Received hh hh hh; | Revised hh hh hh; | Accepted hh hh hh

DOI: xxx/xxxx

SPECIAL ISSUE PAPER

Crank-Nicholson difference scheme for the system of nonlinear parabolic equations observing epidemic models with general nonlinear incidence rate †

Allaberen Ashyralyev[1,2,3] | Bilgen Kaymakamzade[1] | Evren Hincal*[1]

[1]Department of Mathematics, Near East University, Nicosia, Mersin 10, Turkey
[2]Peoples' Friendship University of Russia, RUDN University, Ul Miklukho Maklaya 6, Moscow 117198, Russia
[3]Institute of Mathematics and Mathematical Modeling, 050010, Almaty, Kazakhstan

Correspondence
Evren Hincal, Department of Mathematics, Near East University, Nicosia, Mersin 10, Turkey. Email: evren.hincal@neu.edu.tr

Abstract

Crank-Nicholson difference scheme for the system of nonlinear parabolic equations observing epidemic models with general nonlinear incidence rate is investigated. The theorem on the existence and uniqueness of a bounded solution of Crank-Nicholson difference scheme uniformly with respect to time step τ is established. Applications of the theoretical results are presented for the four systems of one and multidimensional problems with different boundary conditions. Numerical results are given.

KEYWORDS:
System of nonlinear parabolic equations, Epidemic models, Bounded solution, Realization in computer

1 | INTRODUCTION

Classical epidemic SIR, SIS, and SEIR models have been proposed and studied by many authors in (see[1,2,3,4,5,7,8,9,16] and the references given therein). Theorems on existence and uniqueness of the bounded solution of linear and nonlinear systems are established.[6,12] The numerical solutions of the system of linear parabolic equations for observing HIV mother to child transmission epidemic models is studied.[12] In the paper,[6] we consider a bounded solution of the initial-value problem for the system of parabolic equations observing epidemic models with general nonlinear incidence rate

$$\begin{cases} \frac{du^1(t)}{dt} + \mu u^1(t) + Au^1(t) = -f(t,u^1(t),u^2(t)), \\ \frac{du^2(t)}{dt} + (\alpha+\mu)u^2(t) + Au^2(t) = f(t,u^1(t),u^2(t)) - g(t,u^2(t)), \\ \frac{du^3(t)}{dt} + \mu u^3(t) + Au^3(t) = g(t,u^2(t)), \\ 0 < t < T, u^n(0) = \varphi^n, n = 1,2,3 \end{cases} \tag{1}$$

in a Hilbert space H with a self-adjoint positive definite operator A. The main theorem on the existence and uniqueness of a bounded solution of problem (1) is established.

Theorem 1.1. Assume the following hypotheses:

†This research was funded by "Russian Foundation for Basic Research (RFBR) grant number 16–01–00450".

1. $\varphi^n, n = 1, 2, 3$ belongs to $D(A)$ and
$$\|\varphi^n\|_{D(A)} = M_1. \tag{2}$$
2. The function $f : [0,T] \times H \times H \longrightarrow H$ be continuous function, that is
$$\|f(t, u(t), v(t))\|_H \leq M_2 \tag{3}$$
in $[0,T] \times H \times H$ and Lipschitz condition holds uniformly with respect to t
$$\|f(t, u, v) - f(t, z, w)\|_H \leq L_1 \left[\|u - z\|_H + \|v - w\|_H\right]. \tag{4}$$
3. The function $g : [0,T] \times H \longrightarrow H$ be continuous function, that is
$$\|g(t, u(t))\|_H \leq M_3 \tag{5}$$
in $[0,T] \times H$ and Lipschitz condition holds uniformly with respect to t
$$\|g(t, u) - g(t, z)\|_H \leq L_2\|u - z\|_H. \tag{6}$$

Then there exists a unique bounded solution $u(t) = \left(u^1(t), u^2(t), u^3(t)\right)^{\perp}$ to problem (1).

In applications, theorems on the bounded solutions of several systems of nonlinear parabolic equations were established. Moreover the first order of accuracy difference scheme

$$\begin{cases} \frac{u_k^1 - u_{k-1}^1}{\tau} + \mu u_k^1 + A u_k^1 = -f(t_k, u_k^1, u_k^2), \\ \frac{u_k^2 - u_{k-1}^2}{\tau} + (\alpha + \mu) u_k^2 + A u_k^2 = f(t_k, u_k^1, u_k^2) - g(t_k, u_k^2), \\ \frac{u_k^3 - u_{k-1}^3}{\tau} + \mu u_k^3 + A u_k^3 = g(t_k, u_k^2), \\ t_k = k\tau, 1 \leqslant k \leqslant N, N\tau = T, \\ u_0^n = \varphi^n, n = 1, 2, 3 \end{cases} \tag{7}$$

for the approximate solution of problem (1) was studied. The following theorem on the existence and uniqueness of a bounded solution of difference scheme (7) uniformly with respect to time step τ was established

Theorem 1.2. Let the assumptions (2)-(6) be satisfied and $\mu + \delta > 2\left(L_1 + L_2\right)$. Then, there exists a unique solution $u^\tau = \left\{u_k\right\}_{k=0}^N$ of difference scheme (7) which is bounded uniformly with respect to τ.

In applications, bounded solutions of several systems of nonlinear parabolic equations and difference schemes for the approximate solution of these systems were established. Numerical results were given.

In general, it is not possible to get exact solution of nonlinear problems. Therefore, we are interested in finding a high order of accuracy uniformly bounded difference schemes with respect to time stepsize for the approximate solutions initial value problem (1).

In the present paper, the second order of accuracy Crank-Nicholson difference scheme for the approximate solution of problem (1) is investigated. The theorem on the existence and uniqueness of a bounded solution of Crank-Nicholson difference scheme uniformly with respect to time step τ is established. Applications of the theoretical results are presented on four systems of nonlinear parabolic equations to explain how it works on one and multidimensional problems with different boundary conditions. Numerical results are given.

2 | THE MAIN THEOREM ON UNIFORMLY BOUNDEDNESS

For the approximate solution of (1) we will consider the second order of accuracy Crank-Nicholson difference scheme

$$\begin{cases} \frac{u_k^1-u_{k-1}^1}{\tau}+\mu\frac{u_k^1+u_{k-1}^1}{2}+A\frac{u_k^1+u_{k-1}^1}{2}=-f\left(t_k-\frac{\tau}{2},\frac{u_k^1+u_{k-1}^1}{2},\frac{u_k^2+u_{k-1}^2}{2}\right), \\ \frac{u_k^2-u_{k-1}^2}{\tau}+(\alpha+\mu)\frac{u_k^2+u_{k-1}^2}{2}+A\frac{u_k^2+u_{k-1}^2}{2} \\ =f\left(t_k-\frac{\tau}{2},\frac{u_k^1+u_{k-1}^1}{2},\frac{u_k^2+u_{k-1}^2}{2}\right)-g\left(t_k-\frac{\tau}{2},\frac{u_k^2+u_{k-1}^2}{2}\right), \\ \frac{u_k^3-u_{k-1}^3}{\tau}+\mu\frac{u_k^2+u_{k-1}^2}{2}+A\frac{u_k^2+u_{k-1}^2}{2}=g\left(t_k-\frac{\tau}{2},\frac{u_k^2+u_{k-1}^2}{2}\right), \\ t_k=k\tau, 1\leqslant k\leqslant N, N\tau=T, \\ u_0^n=\varphi^n, n=1,2,3 \end{cases} \tag{8}$$

for the approximate solution of the initial value problem (1).

We are interested to study the existence and uniqueness of a bounded solution of Crank-Nicholson difference scheme (8) uniformly with respect to time step τ under the assumptions of Theorem 1.2. We have not been able to obtain such result. Nevertheless, we can establish the such result under assumptions more strong than in the Theorem 1.2.

