d\to\mathbb{R}^d$ are polynomials in components of x. And we make the important assumption that the bounds of the functions $g(t,x)$ are independent of ω.

The paper is organized as follows. In Section 2, we reformulate the system (1.1) into a first-order system with extrinsic oscillation. In Section 3, we construct asymptotic-numerical solver for highly oscillatory linear system, and in this section, we estimate global error of our proposed solver. In Section 4, we derive the asymptotic method and discuss bound on the asymptotic-numerical solver for nonlinear system. In Section 5, numerical experiments are carried out to show the performance of our proposed methods. A conclusion is included in Section 6.

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2. Transformation of the original problem

We first transform (1.1) into a first-order system with extrinsic oscillation.

Let $\dot{x}(t)=\omega y(t)$, and

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = Ue^{\mathrm{i}\omega Bt}u(t), \tag{2.1}$$

then we have

$$u(t)=e^{-\mathrm{i}\omega Bt}U^{*}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{2.2}$$

with

$$U=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}\otimes I_d, \quad B=\begin{pmatrix} I_d & 0 \\ 0 & -I_d \end{pmatrix}, \tag{2.3}$$

I_d is a d dimensional unit matrix, and U^* is the conjugate transpose of U. After the change of variables, we get a system of first-order ordinary differential equations

$$\begin{aligned} \dot{u}(t) &= \frac{1}{\sqrt{2}}e^{-\mathrm{i}\omega Bt}\omega^{-1}B^{-1}\begin{pmatrix} -\mathrm{i}g(t,x(t)) \\ -g(t,x(t)) \end{pmatrix} \\ &= \frac{\omega^{-1}}{\sqrt{2}}\begin{pmatrix} -\mathrm{i}e^{-\mathrm{i}\omega t}g(t,x) \\ e^{\mathrm{i}\omega t}g(t,x) \end{pmatrix}, \end{aligned} \tag{2.4}$$

with initial values $u(0)=u_0=U^*\begin{pmatrix} x_0 \\ \omega^{-1}\dot{x}_0 \end{pmatrix}$.

Since $g(t,x)$ are polynomials in components of x with sufficiently differentiable t-dependent coefficients, and $x(t)=\frac{1}{\sqrt{2}}(e^{\mathrm{i}\omega t}I_d, \mathrm{i}e^{-\mathrm{i}\omega t}I_d)u(t)$, the functions $g(t,x)$ can be expressed as

$$g(t,x)=\sum_k e^{\mathrm{i}k\omega t}a_k(t)f_k(u), \tag{2.5}$$

where $a_k(t):\mathbb{R}^+\to\mathbb{R}$ are sufficiently differentiable functions and $f_k(u):\mathbb{R}^{2d}\to\mathbb{R}^d$ are polynomials in components of u. Then we can arrange (2.4) into the following initial value problem for u in the form of

$$\dot{u}(t)=\omega^{-1}\sum_m e^{im\omega t}G_m(t,u), \quad u(0)=U^*\begin{pmatrix} x_0 \\ \omega^{-1}\dot{x}_0, \end{pmatrix}, \tag{2.6}$$

where $G_m(t,u):R^+\times\mathbb{R}^{2d}\to\mathbb{R}^{2d}, m\in\mathbb{Z}$ are independent of ω, and the components of them are polynomials in components of u with sufficiently differentiable t-dependent coefficients and independent of ω.

Thus, the system (1.1) is reformed into the first order ordinary differential equations (2.6), which are forced oscillations in the sense of Urabe[11]. The efficient and accurate solutions of systems of ordinary differential equations subject to oscillatory forcing terms have received much attention in recent years [2, 3, 4, 5, 6, 7, 12], and an asymptotic method has been derived in [4, 6] for systems of ODEs of the form $y'(t)=h(y(t))+g_\omega(t)f(y(t)), y(0)=y_0$, where $g_\omega(t)$ can be expressed as a modulated Fourier expansion, that is $g_\omega(t)=\sum_{-\infty}^{\infty}a_m(t)e^{im\omega t}$. This motivates us to derive asymptotic method for highly oscillatory second-ordinary differential equations (2.6). Integrating (2.6) from 0 to t, yields

$$\begin{aligned} u(t) &= u(0)+\omega^{-1}\int_0^t \sum_m e^{im\omega s}G_m(s,u(s))ds \\ &= u(0)+\mathscr{O}(\omega^{-1}), \quad \omega\to\infty. \end{aligned} \tag{2.7}$$

This gives us the idea to suppose the solution $u(t)$ of (2.6) admits an expansion in inverse powers of the oscillatory parameter ω

$$u(t)\sim u_0+\sum_{s=1}^{\infty}\omega^{-s}\psi_s(t), \quad \omega\gg 1, \tag{2.8}$$

where the $\psi_s(t)s$ may depend on ω, for $s \in \mathbb{N}$. Moreover, the solution of the equation (1.1) can be expressed as a modulated Fourier expansion $x(t) = y(t) + \sum_{k\neq 0} e^{ik\omega t} z^k(t)$, where $y(t)$ and $z^k(t)$ together with their derivatives are bounded independent of ω [1, 8, 9]. Given the structure of the ordinary differential equations (2.6), it seems reasonable to assume that the solution of (2.6) can be written in the form of

$$u(t) \sim u_0 + \sum_{s=1}^{\infty} \omega^{-s} \sum_{m=-\infty}^{\infty} e^{im\omega t} p_{sm}(t). \tag{2.9}$$

In order to satisfy the initial conditions, we impose

$$\sum_{m=-\infty}^{\infty} p_{sm}(0) = 0, \quad s \geq 1. \tag{2.10}$$

3. Construction of asymptotic-numerical solver for highly oscillatory linear system

In this section we consider the special case when the perturbation $g(t,x)$ do not depend on x, that is $g(t,x) = g(t)$, the system (1.1) is linear,

$$\ddot{x}(t) + \omega^2 x(t) = g(t), \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \tag{3.1}$$

then the transformed equation (2.4) takes the form

$$\begin{aligned} \dot{u}(t) &= \frac{\omega^{-1}}{\sqrt{2}} \begin{pmatrix} -\mathrm{i}e^{-i\omega t} g(t) \\ e^{i\omega t} g(t) \end{pmatrix} \\ &= \frac{\omega^{-1}}{\sqrt{2}} e^{-i\omega t} \begin{pmatrix} -\mathrm{i}g(t) \\ 0 \end{pmatrix} + \frac{\omega^{-1}}{\sqrt{2}} e^{i\omega t} \begin{pmatrix} 0 \\ g(t) \end{pmatrix}, \quad u(0) = U^* \begin{pmatrix} x_0 \\ \omega^{-1}\dot{x}_0, \end{pmatrix}. \end{aligned} \tag{3.2}$$

As explained in Section 2, we assume that the solution $u(t)$ of the above equation can be written in the form

$$u(t) \sim u(0) + \sum_{s=1}^{\infty} \omega^{-s} \sum_{m} e^{\mathrm{i}m\omega t} p_{sm}(t), \tag{3.3}$$

and

$$\sum_{m=-\infty}^{\infty} p_{sm}(0) = 0, \quad s \geq 1, \tag{3.4}$$

in order to match the initial conditions.