The method of proof of the main theorem on the existence and uniqueness of a bounded solution of difference scheme (8) uniformly with respect to τ is based on reducing this difference scheme to an equvalent system of nonlinear equations. An equvalent system of nonlinear equations for the difference scheme (8) is

$$\begin{cases} u_k^1=B^k\varphi^1-\sum_{m=1}^{k}B^{k-m}Rf\left(t_m-\frac{\tau}{2},\frac{u_m^1+u_{m-1}^1}{2},\frac{u_m^2+u_{m-1}^2}{2}\right)\tau, \\ u_k^2=B_1^k\varphi^2+\sum_{m=1}^{k}B_1^{k-m}R_1\left[f\left(t_m-\frac{\tau}{2},\frac{u_m^1+u_{m-1}^1}{2},\frac{u_m^2+u_{m-1}^2}{2}\right)-g\left(t_m-\frac{\tau}{2}\frac{u_m^2+u_{m-1}^2}{2}\right)\right]\tau, \\ u_k^3=B^k\varphi^3+\sum_{m=1}^{k}B^{k-m}Rg\left(t_m-\frac{\tau}{2}\frac{u_m^2+u_{m-1}^2}{2}\right)\tau, 1\le k\le N \end{cases} \tag{9}$$

in $C_\tau(H)\times C_\tau(H)\times C_\tau(H)$ and the use of successive approximations. Here and in future $B=(I-\frac{\tau(\mu I+A)}{2})R$, $R=(I+\frac{\tau(\mu I+A)}{2})^{-1}$, $B_1=(I-\frac{\tau((\mu+\alpha)I+A)}{2})R_1$, $R_1=(I+\frac{\tau((\mu+\alpha)I+A)}{2})^{-1}$ and $C_\tau(H)=C\left([0,T]_\tau,H\right)$ stands for the Banach space of the mesh functions $v^\tau=\left\{v_l\right\}_{l=0}^N$ defined on $[0,T]_\tau$ with values in H,equipped with the norm

$$\| v^\tau \|_{C_\tau(H)}=\max_{0\le l\le N}\left\|v_l\right\|_H .$$

The recursive formula for the solution of difference scheme (8) is

$$\begin{cases} \frac{ju_k^1-ju_{k-1}^1}{\tau}+\mu\frac{ju_k^1+ju_{k-1}^1}{2}+A\frac{ju_k^1+ju_{k-1}^1}{2} \\ =-f\left(t_k-\frac{\tau}{2},\frac{(j-1)u_k^1+(j-1)u_{k-1}^1}{2},\frac{(j-1)u_k^2+(j-1)u_{k-1}^2}{2}\right), \\ \frac{ju_k^2-ju_{k-1}^2}{\tau}+(\alpha+\mu)\frac{ju_k^2+ju_{k-1}^2}{2}+A\frac{ju_k^2+ju_{k-1}^2}{2} \\ =f\left(t_k-\frac{\tau}{2},\frac{(j-1)u_k^1+(j-1)u_{k-1}^1}{2},\frac{(j-1)u_k^2+(j-1)u_{k-1}^2}{2}\right)-g\left(t_k-\frac{\tau}{2},\frac{(j-1)u_k^2+(j-1)u_{k-1}^2}{2}\right), \\ \frac{ju_k^3-ju_{k-1}^3}{\tau}+\mu\frac{ju_k^2+ju_{k-1}^2}{2}+A\frac{ju_k^2+ju_{k-1}^2}{2}=g\left(t_k-\frac{\tau}{2},\frac{(j-1)u_k^2+(j-1)u_{k-1}^2}{2}\right), \\ t_k=k\tau, 1\leqslant k\leqslant N, N\tau=T, \\ ju_0^n=\varphi^n, n=1,2,3, j=1,2,..., \\ 0u_k^n=B^k\varphi^n, n=1,3, 0u_k^2=B^k\varphi^2, 0\le k\le N. \end{cases} \tag{10}$$

From (9) and (10) it follows

$$\begin{cases} ju_k^1=B^k\varphi^1-\sum_{m=1}^{k}B^{k-m}Rf\left(t_k-\frac{\tau}{2},\frac{(j-1)u_k^1+(j-1)u_{k-1}^1}{2},\frac{(j-1)u_k^2+(j-1)u_{k-1}^2}{2}\right)\tau, \\ ju_k^2=B_1^k\varphi^2+\sum_{m=1}^{k}B_1^{k-m}R_1f\left(t_k-\frac{\tau}{2},\frac{(j-1)u_k^1+(j-1)u_{k-1}^1}{2},\frac{(j-1)u_k^2+(j-1)u_{k-1}^2}{2}\right)\tau \\ -\sum_{m=1}^{k}B_1^{k-m}R_1g\left(t_k-\frac{\tau}{2},\frac{(j-1)u_k^2+(j-1)u_{k-1}^2}{2}\right)\tau, \\ ju_k^3=B^k\varphi^3+\sum_{m=1}^{k}B^{k-m}Rg\left(t_k-\frac{\tau}{2},\frac{(j-1)u_k^2+(j-1)u_{k-1}^2}{2}\right),(j-1)\,u_k^2)\tau, \\ 1\le k\le N, j=1,2,..., \\ 0u_k^m=B^k\varphi^m, m=1,3, 0u_k^2=B^k\varphi^2, 0\le k\le N. \end{cases} \tag{11}$$

Theorem 2.1. Let the assumptions (2)-(6) be satisfied and $2\left(L_1+L_2\right)T<1+\frac{\tau(\mu+\delta)}{2}$. Then, there exists a unique solution $u^\tau=\left\{u_k\right\}_{k=0}^N$ of difference scheme (8) which is bounded in $C_\tau(H)\times C_\tau(H)\times C_\tau(H)$ of uniformly with respect to τ.

Proof. Since u_k^3 does not appear in equations for $\frac{u_k^n-u_{k-1}^n}{\tau}, n=1,2,$ it is sufficient to analyze the behaviors of solutions u_k^1 and u_k^2 of (8). According to the method of recursive approximation (11), we get

$$u_k^n=0u_k^n+\sum_{i=0}^{\infty}\left[(i+1)u_k^n-iu_k^n\right], n=1,2, \tag{12}$$

where

$$0u_k^n=\begin{cases} B^k\varphi^n, n=1,3, \\ B_1^k\varphi^2, n=2. \end{cases} \tag{13}$$

Applying formula (13), estimates

$$\|B\|_{H\to H}\le 1, \left\|B_1\right\|_{H\to H}\le 1, \tag{14}$$

we get

$$\left\|0u_k^n\right\|_H \le \|\varphi^n\|_H \le M_1. \tag{15}$$

Applying formula (11), estimates (14) and

$$\|R\|_{H\to H} \le \frac{1}{1+\frac{\tau(\mu+\delta)}{2}}, \|R_1\|_{H\to H} \le \frac{1}{1+\frac{\tau(\mu+\delta+\alpha)}{2}}, \tag{16}$$

we get

$$\|1u_k^1 - 0u_k^1\|_H \le \sum_{m=1}^{k} \left\|B^{k-m}R\right\|_{H\to H} \left\| f\left(t_m - \frac{\tau}{2}, \frac{0u_m^1 + 0u_{m-1}^1}{2}, \frac{0u_m^2 + 0u_{m-1}^2}{2}\right)\right\|_H \tau$$

$$\le M_2 \sum_{m=1}^{k} \frac{\tau}{1+\frac{\tau(\mu+\delta)}{2}} \le M_2 \frac{T}{1+\frac{\tau(\mu+\delta)}{2}},$$

$$\|1u_k^2 - 0u_k^2\|_H \le \sum_{m=1}^{k} \left\|B_1^{k-m}R_1\right\|_{H\to H} \left[\left\| f\left(t_m - \frac{\tau}{2}, \frac{0u_m^1 + 0u_{m-1}^1}{2}, \frac{0u_m^2 + 0u_{m-1}^2}{2}\right)\right\|_H\right.$$

$$\left. + \left\| g\left(t_m - \frac{\tau}{2}, \frac{0u_m^2 + 0u_{m-1}^2}{2}\right)\right\|_H\right] \tau$$

$$\le \left(M_2 + M_3\right) \sum_{m=1}^{k} \frac{\tau}{1+\frac{\tau(\mu+\delta+\alpha)}{2}} \le \left(M_2 + M_3\right) \frac{T}{1+\frac{\tau(\mu+\delta+\alpha)}{2}}$$

for any $k = 1, \cdots, N$. Using the triangle inequality, we get

$$\|1u_k^1\|_H \le M_1 + \left(M_2 + M_3\right) \frac{T}{1+\frac{\tau(\mu+\delta)}{2}},$$

$$\|1u_k^2\|_H \le M_1 + \left(M_2 + M_3\right) \frac{T}{1+\frac{\tau(\mu+\delta)}{2}}$$

for any $k = 1, \cdots, N$. Applying formula (11), and estimates (14), (16), (4),(2) and (3), we get

$$\|2u_k^1 - 1u_k^1\|_H \le \tau \sum_{m=1}^{k} \left\|B^{k-m}R\right\|_{H\to H}$$

$$\times \left\| f\left(t_m - \frac{\tau}{2}, \frac{1u_m^1 + 1u_{m-1}^1}{2}, \frac{1u_m^2 + 1u_{m-1}^2}{2}\right) - f\left(t_m - \frac{\tau}{2}, \frac{0u_m^1 + 0u_{m-1}^1}{2}, \frac{0u_m^2 + 0u_{m-1}^2}{2}\right)\right\|_H$$

$$\le \sum_{m=1}^{k} \frac{L_1\tau}{1+\frac{\tau(\mu+\delta)}{2}} \left[\left\|\frac{1u_m^1 + 1u_{m-1}^1}{2} - \frac{0u_m^1 + 0u_{m-1}^1}{2}\right\|_H + \left\|\frac{1u_m^2 + 1u_{m-1}^2}{2} - \frac{0u_m^2 + 0u_{m-1}^2}{2}\right\|_H\right]$$

$$\le \frac{2L_1\left(M_2 + M_3\right)T}{1+\frac{\tau(\mu+\delta)}{2}} \sum_{m=1}^{k} \frac{\tau}{1+\frac{\tau(\mu+\delta)}{2}} \le \frac{2\left(L_1 + L_2\right)\left(M_2 + M_3\right)T^2}{\left(1+\frac{\tau(\mu+\delta)}{2}\right)^2},$$

$$\|2u_k^2 - 1u_k^2\|_H \le \tau \sum_{m=1}^{k} \left\|B_1^{k-m}R_1\right\|_{H\to H}$$

$$\times \left\| f\left(t_m - \frac{\tau}{2}, \frac{1u_m^1 + 1u_{m-1}^1}{2}, \frac{1u_m^2 + 1u_{m-1}^2}{2}\right) - f\left(t_m - \frac{\tau}{2}, \frac{0u_m^1 + 0u_{m-1}^1}{2}, \frac{0u_m^2 + 0u_{m-1}^2}{2}\right)\right\|_H$$

$$+\tau \sum_{m=1}^{k} \left\|B_1^{k-m}R_1\right\|_{H\to H} \left\| g\left(t_m - \frac{\tau}{2}, \frac{1u_m^2 + 1u_{m-1}^2}{2}\right) - g\left(t_m - \frac{\tau}{2}, \frac{0u_m^2 + 0u_{m-1}^2}{2}\right)\right\|_H$$

$$\le L_1 \sum_{m=1}^{k} \frac{\tau}{1+\frac{\tau(\mu+\delta+\alpha)}{2}}$$

$$\times \left[\left\|\frac{1u_m^1 + 1u_{m-1}^1}{2} - \frac{0u_m^1 + 0u_{m-1}^1}{2}\right\|_H + \left\|\frac{1u_m^2 + 1u_{m-1}^2}{2} - \frac{0u_m^2 + 0u_{m-1}^2}{2}\right\|_H\right]$$

$$+L_2\sum_{m=1}^{k}\frac{\tau}{1+\frac{\tau(\mu+\delta+\alpha)}{2}}\left\|\frac{1u_m^2+1u_{m-1}^2}{2}-\frac{0u_m^2+0u_{m-1}^2}{2}\right\|_H$$

$$\leq\frac{\left(2L_1+L_2\right)\left(M_2+M_3\right)T}{1+\frac{\tau(\mu+\delta)}{2}}\sum_{m=1}^{k}\frac{\tau}{1+\frac{\tau(\mu+\delta+\alpha)}{2}}\leq\frac{2\left(L_1+L_2\right)\left(M_2+M_3\right)T^2}{\left(1+\frac{\tau(\mu+\delta)}{2}\right)^2}$$

for any $k=1,\cdots,N$. Then

$$\|2u_k^n\|_H\leq M_1+\left(M_2+M_3\right)\frac{T}{1+\frac{\tau(\mu+\delta)}{2}}+\frac{2\left(L_1+L_2\right)\left(M_2+M_3\right)T^2}{(\mu+\delta)^2},n=1,2$$

for any $k=1,\cdots,N$. Let

$$\|ju_k^n-(j-1)u_k^n\|_H\leq\frac{2^{j-1}\left(L_1+L_2\right)^{j-1}\left(M_2+M_3\right)T^j}{\left(1+\frac{\tau(\mu+\delta)}{2}\right)^j},n=1,2.$$

Applying formula (11), estimates (14), (4), (2) and (3), we get

$$\|(j+1)u_k^1-ju_k^1\|_H\leq\tau\sum_{m=1}^{k}\left\|B^{k-m}R\right\|_{H\to H}$$

$$\times\left\|f\left(t_m-\frac{\tau}{2},\frac{ju_m^1+ju_{m-1}^1}{2},\frac{ju_m^2+ju_{m-1}^2}{2}\right)\right.$$

$$\left.-f\left(t_m-\frac{\tau}{2},\frac{(j-1)u_m^1+(j-1)u_{m-1}^1}{2},\frac{(j-1)u_m^2+(j-1)u_{m-1}^2}{2}\right)\right\|_H$$

$$\leq\sum_{m=1}^{k}\frac{L_1\tau}{1+\frac{\tau(\mu+\delta)}{2}}$$

$$\times\left[\left\|\frac{ju_m^1+ju_{m-1}^1}{2}-\frac{(j-1)u_m^1+(j-1)u_{m-1}^1}{2}\right\|_H\right.$$

$$\left.+\left\|\frac{ju_m^2+ju_{m-1}^2}{2}-\frac{(j-1)u_m^2+(j-1)u_{m-1}^2}{2}\right\|_H\right]$$

$$\leq\frac{2L_1\cdot2^{j-1}\left(L_1+L_2\right)^{j-1}\left(M_2+M_3\right)T^j}{\left(1+\frac{\tau(\mu+\delta)}{2}\right)^j}\sum_{m=1}^{k}\frac{\tau}{1+\frac{\tau(\mu+\delta)}{2}}\leq\frac{\left(2\left(L_1+L_2\right)\right)^j\left(M_2+M_3\right)T^{j+1}}{\left(1+\frac{\tau(\mu+\delta)}{2}\right)^{j+1}},$$

$$\|(j+1)u_k^2-ju_k^2\|_H\leq\tau\sum_{m=1}^{k}\left\|B_1^{k-m}R_1\right\|_{H\to H}$$

$$\times\left\|f\left(t_m-\frac{\tau}{2},\frac{ju_m^1+ju_{m-1}^1}{2},\frac{ju_m^2+ju_{m-1}^2}{2}\right)\right.$$

$$\left.-f\left(t_m-\frac{\tau}{2},\frac{(j-1)u_m^1+(j-1)u_{m-1}^1}{2},\frac{(j-1)u_m^2+(j-1)u_{m-1}^2}{2}\right)\right\|_H$$

$$+\tau\sum_{m=1}^{k}\left\|B_1^{k-m}R_1\right\|_{H\to H}$$

$$\times\left\|g\left(t_m-\frac{\tau}{2},\frac{ju_m^2+ju_{m-1}^2}{2}\right)-f\left(t_m-\frac{\tau}{2},\frac{(j-1)u_m^2+(j-1)u_{m-1}^2}{2}\right)\right\|_H$$

$$\leq\sum_{m=1}^{k}\frac{L_1\tau}{1+\frac{\tau(\mu+\delta+a)}{2}}$$

$$\times\left[\left\|\frac{ju_m^1+ju_{m-1}^1}{2}-\frac{(j-1)\,u_m^1+(j-1)\,u_{m-1}^1}{2}\right\|_H\right.$$

$$\left.+\left\|\frac{ju_m^2+ju_{m-1}^2}{2}-\frac{(j-1)\,u_m^2+(j-1)\,u_{m-1}^2}{2}\right\|_H\right]$$

$$+\sum_{m=1}^{k}\frac{L_2\tau}{1+\frac{\tau(\mu+\delta+a)}{2}}\left\|\frac{ju_m^2+ju_{m-1}^2}{2}-\frac{(j-1)\,u_m^2+(j-1)\,u_{m-1}^2}{2}\right\|_H$$