Differentiating (3.3) term by term gives, formally

$$u'(t) \sim \sum_{s=1}^{\infty} \omega^{-s} \sum_{m} [e^{\mathrm{i}m\omega t} p'_{sm}(t) + \mathrm{i}m\omega e^{\mathrm{i}m\omega t} p_{sm}(t)],$$

and inserting it into (3.2), then we can obtain

$$\sum_{s=1}^{\infty} \omega^{-s} \sum_{m} [e^{\mathrm{i}m\omega t} p'_{sm}(t) + im\omega e^{\mathrm{i}m\omega t} p_{sm}(t)] = \frac{\omega^{-1}}{\sqrt{2}} e^{-\mathrm{i}\omega t} \begin{pmatrix} -\mathrm{i}g(t) \\ 0 \end{pmatrix} + \frac{\omega^{-1}}{\sqrt{2}} e^{\mathrm{i}\omega t} \begin{pmatrix} 0 \\ g(t) \end{pmatrix}. \tag{3.5}$$

Comparing the coefficients of the same orders of inverse powers of ω, and then $e^{\mathrm{i}m\omega t}$ within each order of ω, we can obtain the following formulas about the coefficients.

For ω^0, we have:

$$\sum_{m} \mathrm{i}me^{\mathrm{i}m\omega t} p_{1m}(t) = 0.$$

Separation of the values of m yields equations

$$p_{1m}(t) = 0, \quad m \neq 0. \tag{3.6}$$

For ω^{-1}, we have

$$\sum_m [e^{\mathrm{i}m\omega t} p_{1m}'(t) + \mathrm{i}me^{\mathrm{i}m\omega t} p_{2m}(t)] = \frac{1}{\sqrt{2}} e^{-\mathrm{i}\omega t} \begin{pmatrix} -\mathrm{i}g(t) \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} e^{\mathrm{i}\omega t} \begin{pmatrix} 0 \\ g(t) \end{pmatrix},$$

comparing the coefficients of $e^{im\omega t}$, we can obtain

$$p_{2,-1}(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} g(t) \\ 0 \end{pmatrix}, \quad p_{2,1}(t) = \frac{-\mathrm{i}}{\sqrt{2}} \begin{pmatrix} 0 \\ g(t) \end{pmatrix}, \tag{3.7}$$

and

$$p_{1m}'(t) + \mathrm{i}mp_{2m}(t) = 0, \quad m \neq -1, 1.$$

Moreover, when $m = 0$, we have $p_{1,0}'(t) = 0$, additionally with the initial condition $p_{10}(0) = -\sum_{m\neq 0} p_{1m}(0) = 0$, which means $p_{10}(t) \equiv 0$, and

$$p_{2m}(t) = \frac{-\mathrm{i}}{m} p_{1m}'(t) \equiv 0, \quad m \neq -1, 0, 1. \tag{3.8}$$

For ω^{-s}, $s \geq 2$, we have

$$p_{sm}'(t) + \mathrm{i}mp_{s+1,m}(t) = 0,$$

then we can get the following recursions:

(i) If $m \neq 0$:

$$p_{s+1,m}(t) = \frac{-\mathrm{i}}{m} p_{sm}'(t); \tag{3.9}$$

(ii) If $m = 0$: $p_{s0}'(t) = 0$, with initial condition $p_{s0}(0) = -\sum_{m\neq 0} p_{sm}(0)$. Obviously, it leads to $p_{s0}(t) \equiv -\sum_{m\neq 0} p_{sm}(0)$.

This is the scheme that we could find for each value $s \geq 1$, involving all the terms $p_{s,m}(t)$ up to any desired value of s. In the following, we will analyze the number of terms that we need within each order of ω.

Theorem 3.1. *Let* $\theta_s = \max\{m \in Z : p_{s,|m|} \neq 0\}$, *we have*

$$\theta_0 = \theta_1 = 0, \quad \theta_s = 1, \quad s \geq 2.$$

Proof. It can be checked obviously from the above analysis that $\theta_0 = \theta_1 = 0$.

Using (3.8), we can obtain $\theta_2 = 1$. For $s > 2$, we shall use formula (3.9)

$$p_{s+1,m}(t) = \frac{-\mathrm{i}}{m} p_{sm}'(t),$$

because $p_{sm}(t) = 0 \Longrightarrow p_{sm}'(t) = 0$, hence $\theta_s = 1, s \geq 2$ hold. □

Based on the above analysis, we present the approximations of $u(t)$ up to Rth term:

$$u_{AR}(t) = u_0 + \sum_{s=2}^{R} \omega^{-s} [e^{-\mathrm{i}\omega t} p_{s,-1}(t) + p_{s,0}(t) + e^{\mathrm{i}\omega t} p_{s,1}(t)], \quad R = 0, 1, \ldots$$

specifically, $u_{A0}(t) = u_{A1}(t) = u_0$. Then we get the an approximation for the solution of (3.1)

$$x_{AR}(t) = \frac{1}{\sqrt{2}} (e^{\mathrm{i}\omega t} I_d, ie^{-\mathrm{i}\omega t} I_d) u_{AR}(t), \quad R = 0, 1, \ldots \tag{3.10}$$

Before presenting the global errors of the asymptotic-numerical solver for (3.1), we first derive bounds of these coefficients $p_{s,-1}(t), p_{s,0}(t), p_{s,1}(t)$. Here, we mention that all norms in this paper are L^∞ norm.

Lemma 3.1. *Suppose* $g(t)$ *and all derivatives of* $g(t)$ *are bounded by* C *in* $[0, T]$, *then*

$$||p_{s,-1}(t)|| \leq \frac{C}{\sqrt{2}}, \quad ||p_{s,1}(t)|| \leq \frac{C}{\sqrt{2}}, \quad ||p_{s,0}(t)|| \leq \sqrt{2}C, \quad s = 2, 3, \ldots$$

Proof. From (3.7),

$$p_{2,-1}(t)=\frac{1}{\sqrt{2}}\begin{pmatrix} g(t)\\ 0\end{pmatrix},\quad p_{2,1}(t)=\frac{-\mathrm{i}}{\sqrt{2}}\begin{pmatrix} 0\\ g(t)\end{pmatrix},$$

then

$$||p_{2,-1}(t)||\leq\frac{C}{\sqrt{2}},\quad ||p_{2,1}(t)||\leq\frac{C}{\sqrt{2}}.$$