$$\leq\frac{\left(2L_1+L_2\right)2^{j-1}\left(L_1+L_2\right)^{j-1}\left(M_2+M_3\right)T^j}{\left(1+\frac{\tau(\mu+\delta)}{2}\right)^j}\sum_{m=1}^{k}\frac{\tau}{1+\frac{\tau(\mu+\delta)}{2}}\leq\frac{\left(2\left(L_1+L_2\right)\right)^j\left(M_2+M_3\right)T^{j+1}}{\left(1+\frac{\tau(\mu+\delta)}{2}\right)^{j+1}}$$

for any $k=1,\cdots,N$. Then

$$\|(j+1)\,u_k^n\|_H\leq M_1+\left(M_2+M_3\right)\frac{T}{1+\frac{\tau(\mu+\delta)}{2}}$$

$$+\frac{2\left(L_1+L_2\right)\left(M_2+M_3\right)T^2}{(\mu+\delta)^2}+\cdots+\frac{\left(2\left(L_1+L_2\right)\right)^j\left(M_2+M_3\right)T^{j+1}}{\left(1+\frac{\tau(\mu+\delta)}{2}\right)^{j+1}},n=1,2$$

for any $k=1,\cdots,N$. Therefore, for any $j,j\geq 1$, we have that

$$\|(j+1)\,u_k^n-ju_k^n\|_H\leq\frac{\left(2\left(L_1+L_2\right)\right)^j\left(M_2+M_3\right)T^{j+1}}{\left(1+\frac{\tau(\mu+\delta)}{2}\right)^{j+1}},n=1,2,$$

and

$$\left\|(j+1)\,u_k^n\right\|_H\leq M_1+\left(M_2+M_3\right)\frac{T}{1+\frac{\tau(\mu+\delta)}{2}}$$

$$+\frac{2\left(L_1+L_2\right)\left(M_2+M_3\right)T^2}{(\mu+\delta)^2}+\cdots+\frac{\left(2\left(L_1+L_2\right)\right)^j\left(M_2+M_3\right)T^{j+1}}{\left(1+\frac{\tau(\mu+\delta)}{2}\right)^{j+1}},n=1,2$$

by mathematical induction. From that and formula (12) it follows that

$$\|u_k^n\|_H\leq\|0u_k^n\|_H+\sum_{i=0}^{\infty}\|(i+1)u_k^n-iu_k^n\|_H$$

$$\leq M_1+\frac{\left(M_2+M_3\right)T}{1+\frac{\tau(\mu+\delta)}{2}}\sum_{i=0}^{\infty}\frac{2^i\left(L_1+L_2\right)^iT^i}{\left(1+\frac{\tau(\mu+\delta)}{2}\right)^i},n=1,2$$

which proves the existence of a bounded solution of difference scheme (8) which is bounded in $C_\tau(H)\times C_\tau(H)\times C_\tau(H)$ of uniformly with respect to τ. Theorem 2.1 is proved.

A study of discretization, over time only, of the initial value problem also permits one to include general difference schemes in applications, if the differential operator A is replaced by the difference operator A_h that act in the Hilbert spaces and are uniformly self-adjoint positive definite in h for $0<h\leq h_0$.

3 | APPLICATIONS

First, we consider the initial-boundary value problem for one dimensional system of nonlinear partial differential equations

$$\begin{cases} \frac{\partial u^1(t,x)}{\partial t} - \left(a(x)u_x^1(t,x)\right)_x + (\delta+\mu)\, u^1(t,x) = -f(t,x;u^1(t,x),u^2(t,x)), \\ \frac{\partial u^2(t,x)}{\partial t} - \left(a(x)u_x^2(t,x)\right)_x + (\delta+\mu+\alpha)\, u^2(t,x) \\ = f(t,x;u^1(t,x),u^2(t,x)) - g(t,x;u^2(t,x)), \\ \frac{\partial u^3(t,x)}{\partial t} - \left(a(x)u_x^3(t,x)\right)_x + (\delta+\mu)\, u^3(t,x) = g(t,x;u^2(t,x)), \\ 0<t<T, 0<x<l, \\ u^n(0,x)=\varphi^n(x), \varphi^n(0)=\varphi^n(l), \varphi_x^m(0)=\varphi_x^n(l), x\in[0,l], n=1,2,3, \\ u^n(t,0)=u^n(t,l), u_x^n(t,0)=u_x^n(t,l), 0\le t\le T, n=1,2,3, \end{cases} \tag{17}$$

where $a(x), \varphi(x)$ are given sufficiently smooth functions and $\delta>0$ is the sufficiently large number. We will assume that $a(x)\ge a>0$ and $a(l)=a(0)$.

Assume the following hypotheses:

1. $\varphi^n, n=1,2,3$ belongs to $W_2^2[0,l]$ and
$$\|\varphi^n\|_{W_2^2[0,l]} \le M_1. \tag{18}$$
2. The function $f : [0,T]\times[0,l]\times L_2[0,l]\times L_2[0,l] \to L_2[0,l]$ be continuous function in t, that is
$$\|f(t,\cdot,u(t,\cdot),v(t,\cdot))\|_{L_2[0,l]} \le M_2 \tag{19}$$
in $[0,T]\times[0,l]\times L_2[0,l]\times L_2[0,l]$ and Lipschitz condition holds uniformly with respect to t
$$\|f(t,\cdot,u,v)-f(t,\cdot,z,w)\|_{L_2[0,l]} \le L_1\left[\|u-z\|_{L_2[0,l]}+\|v-w\|_{L_2[0,l]}\right]. \tag{20}$$
3. The function $g : [0,T]\times[0,l]\times L_2[0,l] \to L_2[0,l]$ be continuous function in t, that is
$$\|g(t,\cdot,u(t,\cdot))\|_{L_2[0,l]} \le M_3 \tag{21}$$
in $[0,T]\times[0,l]\times L_2[0,l]$ and Lipschitz condition holds uniformly with respect to t
$$\|g(t,\cdot,u)-g(t,\cdot,z)\|_{L_2[0,l]} \le L_2\|u-z\|_{L_2[0,l]}. \tag{22}$$

Here and in future, $L_m, m=1,2, M_m, m=1,2,3$ are positive constants.

The discretization of problem (17) is carried out in two steps. In the first step, let us define the grid space

$$[0,l]_h = \{x : x_r = rh, 0\le r\le K, Kh=l\}.$$

We introduce the Hilbert spaces $L_{2h}=L_2([0,l]_h)$ and $W_{2h}^2=W_2^2([0,l]_h)$ of the grid functions $\varphi^h(x)=\{\varphi^r\}_0^K$ defined on $[0,l]_h$, equipped with the norms

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x\in[0,l]_h}\left|\varphi^h(x)\right|^2 h\right)^{1/2}$$

and

$$\left\|\varphi^h\right\|_{W_{2h}^2} = \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x\in[0,l]_h}\left|\left(\varphi^h\right)_{x\bar{x},j}\right|^2 h\right)^{1/2}$$

respectively. To the differential operator A generated by problem (17), we assign the difference operator A_h^x by the formula

$$A_h^x \varphi^h(x) = \{-(a(x)\varphi_{\bar{x}})_{x,r} + \delta\varphi^r\}_1^{K-1}, \tag{23}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi^r\}_0^K$ satisfying the conditions $\varphi^0 = \varphi^K,\ \varphi^1 - \varphi^0 = \varphi^K - \varphi^{K-1}$. With the help of A_h^x, we arrive at the initial value problem

$$\begin{cases} \frac{du^{1h}(t,x)}{dt} + \mu u^{1h}(t,x) + A_h^x u^{1h}(t,x) = -f^h(t,x;u^{1h}(t,x),u^{2h}(t,x)), \\ \frac{du^{2h}(t,x)}{dt} + (\mu+\alpha)\, u^{2h}(t,x) + A_h^x u^{2h}(t,x) \\ = f^h(t,x;u^{1h}(t,x),u^{2h}(t,x)) - g^h(t,x;u^{2h}(t,x)), \\ \frac{du^{3h}(t,x)}{dt} + \mu u^{3h}(t,x) + A_h^x u^{3h}(t,x) = g^h(t,x;u^{2h}(t,x)), \\ 0 < t < T, x \in [0,l]_h, \\ u^{nh}(0,x) = \varphi^n(x), n = 1,2,3, x \in [0,l]_h \end{cases} \tag{24}$$

for an infinite system of nonlinear ordinary differential equations. In the second step, we replace problem (24) by difference scheme (8)