For $s\geq 2$, by (3.8), we can obtain

$$p_{s+1,-1}(t)=\frac{\mathrm{i}^{s-1}}{\sqrt{2}}\begin{pmatrix} g^{(s-1)}(t)\\ 0\end{pmatrix},\quad p_{s+1,1}(t)=\frac{(-\mathrm{i})^{s-1}}{\sqrt{2}}\begin{pmatrix} 0\\ g^{(s-1)}(t)\end{pmatrix},$$

thus

$$||p_{s+1,-1}(t)||\leq\frac{C}{\sqrt{2}},\quad ||p_{s+1,1}(t)||\leq\frac{C}{\sqrt{2}},\quad s=2,3,\ldots$$

For $p_{s,0}$, observing $p_{s0}(t)\equiv-\sum_{m\neq0}p_{s,m}(0)=-p_{s,-1}(0)-p_{s,1}(0)$, we have

$$||p_{s0}(t)||\leq||p_{s,-1}(0)||+||p_{s,1}(0)||\leq\sqrt{2}C,\quad s=2,3,\ldots$$

□

Theorem 3.2. *Suppose $g(t)$ and all derivatives of $g(t)$ are bounded by C for $t\in[0,T]$, then the global errors of the asymptotic-numerical solver up to Rth term for* (3.1) *are*

$$||x(t)-x_{AR}(t)||\leq\begin{cases}4C\frac{\omega^{-2}}{1-\omega^{-1}}, & R=0\\ 4C\frac{\omega^{-R-1}}{1-\omega^{-1}}, & R=1,2\ldots\end{cases}$$

Proof. By Lemma 3.1, the global errors of the asymptotic-numerical solver up to Rth term for (3.2) satisfy

$$\begin{aligned}||u(t)-u_{AR}(t)||&=||\sum_{s=R+1}^{\infty}\omega^{-s}[e^{-\mathrm{i}\omega t}p_{s,-1}(t)+p_{s,0}(t)+e^{\mathrm{i}\omega t}p_{s,1}(t)]||\\ &\leq\sum_{s=R+1}^{\infty}\omega^{-s}(||p_{s,-1}(t)||+||p_{s,0}(t)||+||p_{s,1}(t)||)\\ &\leq 2\sqrt{2}C\frac{\omega^{-R-1}}{1-\omega^{-1}},\quad R=1,2,\ldots\end{aligned}\tag{3.11}$$

for $R=0$,

$$||u(t)-u_{A0}(t)||=||u(t)-u_{A1}(t)||\leq 2\sqrt{2}C\frac{\omega^{-2}}{1-\omega^{-1}}.\tag{3.12}$$

Thus from (3.10), (3.11) and (3.12), we can obtain

$$\begin{aligned}||x(t)-x_{AR}(t)||&=||\frac{1}{\sqrt{2}}(e^{\mathrm{i}\omega t}I_d,ie^{-\mathrm{i}\omega t}I_d)(u(t)-u_{AR}(t))||\\ &\leq 4C\frac{\omega^{-R-1}}{1-\omega^{-1}},\quad R=1,2,\ldots\end{aligned}\tag{3.13}$$

and for $R=0$,

$$||x(t)-x_{A0}(t)||=||x(t)-x_{A1}(t)||\leq 4C\frac{\omega^{-2}}{1-\omega^{-1}}.$$

□

This theorem shows that the accuracy of the asymptotic-numerical solver for highly oscillatory second-order linear system (3.1) improves as the frequency ω grows, moreover, when R is chosen as a larger natural number, the method is expected to behave better.

4. Construction of the asymptotic-numerical solvers for nonlinear system

In this section, we construct the asymptotic-numerical solver for the general nonlinear system (1.1). Suppose that the system (1.1) has been transformed into the form of (2.6). And as explained in Section 2, we assume that the solution $u(t)$ admits an asymptotic expansion in inverse powers of the oscillatory parameter ω

$$u(t) \sim u_0(t) + \sum_{s=1}^{\infty} \omega^{-s} \sum_{m=-\infty}^{\infty} e^{\mathrm{i}m\omega t} p_{sm}(t). \tag{4.1}$$

Since $G_m(t,u) : \mathbb{R}^+ \times \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ are sufficiently differential functions, we can expand them into Taylor series around $u(t) = u_0(t)$, denote

$$\begin{aligned}
G_m^0(u_0) &= G_m(t,u_0), \\
G_m^1(u_0,\theta) &= \frac{\partial G_m(t,u_0)}{\partial u}\theta, \\
(G_m^2(u_0,\theta,\theta))_r &= \sum_{i=1}^{2d}\sum_{j=1}^{2d} \theta_i \frac{\partial^2 G_{m,r}(t,u_0)}{\partial u_i \partial u_j}\theta_j, \quad r = 1,2,\cdots 2d, \\
&\vdots \\
(G_m^n(u_0,\theta,\dots\theta))_r &= \sum_{i_1=1}^{2d}\cdots\sum_{i_n=1}^{2d} \frac{\partial^n G_{m,r}(t,u_0)}{\partial u_{i_1}\cdots\partial u_{i_n}}\theta_{i_1}\cdots\theta_{i_n}, \quad r = 1,2,\cdots 2d, \\
&\vdots
\end{aligned} \tag{4.2}$$

Inserting (4.1), and grouping all those terms that multiply equal powers of ω, we can obtain

$$G_m(t,u) \sim G_m(u_0) + \sum_{s=1}^{\infty} \omega^{-s} \sum_{n=1}^{s} \frac{1}{n!} \sum_{k\in\mathbb{I}_{n,s}} G_m^n(u_0, \chi_{k_1}, \cdots, \chi_{k_n}),$$

where

$$\chi_k(t) = \sum_{m=-\infty}^{\infty} e^{\mathrm{i}m\omega t} p_{k,m},$$

and

$$\mathbb{I}_{n,s} = \{(k_1,\cdots,k_n) \in \mathbb{N}^n : |k| = s\},$$

with the standard notation for multi-indices $|k| = k_1 + \cdots + k_n$. Then collecting all those terms that have the same frequency(that is, those terms that multiply $e^{\mathrm{i}m\omega t}$) within each level, we will have the following notation,

$$G_m(t,u) \sim G_m(u_0) + \sum_{s=1}^{\infty} \omega^{-s} \sum_{n=1}^{s} \frac{1}{n!} \sum_{r=-\infty}^{\infty} e^{\mathrm{i}r\omega t} \sum_{k\in\mathbb{I}_{n,s}} \sum_{l\in\mathbb{K}_{n,r}} G_m^n(u_0, p_{k_1,l_1}, \cdots, p_{k_n,l_n}), \tag{4.3}$$

where

$$\mathbb{K}_{n,r} = \{(l_1,\cdots,l_n) \in \mathbb{Z}^n : |l| = r\}.$$

Differentiating the right side of (4.1) term by term and plugging (4.3) into (2.6), we can equate both sides of the differential equation:

$$\begin{aligned}
&\sum_{s=1}^{\infty} \omega^{-s} \sum_{m=-\infty}^{\infty} [e^{\mathrm{i}m\omega t} p'_{s,m}(t) + \mathrm{i}m\omega e^{\mathrm{i}m\omega t} p_{s,m}(t)] \\
=&\omega^{-1} \sum_{m=-\infty}^{\infty} e^{\mathrm{i}m\omega t} [G_m(u_0) + \sum_{s=1}^{\infty} \omega^{-s} \sum_{n=1}^{s} \frac{1}{n!} \sum_{r=-\infty}^{\infty} e^{\mathrm{i}r\omega t} \sum_{k\in\mathbb{I}_{n,s}} \sum_{l\in\mathbb{K}_{n,r}} G_m^n(u_0, p_{k_1,l_1}, \cdots, p_{k_n,l_n})].
\end{aligned} \tag{4.4}$$

As the same analysis process in Section 3, we can obtain the following scheme.

For ω^0, we have

$$\sum_{m=-\infty}^{\infty} \mathrm{i}me^{\mathrm{i}m\omega t} p_{1,m}(t) = 0,$$

separation of the values of m yields

$$p_{1,m}(t) = 0, \quad m \neq 0. \tag{4.5}$$

For ω^{-1}, $m \in \mathbb{Z}$, we have

$$\mathrm{i}mp_{2,m} + p'_{1,m} = G_m(p_{0,0}),$$

Then the following formulas hold

$$p_{2,m} = \frac{-\mathrm{i}}{m} G_m(u_0), \quad m \neq 0, \tag{4.6}$$

$$p'_{1,0}(t) = G_0(u_0), \tag{4.7}$$

with the initial condition $p_{1,0}(0) = -\sum_{m\neq 0} p_{1,m}(0) = 0$ because of (4.5).

For the general ω^{-s}, $s \geq 2$, and $m \in \mathbb{Z}$, we obtain

$$p'_{s,m}(t) + \mathrm{i}mp_{s+1,m}(t) = \sum_{r=-\infty}^{\infty} \sum_{n=1}^{s-1} \frac{1}{n!} \sum_{k\in\mathbb{I}_{n,s-1}} \sum_{l\in\mathbb{K}_{n,m-r}} G_r^n(u_0, p_{k_1,l_1}.\cdots, p_{k_n,l_n}),$$

Then we have on the one hand

$$p'_{s,0}(t) = \sum_{r=-\infty}^{\infty} \sum_{n=1}^{s-1} \frac{1}{n!} \sum_{k\in\mathbb{I}_{n,s-1}} \sum_{l\in\mathbb{K}_{n,-r}} G_r^n(u_0, p_{k_1,l_1}, \cdots, p_{k_n,l_n}), \tag{4.8}$$

with initial condition$p_{s,0}(0) = -\sum_{m\neq 0} p_{s,m}(0)$. And on the other hand, we get recursions

$$p_{s+1,m}(t) = \frac{-\mathrm{i}}{m}[-p'_{s,m}(t) + \sum_{r=-\infty}^{\infty} \sum_{n=1}^{s-1} \frac{1}{n!} \sum_{k\in\mathbb{I}_{n,s-1}} \sum_{l\in\mathbb{K}_{n,m-r}} G_r^n(u_0, p_{k_1,l_1} \cdots, p_{k_n,l_n})], \tag{4.9}$$

for $m \neq 0$. This is the general scheme that we are going to use to deduce the coefficients in the asymptotic expansion (4.1) up to any desired value of s. From a computational perspective, the scheme only needs very simple computations, solving nonoscillatory first-order ordinary differential equations. After the approximation of $u(t)$ is obtained, the asymptotic-numerical solver of $x(t)$ is defined by

$$\begin{aligned} x_{AR}(t) &= \frac{1}{\sqrt{2}}(e^{\mathrm{i}\omega t} I_d, ie^{-\mathrm{i}\omega t} I_d) u_{AR}(t) \\ &= \frac{1}{\sqrt{2}}(e^{\mathrm{i}\omega t} I_d, ie^{-\mathrm{i}\omega t} I_d)[u_0 + \sum_{s=1}^{R} \omega^{-s} \sum_{m=-\infty}^{\infty} e^{\mathrm{i}m\omega t} p_{sm}(t)], \quad R = 0, 1, ... \end{aligned} \tag{4.10}$$

The following lemma give us the answer how many terms $p_{s,m}$ we need to compute within each order of ω.

Lemma 4.1. *Suppose there exists $\rho \in \mathbb{N}$, such that $G_m(t,u) \equiv \mathbf{0}, |m| \geq \rho + 1$, and denote $\theta_s = \max\{m \in Z : p_{s,|m|} \neq 0\}$. Then we have*

$$\theta_s = \begin{cases} \frac{s}{2}\rho, & if \quad s \quad is \quad even, \\ \frac{s-1}{2}\rho, & if \quad s \quad is \quad odd. \end{cases}$$

Proof. It can be seen obviously that $\theta_0 = 0$, $\theta_1 = 0$, and $\theta_2 = \rho$. For $s \geq 2$, we use the formula (4.9),

$$p_{s+1,m}(t) = \frac{-\mathrm{i}}{m}[-p'_{s,m}(t) + \sum_{r=-\infty}^{\infty} \sum_{n=1}^{s-1} \frac{1}{n!} \sum_{k\in\mathbb{I}_{n,s-1}} \sum_{l\in\mathbb{K}_{n,m-r}} G_r^n(u_0, p_{k_1,l_1} \cdots, p_{k_n,l_n})].$$

We denote

$$b_{s,m}[G_r](t)=\sum_{n=1}^{s}\frac{1}{n!}\sum_{k\in\mathbb{I}_{n,s}}\sum_{l\in\mathbb{K}_{n,m}}G_r^n(u_0,p_{k_1,l_1}\cdots,p_{k_n,l_n})],$$

then

$$p_{s+1,m}(t)=\frac{-\mathrm{i}}{m}[-p'_{s,m}(t)+\sum_{r=-\infty}^{\infty}b_{s-1,m-r}],$$

we note that the differentiation above does not alter the bandwidth. Let's make the assumption that s is odd(when s is even, we can use a similar argument), when $m=\frac{s+1}{2}\rho$, $r=\rho$, then

$$b_{s-1,\frac{s-1}{2}\rho}[G_r](t)=\sum_{n=1}^{s-1}\frac{1}{n!}\sum_{k\in\mathbb{I}_{n,s-1}}\sum_{l\in\mathbb{K}_{n,m-r}}G_r^n(u_0,p_{k_1,l_1}\cdots,p_{k_n,l_n})]\neq 0,$$

since $G_\rho^{\frac{s-1}{2}}(p_{2,\rho},\cdots p_{2,\rho})\neq 0$.