$$\begin{cases} \frac{u_k^1 - u_{k-1}^1}{\tau} + \mu\frac{u_k^1 + u_{k-1}^1}{2} + A_h^x \frac{u_k^1 + u_{k-1}^1}{2} = -f^h\left(t_k - \frac{\tau}{2}, x, \frac{u_k^1 + u_{k-1}^1}{2}, \frac{u_k^2 + u_{k-1}^2}{2}\right), \\ \frac{u_k^2 - u_{k-1}^2}{\tau} + (\alpha+\mu)\frac{u_k^2 + u_{k-1}^2}{2} + A_h^x \frac{u_k^2 + u_{k-1}^2}{2} \\ = f^h\left(t_k - \frac{\tau}{2}, x, \frac{u_k^1 + u_{k-1}^1}{2}, \frac{u_k^2 + u_{k-1}^2}{2}\right) - g^h\left(t_k - \frac{\tau}{2}, x, \frac{u_k^2 + u_{k-1}^2}{2}\right), \\ \frac{u_k^3 - u_{k-1}^3}{\tau} + \mu\frac{u_k^3 + u_{k-1}^3}{2} + A_h^x \frac{u_k^3 + u_{k-1}^3}{2} = g^h\left(t_k - \frac{\tau}{2}, x, \frac{u_k^2 + u_{k-1}^2}{2}\right), \\ t_k = k\tau, 1 \leqslant k \leqslant N, N\tau = T, x \in [0,l]_h, \\ u_0^n = \varphi^n, n = 1,2,3. \end{cases} \tag{25}$$

Theorem 3.1. Let the assumptions (18)-(22) be satisfied and $2\left(L_1 + L_2\right)T < 1 + \frac{\tau(\mu+\delta)}{2}$. Then, there exists a unique solution $u^\tau = \left\{u_k\right\}_{k=0}^N$ of difference scheme (25) which is bounded in $C_\tau\left(L_{2h}\right) \times C_\tau\left(L_{2h}\right) \times C_\tau\left(L_{2h}\right)$ of uniformly with respect to τ and h.

The proof of Theorem 3.1 is based on the abstract Theorem 2.1 and symmetry properties of the difference operator A_h^x defined by formula (23).[13]

Second, we consider the initial-boundary value problem for one dimensional system of nonlinear partial differential equations with involution

$$\begin{cases} \frac{\partial u^1(t,x)}{\partial t} - \left(a(x)u_x^1(t,x)\right)_x - \beta\left(a(-x)u_x(t,-x)\right)_x + (\delta+\mu)u^1(t,x) \\ = -f(t,x;u^1(t,x),u^2(t,x)), \\ \frac{\partial u^2(t,x)}{\partial t} - \left(a(x)u_x^2(t,x)\right)_x - \beta\left(a(-x)u_x(t,-x)\right)_x + (\delta+\mu+\alpha)u^2(t,x) \\ = f(t,x;u^1(t,x),u^2(t,x)) - g(t,x;u^2(t,x)), \\ \frac{\partial u^3(t,x)}{\partial t} - \left(a(x)u_x^3(t,x)\right)_x - \beta\left(a(-x)u_x(t,-x)\right)_x + (\delta+\mu)u^3(t,x) \\ = g(t,x;u^2(t,x)), 0<t<T, -l<x<l, \\ u^n(0,x)=\varphi^n(x), \varphi^n(-l)=\varphi^n(l)=0, x\in[-l,l], n=1,2,3, \\ u^n(t,-l)=u^n(t,l)=0, 0\le t\le T, n=1,2,3, \end{cases} \tag{26}$$

where $a(x),\varphi(x)$ are given sufficiently smooth functions and $\delta>0$ is the sufficiently large number. We will assume that $a \geq a(x)=a(-x)\geq\delta>0$, $\delta - a|\beta|\geq 0$.

Assume the following hypotheses:

1. $\varphi^n, n=1,2,3$ belongs to $W_2^2[-l,l]$ and
$$\|\varphi^n\|_{W_2^2[-l,l]} \leq M_1. \tag{27}$$
2. The function $f : [0,T]\times[-l,l]\times L_2[-l,l]\times L_2[-l,l] \to L_2[-l,l]$ be continuous function in t, that is
$$\|f(t,\cdot,u(t,\cdot),v(t,\cdot))\|_{L_2[-l,l]} \leq M_2 \tag{28}$$
in $[0,T]\times[-l,l]\times L_2[-l,l]\times L_2[-l,l]$ and Lipschitz condition holds uniformly with respect to t
$$\|f(t,\cdot,u,v)-f(t,\cdot,z,w)\|_{L_2[-l,l]} \leq L_1\left[\|u-z\|_{L_2[-l,l]}+\|v-w\|_{L_2[-l,l]}\right]. \tag{29}$$
3. The function $g : [0,T]\times[-l,l]\times L_2[-l,l] \to L_2[-l,l]$ be continuous function in t, that is
$$\|g(t,\cdot,u(t,\cdot))\|_{L_2[0,l]} \leq M_3 \tag{30}$$
in $[0,T]\times[-l,l]\times L_2[-l,l]$ and Lipschitz condition holds uniformly with respect to t
$$\|g(t,\cdot,u)-g(t,\cdot,z)\|_{L_2[-l,l]} \leq L_2\|u-z\|_{L_2[-l,l]}. \tag{31}$$

The discretization of problem (26) is carried out in two steps. In the first step, let us define the grid space

$$[-l,l]_h = \{x : x_r = rh, -K\leq r\leq K, Kh=l\}.$$

We introduce the Hilbert spaces $L_{2h}=L_2([-l,l]_h)$ and $W_{2h}^2=W_2^2([-l,l]_h)$ of the grid functions $\varphi^h(x)=\{\varphi^r\}_{-K}^K$ defined on $[-l,l]_h$, equipped with the norms

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x\in[-l,l]_h}\left|\varphi^h(x)\right|^2 h\right)^{1/2}$$

and

$$\left\|\varphi^h\right\|_{W_{2h}^2} = \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x\in[-l,l]_h}\left|\left(\varphi^h\right)_{x\bar{x},j}\right|^2 h\right)^{1/2}$$

respectively. To the differential operator A generated by problem (26), we assign the difference operator A_h^x by the formula

$$A_h^x\varphi^h(x) = \{-(a(x)\varphi_{\bar{x}}(x))_{x,r} - \beta\left(a(-x)\varphi_{\bar{x}}(-x)\right)_{x,r} + \delta\varphi^r\}_{-K+1}^{K-1}, \tag{32}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi^r\}_{-K}^{K}$ satisfying the conditions $\varphi^{-K} = \varphi^K = 0$. With the help of A_h^x, we arrive at the initial value problem

$$\begin{cases} \frac{du^{1h}(t,x)}{dt} + \mu u^{1h}(t,x) + A_h^x u^{1h}(t,x) = -f^h(t,x;u^{1h}(t,x),u^{2h}(t,x)), \\ \frac{du^{2h}(t,x)}{dt} + (\mu+\alpha)\, u^{2h}(t,x) + A_h^x u^{2h}(t,x) \\ = f^h(t,x;u^{1h}(t,x),u^{2h}(t,x)) - g^h(t,x;u^{2h}(t,x)), \\ \frac{du^{3h}(t,x)}{dt} + \mu u^{3h}(t,x) + A_h^x u^{3h}(t,x) = g^h(t,x;u^{2h}(t,x)), 0<t<T, x \in [-l,l]_h, \\ u^{nh}(0,x) = \varphi^n(x), n=1,2,3, x\in[-l,l]_h \end{cases} \tag{33}$$

for an infinite system of nonlinear ordinary differential equations. In the second step, we replace problem (33) by difference scheme (8)

$$\begin{cases} \frac{u_k^1-u_{k-1}^1}{\tau} + \mu\frac{u_k^1+u_{k-1}^1}{2} + A_h^x \frac{u_k^1+u_{k-1}^1}{2} = -f^h\left(t_k - \frac{\tau}{2}, x, \frac{u_k^1+u_{k-1}^1}{2}, \frac{u_k^2+u_{k-1}^2}{2}\right), \\ \frac{u_k^2-u_{k-1}^2}{\tau} + (\alpha+\mu)\,\frac{u_k^2+u_{k-1}^2}{2} + A_h^x \frac{u_k^2+u_{k-1}^2}{2} \\ = f^h\left(t_k-\frac{\tau}{2}, x, \frac{u_k^1+u_{k-1}^1}{2}, \frac{u_k^2+u_{k-1}^2}{2}\right) - g^h\left(t_k-\frac{\tau}{2}, x, \frac{u_k^2+u_{k-1}^2}{2}\right), \\ \frac{u_k^3-u_{k-1}^3}{\tau} + \mu\frac{u_k^3+u_{k-1}^3}{2} + A_h^x \frac{u_k^3+u_{k-1}^3}{2} = g^h\left(t_k-\frac{\tau}{2}, x, \frac{u_k^2+u_{k-1}^2}{2}\right), \\ t_k = k\tau, 1\leqslant k\leqslant N, N\tau = T, x\in[-l,l]_h, \\ u_0^n = \varphi^n, n=1,2,3. \end{cases} \tag{34}$$

Theorem 3.2. Let the assumptions (27)-(31) be satisfied and $2\left(L_1+L_2\right)T < 1+\frac{\tau(\mu+\delta)}{2}$. Then, there exists a unique solution $u^\tau = \left\{u_k\right\}_{k=0}^N$ of difference scheme (34) which is bounded in $C_\tau\left(L_{2h}\right)\times C_\tau\left(L_{2h}\right)\times C_\tau\left(L_{2h}\right)$ of uniformly with respect to τ and h.

The proof of Theorem 3.2 is based on the abstract Theorem 2.1 and symmetry properties of the difference operator A_h^x defined by formula (32).[14]

Third, let $\Omega \subset R^n$ be a bounded open domain with smooth boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,T] \times \Omega$ we consider the initial-boundary value problem for multidimensional system of nonlinear partial differential equations

$$\begin{cases} \frac{\partial u^1(t,x)}{\partial t} - \sum\limits_{r=1}^{n} (a_r(x) u^1_{x_r})x_r + (\delta+\mu)\, u^1(t,x) \\ = -f(t,x;u^1(t,x),u^2(t,x)), \\ \frac{\partial u^2(t,x)}{\partial t} - \sum\limits_{r=1}^{n} (a_r(x) u^2_{x_r})x_r + (\delta+\mu+\alpha)\, u^2(t,x) \\ = f(t,x;u^1(t,x),u^2(t,x)) - g(t,x;u^2(t,x)), \\ \frac{\partial u^3(t,x)}{\partial t} - \sum\limits_{r=1}^{n} (a_r(x) u^3_{x_r})x_r + (\delta+\mu)\, u^3(t,x) \\ = g(t,x;u^2(t,x)), 0 < t < T, x = (x_1,...,x_n) \in \Omega, \\ u^m(0,x) = \varphi^m(x), x \in \overline{\Omega}, m = 1,2,3, \\ u^m(t,x) = 0, 0 \le t \le T, x \in S, m = 1,2,3, \end{cases} \tag{35}$$

where $a_r(x)$ and $\varphi^m(x)$ are given sufficiently smooth functions and $\delta > 0$ is the sufficiently large number and $a_r(x) > 0$.

Assume the following hypotheses:

1. $\varphi^m, m = 1,2,3$ belongs to $L_2(\overline{\Omega})$ and
$$\|\varphi^m\|_{W_2^2(\overline{\Omega})} \le M_1. \tag{36}$$
2. The function $f : [0,T] \times [0,l] \times L_2(\overline{\Omega}) \times L_2(\overline{\Omega}) \to L_2(\overline{\Omega})$ be continuous function in t, that is
$$\|f(t,\cdot,u(t,\cdot),v(t,\cdot))\|_{L_2(\overline{\Omega})} \le M_2 \tag{37}$$
in $[0,T] \times [0,l] \times L_2(\overline{\Omega}) \times L_2(\overline{\Omega})$ and Lipschitz condition holds uniformly with respect to t
$$\|f(t,\cdot,u,v) - f(t,\cdot,z,w)\|_{L_2(\overline{\Omega})} \le L_1 \left[\|u-z\|_{L_2(\overline{\Omega})} + \|v-w\|_{L_2(\overline{\Omega})} \right]. \tag{38}$$
3. The function $g : [0,T] \times [0,l] \times L_2(\overline{\Omega}) \to L_2(\overline{\Omega})$ be continuous function in t, that is
$$\|g(t,\cdot,u(t,\cdot))\|_{L_2(\overline{\Omega})} \le M_3 \tag{39}$$
in $[0,T] \times [0,l] \times L_2(\overline{\Omega})$ and Lipschitz condition holds uniformly with respect to t
$$\|g(t,\cdot,u) - g(t,\cdot,z)\|_{L_2(\overline{\Omega})} \le L_2 \|u-z\|_{L_2(\overline{\Omega})}. \tag{40}$$

The discretization of problem (35) is also carried out in two steps. In the first step, let us define the grid sets

$$\begin{gathered} \overline{\Omega}_h = \{ x = x_r = (h_1 r_1, ..., h_m r_m),\ r = (r_1,...,r_m), \\ 0 \le r_j \le N_j,\ h_j N_j = 1,\ j = 1,...,m, \}, \\ \Omega_h = \overline{\Omega}_h \cap \Omega,\ S_h = \overline{\Omega}_h \cap S. \end{gathered}$$

We introduce the Banach spaces $L_{2h} = L_2(\overline{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\overline{\Omega}_h)$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 r_1, ..., h_m r_m)\}$ defined on $\overline{\Omega}_h$, equipped with the norms

$$\left\|\varphi^h\right\|_{L_{2h}} = \left(\sum_{x \in \overline{\Omega}_h} \left|\varphi^h(x)\right|^2 h_1 \cdots h_m \right)^{1/2}$$

and

$$\left\|\varphi^h\right\|_{W_{2h}} = \left\|\varphi^h\right\|_{L_{2h}} + \left(\sum_{x\in\overline{\Omega}_h}\sum_{r=1}^{m}\left|\left(\varphi^h\right)_{x_r\overline{x}_r,j_r}\right|^2 h_1\cdots h_m\right)^{1/2}$$

respectively. To the differential operator A generated by problem (35), we assign the difference operator A_h^x by the formula

$$A_h^x u_x^h = -\sum_{r=1}^{m}\left(a_r(x)u_{\overline{x}_r}^h\right)_{x_r,j_r} \tag{41}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\overline{\Omega}_h)$. With the help of A_h^x, we arrive at the initial value problem

$$\begin{cases} \frac{du^{1h}(t,x)}{dt} + \mu u^{1h}(t,x) + A_h^x u^{1h}(t,x) = -f^h(t,x;u^{1h}(t,x),u^{2h}(t,x)), \\ \frac{du^{2h}(t,x)}{dt} + (\mu+\alpha)u^{2h}(t,x) + A_h^x u^{2h}(t,x) \\ = f^h(t,x;u^{1h}(t,x),u^{2h}(t,x)) - g^h(t,x;u^{2h}(t,x)), \\ \frac{du^{3h}(t,x)}{dt} + \mu u^{3h}(t,x) + A_h^x u^{3h}(t,x) = g^h(t,x;u^{2h}(t,x)), 0<t<T, x\in\overline{\Omega}_h, \\ u^{mh}(0,x) = \varphi^m(x), m=1,2,3, x\in\overline{\Omega}_h \end{cases} \tag{42}$$

for an infinite system of nonlinear ordinary differential equations. In the second step, we replace problem (42) by difference scheme (8)

$$\begin{cases} \frac{u_k^1-u_{k-1}^1}{\tau} + \mu\frac{u_k^1+u_{k-1}^1}{2} + A_h^x\frac{u_k^1+u_{k-1}^1}{2} = -f^h\left(t_k-\frac{\tau}{2},x,\frac{u_k^1+u_{k-1}^1}{2},\frac{u_k^2+u_{k-1}^2}{2}\right), \\ \frac{u_k^2-u_{k-1}^2}{\tau} + (\alpha+\mu)\frac{u_k^2+u_{k-1}^2}{2} + A_h^x\frac{u_k^2+u_{k-1}^2}{2} \\ = f^h\left(t_k-\frac{\tau}{2},x,\frac{u_k^1+u_{k-1}^1}{2},\frac{u_k^2+u_{k-1}^2}{2}\right) - g^h\left(t_k-\frac{\tau}{2},x,\frac{u_k^2+u_{k-1}^2}{2}\right), \\ \frac{u_k^3-u_{k-1}^3}{\tau} + \mu\frac{u_k^3+u_{k-1}^3}{2} + A_h^x\frac{u_k^3+u_{k-1}^3}{2} = g^h\left(t_k-\frac{\tau}{2},x,\frac{u_k^2+u_{k-1}^2}{2}\right), \\ t_k = k\tau, 1\leqslant k\leqslant N, N\tau = T, x\in\overline{\Omega}_h, \\ u_0^m = \varphi^m, m=1,2,3. \end{cases} \tag{43}$$

Theorem 3.3. Let the assumptions (36)-(40) be satisfied and $2\left(L_1+L_2\right)T < 1+\frac{\tau(\mu+\delta)}{2}$. Then, there exists a unique solution $u^\tau = \left\{u_k\right\}_{k=0}^N$ of difference scheme (43) which is bounded in $C_\tau\left(L_{2h}\right)\times C_\tau\left(L_{2h}\right)\times C_\tau\left(L_{2h}\right)$ of uniformly with respect to τ and h.