If $m>\frac{s+1}{2}\rho$, then $m-r>\frac{s-1}{2}\rho$, $p'_{s,m}(t)=0$, and we have $b_{s-1,m-r}=0$ according to the analysis above. We can prove the conclusion in the same way when m, and r are negative numbers. Therefore, this completes the proof. □

In the following, we will show that the coefficients of (4.1) are all bounded under some conditions for $0\leq t\leq T$. Then we analyze the global error bounds of the asymptotic-numerical solver for nonlinear system (1.1). For simplicity, we take the case of $d=1$ in (1.1) in the proof procedure.

Lemma 4.2. *Under the assumption $g(t,x)$ are polynomials in components of x, we suppose that there exists $N\in\mathbb{N}$ such that $\frac{\partial^n g(t,0)}{\partial x^n}\equiv 0$, $n\geq N+1$, then*

$$\theta_s=\begin{cases}\frac{s}{2}(N+1), & if\quad s\quad is\quad even,\\ \frac{s-1}{2}(N+1), & if\quad s\quad is\quad odd.\end{cases}$$

Proof. Expanding $g(t,x)$ into Taylor series around $x=0$, we have

$$\begin{aligned}g(t,x)&=\sum_{n=0}^{N}\frac{1}{n!}\frac{\partial^n g(t,0)}{\partial x^n}x^n\\&=\sum_{n=0}^{N}\frac{1}{n!}\frac{\partial^n g(t,0)}{\partial x^n}2^{-\frac{n}{2}}\sum_{m=0}^{n}C_n^m\mathrm{i}^m e^{\mathrm{i}(n-2m)\omega t}u_1^{n-m}u_2^m\\&=\sum_{k=-N}^{0}e^{\mathrm{i}k\omega t}\sum_{n=-k}^{N}\frac{1}{n!}\frac{\partial^n g(t,0)}{\partial x^n}2^{-\frac{n}{2}}C_n^{\frac{n-k}{2}}\mathrm{i}^{\frac{n-k}{2}}u_1^{\frac{n+k}{2}}u_2^{\frac{n-k}{2}}\\&+\sum_{k=0}^{N}e^{\mathrm{i}k\omega t}\sum_{n=k}^{N}\frac{1}{n!}\frac{\partial^n g(t,0)}{\partial x^n}2^{-\frac{n}{2}}C_n^{\frac{n-k}{2}}\mathrm{i}^{\frac{n-k}{2}}u_1^{\frac{n+k}{2}}u_2^{\frac{n-k}{2}}.\end{aligned}$$

Let

$$Q_{nk}(t,u)=\frac{1}{n!}\frac{\partial^n g(t,0)}{\partial x^n}2^{-\frac{n}{2}}C_n^{\frac{n-k}{2}}\mathrm{i}^{\frac{n-k}{2}}u_1^{\frac{n+k}{2}}u_2^{\frac{n-k}{2}},\quad 0\leq n\leq N,-N\leq k\leq N,\tag{4.11}$$

then we have

$$\begin{aligned}g(t,x)&=\sum_{k=-N}^{0}e^{\mathrm{i}k\omega t}\sum_{n=-k}^{N}Q_{nk}(t,u)+\sum_{k=0}^{N}e^{\mathrm{i}k\omega t}\sum_{n=k}^{N}Q_{nk}(t,u)\\&=\sum_{k=-N}^{N}e^{\mathrm{i}k\omega t}L_k,\end{aligned}\tag{4.12}$$

where

$$L_k(t,u)=\begin{cases}\sum_{n=-k}^{N}Q_{nk}(t,u), & -N\leq k<0,\\ \sum_{n=k}^{N}Q_{nk}(t,u), & 0\leq k\leq N.\end{cases}$$

Inserting (4.12) into (2.4), we obtain

$$\begin{aligned}\dot{u} &= \omega^{-1}\sum_{k=-N-1}^{N+1} e^{\mathrm{i}k\omega t}\begin{pmatrix} E_k(t,u) \\ F_k(t,u)\end{pmatrix} \\ &= \omega^{-1}\sum_{k=-N-1}^{N+1} e^{\mathrm{i}k\omega t} G_k(t,u),\end{aligned} \tag{4.13}$$

with $G_k(t,u) = \begin{pmatrix} E_k(t,u) \\ F_k(t,u)\end{pmatrix}$, and

$$E_k(t,u) = \begin{cases} \frac{iL_{k+1}(t,u)}{\sqrt{2}}, & -N-1 \le k < N-1, \\ 0, & N-1 \le k \le N+1. \end{cases} \qquad F_k(t,u) = \begin{cases} 0, & -N-1 \le k < -N, \\ \frac{L_{k-1}(t,u)}{\sqrt{2}}, & -N+1 \le k \le N+1. \end{cases}$$

By Lemma (4.1), the conclusion is proved. □

Lemma 4.3. *Suppose that there exists $N \in \mathbb{N}$ such that $\frac{\partial^n g(t,0)}{\partial x^n} \equiv 0$, when $n \ge N+1$, and $\frac{\partial^{m+n} g(t,0)}{\partial t^m \partial x^n}$ are uniformly bounded for $m = 0,1,2...$, $n = 0,1,2...N$. Then $||p_{s,m}^{(q)}(t)||$ are bounded for any $s \ge 1$, $q \ge 0$, $0 \le t \le T$.*

Proof. Based on the assumptions, it follows from (4.11) that any order derivatives of $Q_{nk}(t,u)$, $E_k(t,u)$ and $F_k(t,u)$ with respect to u or t are all uniformly bounded. Moreover, it means that any order derivatives of $G_k(t,u)$ with respect to u or t are uniformly bounded for any $-N-1 \le k \le N+1$. Then we prove the theorem by induction in s.

First we show that the result is true for $s = 1$. For $m \ne 0$, $||p_{1,m}^{(q)}(t)|| \equiv 0$, for $m = 0$, it follows from (4.7) that $||p_{1,0}^{(q)}(t)||$ satisfy the result.

Then we consider the case of $s = 2$. It is easy to check that $||p_{2,m}^{(q)}(t)||$ are bounded for $m \ne 0$. For $m = 0$, form (4.8), we have

$$p'_{2,0}(t) = G_0^1(u_0, p_{1,0}),$$

and obviously the result is true for $||p_{2,0}^{(q)}(t)||$.

Next, we suppose that the result holds for $||p_{l,m}^{(q)}(t)||$ with any $1 \le l \le s-1, q \ge 0$, $m \in \mathbb{Z}$ and we prove it for $||p_{s,m}^{(q)}(t)||$. Taking into account of the assumptions, it follows from (4.8) and (4.9) immediately that $||p_{s,m}^{(q)}(t)||$ are bounded for any $q \ge 0$. The proof is completed. □

Under the assumptions in Lemma 4.3, we estimate the error $x(t) - x_{AR}(t)$.

Theorem 4.1. *Suppose that there exists $N \in \mathbb{N}$ such that $\frac{\partial^n g(t,0)}{\partial x^n} \equiv 0$, when $n \ge N+1$, and $\frac{\partial^{m+n} g(t,0)}{\partial t^m \partial x^n}$ are uniformly bounded for $m = 0,1,2...$, $n = 0,1,2...N$. Then the global error bounds of the asymptotic-numerical solver up to Rth term for nonlinear system* (1.1) *is*

$$||x(t) - x_{AR}(t)|| = \mathscr{O}(\omega^{-R-1}),$$

where $R \ge 0$.

Proof. By construction, the functions $p_{sm}(t)$ of $u_{AR}(t)$ satisfy the equation (4.4) up to a defect $\mathscr{O}(\omega^{-R-1})$ by the conclusion of Lemma 4.3. This gives a defect of size $\mathscr{O}(\omega^{-R-1})$ when $u_{AR}(t)$ is inserted into (2.6), we denote the defect ε, i.e.,

$$\dot{u}_{AR}(t) = \omega^{-1}\sum_m e^{\mathrm{i}m\omega t} G_m(t, u_{AR}) + \varepsilon, \tag{4.14}$$

where $\varepsilon = \mathscr{O}(\omega^{-R-1})$. We make (4.14)-(2.6), and use mean value theorem, we have

$$\begin{aligned}\dot{u}_{AR}(t) - \dot{u}(t) &= \omega^{-1}\sum_m e^{\mathrm{i}m\omega t}[G_m(t,u_{AR}) - G_m(t,u)] + \varepsilon \\ &= \omega^{-1}\sum_m e^{\mathrm{i}m\omega t} G_m^1(u_{AR} + \theta(u_{AR} - u))[u_{AR}(t) - u(t)] + \varepsilon,\end{aligned}$$

where $0 < \theta < 1$. Let $Z(t) = u_{AR}(t) - u(t)$, then

$$Z'(t) = \omega^{-1}\sum_m e^{\mathrm{i}m\omega t}G_m^1(u_{AR} + \theta(u_{AR} - u), Z(t)) + \varepsilon. \tag{4.15}$$

Integrating (4.15), we have

$$\begin{aligned} Z(t) &= Z(0) + \int_0^t [\omega^{-1}\sum_m e^{\mathrm{i}m\omega s}G_m^1(u_{AR} + \theta(u_{AR} - u), Z(s)) + \varepsilon]ds \\ &= \int_0^t [\omega^{-1}\sum_m e^{\mathrm{i}m\omega s}G_m^1(u_{AR} + \theta(u_{AR} - u), Z(s))ds + \varepsilon t, \end{aligned} \tag{4.16}$$

because $Z(0) = u_{AR}(0) - u(0) = 0$. Under the assumption in Lemma 4.3, which means $|m| \leq N+1$ and any order derivatives of $G_k(t,u)$ with respect to u or t are uniformly bounded, we have

$$||Z(t)|| \leq \omega^{-1}M\int_0^t ||Z(s)||ds + ||\varepsilon||t, \tag{4.17}$$

where M is a constant independent of ω. Then using Gronwall inequality,

$$||Z(t)|| \leq ||\varepsilon||t + ||\varepsilon||\omega^{-1}M\int_0^t s\cdot e^{\omega^{-1}M(t-s)}ds$$

Hence on a finite time interval $[0,T]$, $||Z(t)|| = ||u_{AR}(t) - u(t)|| = \mathscr{O}(\omega^{-R-1})$. It means $||x(t) - x_{AR}(t)|| = \mathscr{O}(\omega^{-R-1})$. □

The theorem above shows that the accuracy of the asymptotic-numerical solver for highly oscillatory second-order nonlinear system (1.1) improves as the frequency ω and R grows, as illustrated in the numerical experiments.

5. Numerical experiments

In this section, we present three examples that illustrate the construction and properties of the expansion that we have presented in previous sections. We denote the approximation for $u(t)$ in (2.6) using up to the Rth term by u_{AR}, the corresponding approximation for $x(t)$ of the original problems (1.1) by x_{AR}, and the asymptotic-numerical solver defined by (3.10) or (4.10) using up to the Rth term by AsR. Experiments in this section illustrate the effectiveness of the methods. In all cases, we will compare the approximation given by the first few terms of the asymptotic-numerical solver with the exact solution. We use the notation $e_R = ||x(t) - x_{AR}||, R \geq 0$, for the global errors.

Example 1. *Consider the highly oscillatory linear system*

$$\begin{cases} \ddot{x}(t) + \omega^2 x(t) = -\cos(t), & t > 0, \\ x(0) = 1, \ x'(0) = 0. \end{cases} \tag{5.1}$$

The exact solution is

$$x(t) = \cos(\omega t) + \frac{\cos(t) - \cos(\omega t)}{1 - \omega^2}.$$

The system is transformed into a first ordinary differential equation of $u(t)$:

$$\dot{u}(t) = \frac{\omega^{-1}}{\sqrt{2}}e^{-\mathrm{i}\omega t}\begin{pmatrix} \mathrm{i}\cos(t) \\ 0 \end{pmatrix} + \frac{\omega^{-1}}{\sqrt{2}}e^{\mathrm{i}\omega t}\begin{pmatrix} 0 \\ -\cos(t) \end{pmatrix}, \quad u(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}. \tag{5.2}$$