The proof of Theorem 3.4 is based on the abstract Theorem 3.1 and symmetry properties of the difference operator A_h^x defined by formula (41) and the following theorem on coercivity inequality for the solution of the elliptic problem in L_{2h}.

Theorem 3.4. For the solutions of the elliptic difference problem

$$\begin{cases} A_h^x v^h(x) = g^h(x),\ x\in\Omega_h, \\ v^h(x) = 0,\ x\in S_h \end{cases}$$

the following coercivity inequality

$$\sum_{r=1}^{m}\left\|v_{x_r\overline{x}_r,j_r}^h\right\|_{L_{2h}} \leq M\left\|g^h\right\|_{L_{2h}}.$$

holds.[15]

Fourth, in $[0,T]\times\Omega$ we consider the initial-boundary value problem for multidimensional system of nonlinear partial differential equations

$$\begin{cases} \frac{\partial u^1(t,x)}{\partial t} - \sum\limits_{r=1}^{n}(a_r(x)u^1_{x_r})x_r + (\delta+\mu)\,u^1(t,x) = -f(t,x;u^1(t,x),u^2(t,x)), \\ \frac{\partial u^2(t,x)}{\partial t} - \sum\limits_{r=1}^{n}(a_r(x)u^2_{x_r})x_r + (\delta+\mu+\alpha)\,u^2(t,x) \\ = f(t,x;u^1(t,x),u^2(t,x)) - g(t,x;u^2(t,x)), \\ \frac{\partial u^3(t,x)}{\partial t} - \sum\limits_{r=1}^{n}(a_r(x)u^3_{x_r})x_r + (\delta+\mu)\,u^3(t,x) \\ = g(t,x;u^2(t,x)), 0<t<T, x=(x_1,...,x_n)\in\Omega, \\ u^m(0,x)=\varphi^m(x), x\in\overline{\Omega}, m=1,2,3, \\ \frac{\partial u}{\partial\vec{p}}(t,x)=0, 0\le t\le T, x\in S, m=1,2,3, \end{cases} \tag{44}$$

where $a_r(x)$ and $\varphi^m(x)$ are given sufficiently smooth functions and $\delta>0$ is the sufficiently large number and $a_r(x)>0$. Here, $\vec{p}$ is the normal vector to Ω.

The discretization of problem (44) is also carried out in two steps. In the first step, to the differential operator A generated by problem (44), we assign the difference operator A_h^x by the formula

$$A_h^x u_x^h = -\sum_{r=1}^{m}\left(a_r(x)u^h_{\overline{x}_r}\right)_{x_r,j_r} + \delta u^h(x) \tag{45}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $D^h u^h(x)=0$ for all $x\in S_h$. Here D^h is the approximation of operator $\dfrac{\partial}{\partial\vec{p}}$. It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\overline{\Omega}_h)$. With the help of A_h^x, we arrive at the initial value problem (42) for an infinite system of nonlinear ordinary differential equations. In the second step, we replace problem (42) by difference scheme (8), we get difference scheme (43).

Theorem 3.5. Let the assumptions (36)-(40) be satisfied and $2\left(L_1+L_2\right)T<1+\frac{\tau(\mu+\delta)}{2}$. Then, there exists a unique solution $u^\tau=\left\{u_k\right\}_{k=0}^N$ of difference scheme (43) which is bounded in $C_\tau\left(L_{2h}\right)\times C_\tau\left(L_{2h}\right)\times C_\tau\left(L_{2h}\right)$ of uniformly with respect to τ and h.

The proof of Theorem 3.5 is based on the abstract Theorem 2.1 and symmetry properties of the difference operator A_h^x defined by formula (45) and the following theorem on coercivity inequality for the solution of the elliptic problem in L_{2h}.

Theorem 3.6. For the solutions of the elliptic difference problem

$$\begin{cases} A_h^x v^h(x)=g^h(x),\ x\in\Omega_h, \\ D^h v^h(x)=0,\ x\in S_h \end{cases}$$

the following coercivity inequality holds.[15]

$$\sum_{r=1}^{m}\left\|v^h_{x_r\overline{x}_r,j_r}\right\|_{L_{2h}} \le M\left\|g^h\right\|_{L_{2h}}.$$

4 | NUMERICAL RESULTS

In present section, we consider the initial-boundary value problem

$$\begin{cases} \frac{\partial u^1(t,x)}{\partial t} - \lambda + \mu u^1(t,x) - \beta \frac{\partial^2 u^1(t,x)}{\partial x^2} \\ = -\lambda + (-1+\mu+\beta) e^{-t} \sin x - \sin\left(u^1(t,x)u^2(t,x) - e^{-2t}\sin^2 x\right), \\ \frac{\partial u^2(t,x)}{\partial t} + (\mu+\alpha)u^2(t,x) - d\frac{\partial^2 u^2(t,x)}{\partial x^2} = (-1+\mu+\alpha+d)e^{-t}\sin x \\ +\sin\left(u^1(t,x)u^2(t,x) - e^{-2t}\sin^2 x\right) - \cos\left(u^2(t,x) - e^{-t}\sin x\right), \\ \frac{\partial u^3(t,x)}{\partial t} + \mu u^1(t,x) - \gamma\frac{\partial^2 u^1(t,x)}{\partial x^2} = (-1+\mu+\gamma)e^{-t}\sin x \\ +\cos\left(u^2(t,x) - e^{-t}\sin x\right), 0 < t < 1, 0 < x < \pi, \\ u^m(0,x) = \sin(x), 0 \le x \le \pi, m = 1,2,3, \\ u^m(t,0) = u^m(t,\pi) = 0, 0 \le t \le 1, m = 1,2,3 \end{cases} \tag{46}$$

for the system of nonlinear partial differential equations. The spatial factor, x, can be spatially discrete or spatially continuous. In either case, the spatial factor is used to describe the mobility of the population. This mobility can be due to travel and migration, and it could be between cities, towns or even countries, depending on the studied case. The exact solution of problem (46) is $u^m(t,x) = e^{-t}\sin x, m = 1,2,3$.

We consider the first order of accuracy iterative difference scheme

$$\begin{cases} \frac{ju_n^{1,k} - ju_n^{1,k-1}}{\tau} + \mu j u_n^{1,k} - \beta\frac{ju_{n+1}^{1,k} - 2(ju_n^{1,k}) + ju_{n-1}^{1,k}}{h^2} \\ = (-1+\mu+\beta)e^{-t_k}\sin x_n - \sin\left((j-1)u_n^{1,k}(j-1)u_n^{2,k} - e^{-2t_k}\sin^2 x_n\right), \\ \frac{ju_n^{2,k} - ju_n^{2,k-1}}{\tau} + (\alpha+\mu) ju_n^{2,k} - d\frac{ju_{n+1}^{2,k} - 2(ju_n^{2,k}) + ju_{n-1}^{2,k}}{h^2} = (-1+\mu+\alpha+d)e^{-t_k}\sin x_n \\ +\sin\left((j-1)u_n^{1,k}(j-1)u_n^{2,k} - e^{-2t_k}\sin^2 x_n\right) - \cos\left((j-1)u_n^{2,k} - e^{-t_k}\sin x_n\right), \\ \frac{ju_n^{3,k} - ju_n^{3,k-1}}{\tau} + \mu j u_n^{3,k} - \gamma\frac{ju_{n+1}^{3,k} - 2(ju_n^{3,k}) + ju_{n-1}^{3,k}}{h^2} = (-1+\mu+\gamma)e^{-t_k}\sin x_n \\ +\cos\left((j-1)u_n^{2,k} - e^{-t_k}\sin x_n\right), \\ t_k = k\tau, 1 \le k \le N, N\tau = 1, x_n = nh, 1 \le n \le M-1, Mh = \pi, \\ ju_n^{m,0} = \varphi^m(x_n), ju_0^{m,k} = ju_M^{m,k} = 0, 0 \le k \le N, m = 1,2,3, j = 1,2,\cdots, \\ 0u_n^{m,k}, 0 \le k \le N, 0 \le n \le M, m = 1,2,3 \text{ is given} \end{cases} \tag{47}$$