According to the analysis in Section 3, after brief computation, we have

$$p_{1m}(t) = 0, m \in \mathbb{Z}, \tag{5.3}$$

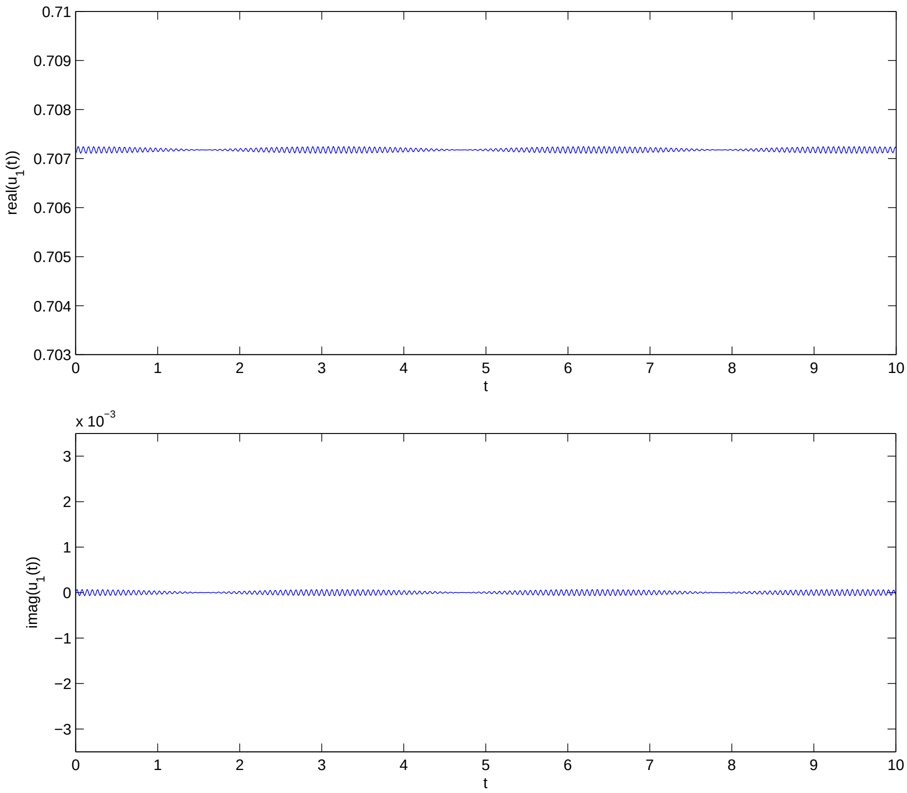

Figure 1: The first component of the solution $u(t)$ of the equation (5.2) with $\omega = 100$.

and

$$p_{2,-1}(t) = \begin{pmatrix} \frac{-cos(t)}{\sqrt{2}} \\ 0 \end{pmatrix}, p_{2,1}(t) = \begin{pmatrix} 0 \\ \frac{\mathrm{i}cos(t)}{\sqrt{2}} \end{pmatrix}, p_{2,0}(t) = -[p_{2,-1}(0) + p_{2,1}(0)] = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-\mathrm{i}}{\sqrt{2}} \end{pmatrix}. \tag{5.4}$$

We can see from (5.3) and (5.4) that, the first component of the solution $u(t)$ of the equation (5.2) is superimposed with tiny oscillatory oscillation of amplitude $\mathscr{O}(\omega^{-2})$, this is consistent with what can be seen in Figure1. Obviously, asymptotic solvers As0, As1 are the same because of $p_{1m}(t) = 0, m \in \mathbb{Z}$. We apply the asymptotic-numerical solvers As0 and As2, respectively to the problem (5.1) in the interval [0,100] and the global errors are shown in Figure 2 for $\omega = 10^2, 10^3, 10^4$.

Example 2. *Consider the nonlinear Duffing equation*

$$\begin{cases} \ddot{x}(t) + \omega^2 x(t) = 2k^2 x(t)^3 - k^2 x(t), \quad t > 0, \\ x(0) = 0, \quad x'(0) = \omega. \end{cases} \tag{5.5}$$

The analytic solution of this initial problem is given by $x(t) = sn(\omega t; k/\omega)$, which represents a periodic motion in term of the Jacobian elliptic function sn.

The system is transformed into

$$u'(t) = \omega^{-1}[e^{4\mathrm{i}\omega t} G_4(u) + e^{2\mathrm{i}\omega t} G_2(u) + G_0(u) + e^{-2\mathrm{i}\omega t} G_{-2}(u) + e^{-4\mathrm{i}\omega t} G_{-4}(u)], \tag{5.6}$$

with

$$G_4(u) = \frac{k^2}{2} \begin{pmatrix} 0 \\ u_1^3 \end{pmatrix}, G_2(u) = \frac{k^2}{2} \begin{pmatrix} -\mathrm{i}u_1^3 \\ 3\mathrm{i}u_1^2 u_2 - u_1 \end{pmatrix}, G_0(u) = \frac{k^2}{2} \begin{pmatrix} 3u_1^2 u_2 + \mathrm{i}u_1 \\ -3\mathrm{i}u_1 u_2^2 - \mathrm{i}u_2 \end{pmatrix},$$

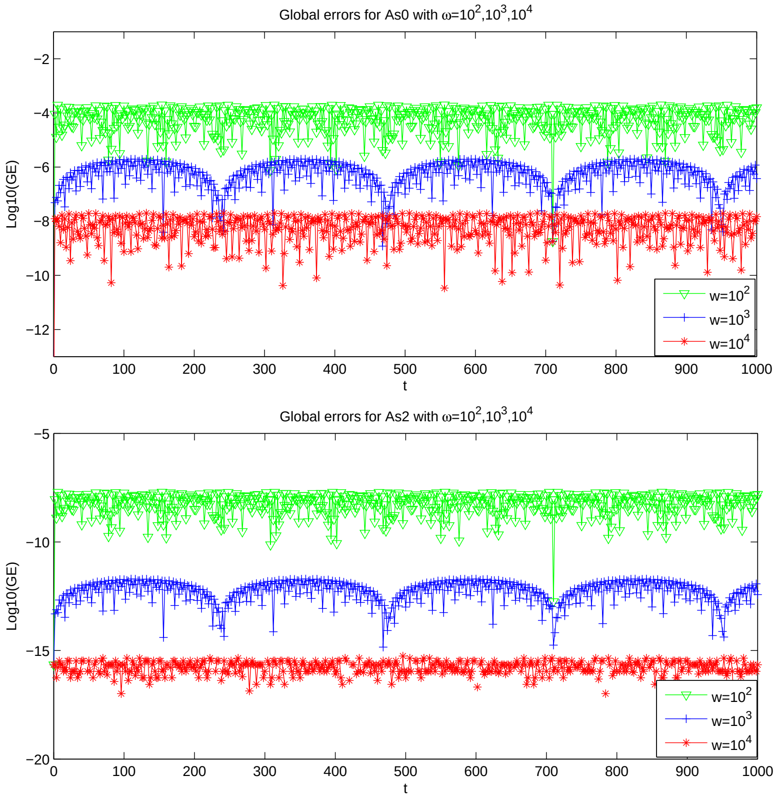

Figure 2: Logarithmic errors of the methods As0, As2 for the equation (5.1) with different ω.

and

$$G_{-2}(u) = \frac{k^2}{2}\begin{pmatrix} 3\mathrm{i}u_1u_2^2 - u_2 \\ -\mathrm{i}u_2^3 \end{pmatrix}, G_{-4}(u) = \frac{k^2}{2}\begin{pmatrix} -u_2^3 + iu_1 \\ 0 \end{pmatrix}.$$

Then the first few terms of asymptotic-numerical solver for (5.6) are as follows:

$$p_{00}(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} -\mathrm{i} \\ 1 \end{pmatrix}, \quad p_{10}(t) = \frac{k^2 t}{4\sqrt{2}}\begin{pmatrix} -1 \\ \mathrm{i} \end{pmatrix}, \quad p_{1m} = 0, m \neq 0,$$
$$p_{2,m}(t) = \frac{-\mathrm{i}}{m} G_m(p_{0,0}), \quad m = -4, -2, 2, 4,$$
$$p_{20}(t) = \frac{k^4 t^2}{32\sqrt{2}}\begin{pmatrix} -\mathrm{i} \\ 1 \end{pmatrix} - [p_{2,-4}(0) + p_{2,-2}(0) + p_{2,2}(0) + p_{2,4}(0)].$$

We apply the asymptotic-numerical solvers As0, As1 and As2, respectively to problem (5.5) in the interval [0,100]. The global errors are shown in Figures 3, 4 and 5 for $\omega = 100, 1000$ and $k = 0.01$.

From the numerical results of the experiments, it can be observed that a larger values of ω will yield a more accurate expansion with the same number of terms, and the accuracy improves as the number of terms increases. Furthermore, the cost of the asymptotic-numerical solvers is essentially independent of ω.

6. Conclusions

We derive asymptotic-numerical solvers for highly oscillatory second-order differential equations (1.1). We first transform the original problems into first-order differential equations with oscillatory forcing terms. And then derive

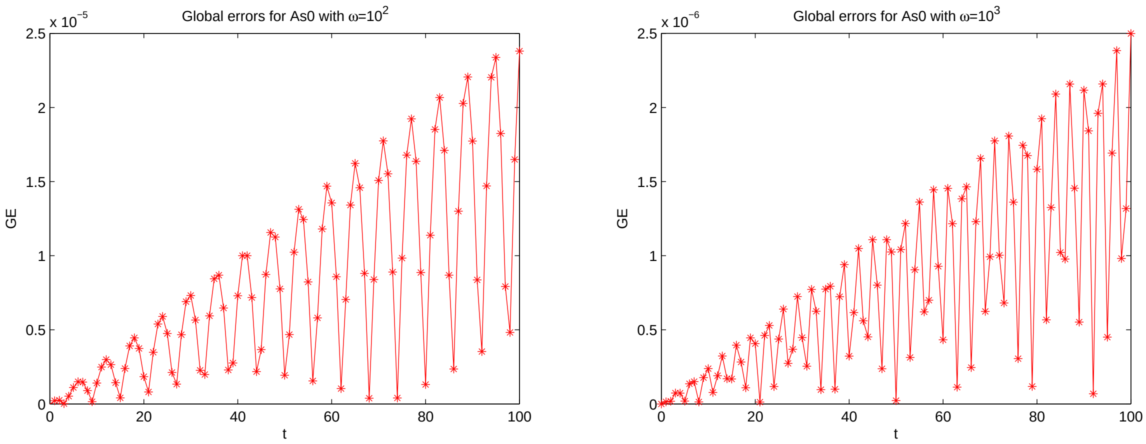

Figure 3: Global errors of the method As0 for the equation (5.5) with $\omega = 10^2, 10^3$.

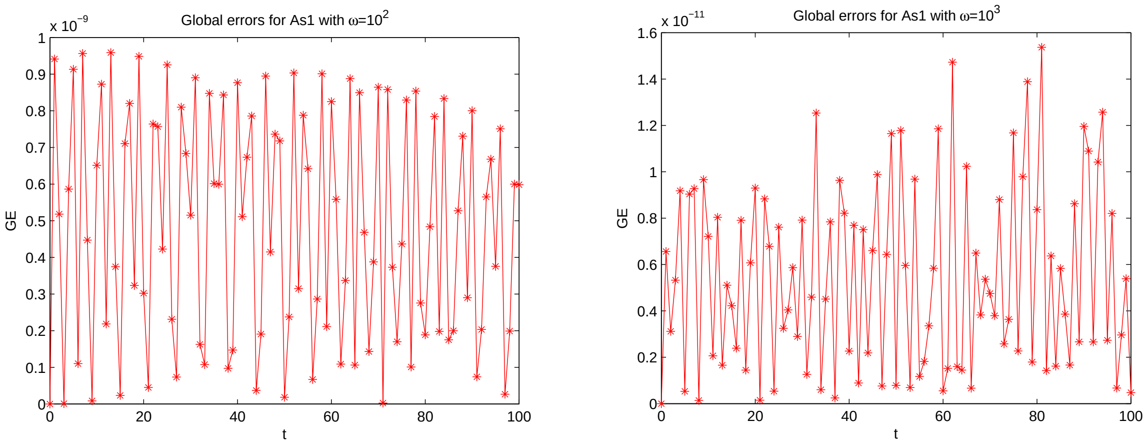

Figure 4: Global errors of the method As1 for the equation (5.5) with $\omega = 10^2, 10^3$.

the asymptotic-numerical scheme for the system based on asymptotic expansions in inverse powers of the oscillatory parameter ω, featuring modulated Fourier series in the expansion coefficients. Our proposed methods benefit the advantages that

- firstly, the coefficients $p_{s,m}(t)$ are easily obtained without solving highly oscillatory equations and highly oscillatory integrals.
- secondly, the computational cost of the methods is independent of the size of ω, and the accuracy improves as ω grows.
- thirdly, once the coefficients have been computed, the equation can be solved easily for different frequencies ω.

The results of the numerical experiments show the outstanding performance of our methods.

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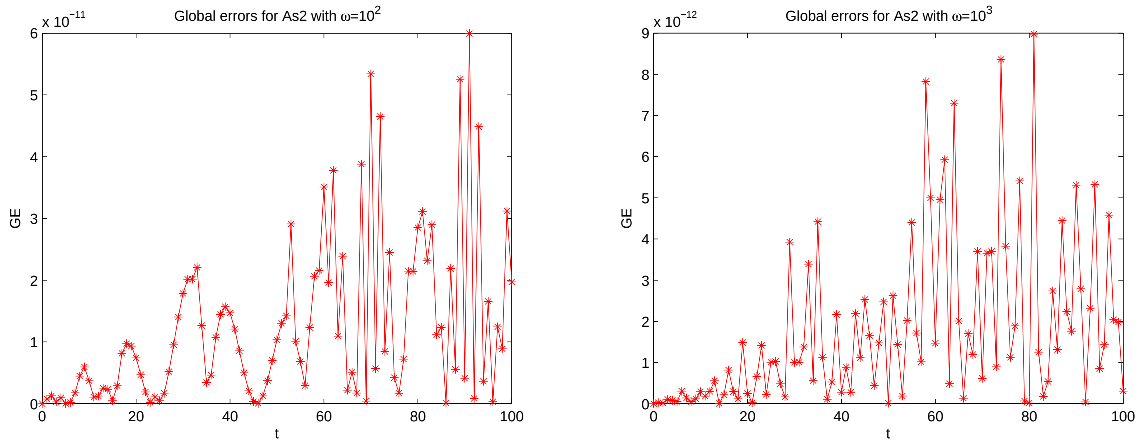

Figure 5: Global errors of the method As2 for the equation (5.5) with $\omega = 10^2, 10^3$.

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