and the second order of accuracy iterative Crank-Nicholson difference scheme

$$\begin{cases} \frac{{}_ju_n^{1,k}-{}_ju_n^{1,k-1}}{\tau}+\mu\frac{{}_ju_n^{1,k}+{}_ju_n^{1,k-1}}{2}-\beta\frac{{}_ju_{n+1}^{1,k}-2({}_ju_n^{1,k})+{}_ju_{n-1}^{1,k}}{2h^2}-\beta\frac{{}_ju_{n+1}^{1,k-1}-2({}_ju_n^{1,k-1})+{}_ju_{n-1}^{1,k-1}}{2h^2} \\ =(-1+\mu+\beta)e^{-\left(t_k-\frac{\tau}{2}\right)}\sin x_n \\ -\sin\left(\frac{(j-1)u_n^{1,k}+(j-1)u_n^{1,k-1}}{2}\frac{(j-1)u_n^{2,k}+(j-1)u_n^{2,k-1}}{2}-e^{-2\left(t_k-\frac{\tau}{2}\right)}\sin^2 x_n\right), \\ \frac{{}_ju_n^{2,k}-{}_ju_n^{2,k-1}}{\tau}+(\alpha+\mu)\frac{{}_ju_n^{2,k}+{}_ju_n^{2,k-1}}{2}-d\frac{{}_ju_{n+1}^{2,k}-2({}_ju_n^{2,k})+{}_ju_{n-1}^{2,k}}{2h^2}-d\frac{{}_ju_{n+1}^{2,k-1}-2({}_ju_n^{2,k-1})+{}_ju_{n-1}^{2,k-1}}{2h^2} \\ =(-1+\mu+\alpha+d)e^{-\left(t_k-\frac{\tau}{2}\right)}\sin x_n \\ +\sin\left(\frac{(j-1)u_n^{1,k}+(j-1)u_n^{1,k-1}}{2}\frac{(j-1)u_n^{2,k}+(j-1)u_n^{2,k-1}}{2}-e^{-2\left(t_k-\frac{\tau}{2}\right)}\sin^2 x_n\right) \\ -\cos\left(\frac{(j-1)u_n^{2,k}+(j-1)u_n^{2,k-1}}{2}-e^{-\left(t_k-\frac{\tau}{2}\right)}\sin x_n\right), \\ \frac{{}_ju_n^{3,k}-{}_ju_n^{3,k-1}}{\tau}+\mu\frac{{}_ju_n^{3,k}+{}_ju_n^{3,k-1}}{2}-\gamma\frac{{}_ju_{n+1}^{3,k}-2({}_ju_n^{3,k})+{}_ju_{n-1}^{3,k}}{2h^2}-\gamma\frac{{}_ju_{n+1}^{3,k-1}-2({}_ju_n^{3,k-1})+{}_ju_{n-1}^{3,k-1}}{2h^2} \\ =(-1+\mu+\gamma)e^{-\left(t_k-\frac{\tau}{2}\right)}\sin x_n+\cos\left(\frac{(j-1)u_n^{2,k}+(j-1)u_n^{2,k-1}}{2}-e^{-\left(t_k-\frac{\tau}{2}\right)}\sin x_n\right), \\ t_k=k\tau, 1\le k\le N, N\tau=1, x_n=nh, 1\le n\le M-1, Mh=\pi, \\ ju_n^{m,0}=\varphi^m(x_n), ju_0^{m,k}=ju_M^{m,k}=0, 0\le k\le N, m=1,2,3, j=1,2,\cdots, \\ 0u_n^{m,k}, 0\le k\le N, 0\le n\le M, m=1,2,3 \text{ is given} \end{cases} \tag{48}$$

for the approximate solution of the initial-boundary value problem (46) for the system of nonlinear parabolic equations. Here and in future j denotes the iteration index and an initial guess ${}_0u_n^k, k\ge 1, 0\le n\le M$ is to be made. For solving difference schemes (48), the numerical steps are given below. For $0\le k<N, 0\le n\le M$ the algorithm is as follows:[10]

1. $j=1$.
2. ${}_{j-1}u_n^k$ is known.
3. ${}_ju_n^k$ is calculated.
4. If the max absolute error between ${}_{j-1}u_n^k$ and ${}_ju_n^k$ is greater than the given tolerance value, take $j=j+1$ and go to step 2. Otherwise, terminate the iteration process and take ${}_ju_n^k$ as the result of the given problem. The errors are computed by

$$(jE^m)_M^N=\max_{1\le k\le N, 1\le n\le M-1}\left|u^m(t_k,x_n)-(ju^m)_n^k\right|, m=1,2,3 \tag{49}$$

of the numerical solutions, where $u^m(t_k,x_n)$, $m=1,2,3$ represents the exact solution and $(ju^m)_n^k$, $m=1,2,3$ represents the numerical solution at (t_k,x_n) and the results are given in the following table

$(jE^m)_M^N$	$N=M=20$	$N=M=40$	$N=M=80$
$m=1$	0.0068, j=6	0.0032, j=6	0.0016, j=6
$m=2$	0.0071, j=6	0.0033, j=6	0.0016, j=6
$m=3$	0.0073, j=6	0.0034, j=6	0.0017, j=6

(50)

$(jE^m)_M^N$	$N=M=20$	$N=M=40$	$N=M=80$
$m=1$	3.4708e-6, j=7	5.4645e-6, j=7	6.9510e-6, j=7
$m=2$	1.3882e-5, j=7	2.1857e-5, j=7	2.7803e-5, j=7
$m=3$	5.5516e-5, j=7	8.7420e-5, j=7	1.1120e-4, j=7

(51)

As it is seen in Table 50 and Table 51, we get some numerical results. If N and M are doubled, the value of errors in the first order of accuracy difference scheme decrease by a factor of $1/2$, the errors in the second order of accuracy difference schemes (48) decrease approximately by a factor of $1/4$. The errors presented in the tables indicate the stability of the difference schemes and the accuracy of the results. Thus, the second order of accuracy difference scheme increases faster than the first order of accuracy difference scheme.

5 | CONCLUSIONS

In the present paper, the initial boundary value problem for the nonlinear system of parabolic equations observing epidemic models with general nonlinear incidence rate is investigated.

The main theorem on the existence and uniqueness of a bounded solution of Crank-Nicholson difference scheme uniformly with respect to time step τ is established. Applications of the theoretical results are presented for the four systems of one and multidimensional problems with different boundary conditions. Numerical results are given.

ACKNOWLEDGMENTS

The publication has been prepared with the support of the "RUDN University Program 5-100" and published under target program BR05236656 of the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan.

References

1. Li M, Liu X. An SIR epidemic model with time delay and general nonlinear incidence rate. *Abstract and Applied Analysis.* 2014;2014:7.
2. Kirane M, Tatar N. Convergence rates for a reaction-diffusion system. *Zeitschrift für Analysis und ihre Anwendungen.* 2001;20(2):347–357.
3. Kaymakamzade B, Hincal E. Delay epidemic model with and without vaccine. *Quality and Quantity.* 2018;52(2):695–709.
4. Jun-Jie W, Reilly KH, Hua H, Zhi-Hang P, Ning W. Dynamic characteristic analysis of HIV mother to child transmission in China. *Biomedical and Environmental Sciences.* 2010;23(5):402–408.
5. Ashyralyev A, Hincal E, Kaymakamzade B. Numerical solutions of the system of partial differential equations for observing epidemic models. *AIP Conference Proceedings.* 2018;1997(1):020050.
6. Ashyralyev A, Hincal E, Kaymakamzade B. A Crank Nicolson difference scheme for system of nonlinear observing epidemic models. *AIP Conference Proceedings.* 2019;2183(1):070013.
7. Lotfi EM, Maziane M, Hattaf K, Yousfi N. Partial differential equations of an epidemic model with spatial diffusion. *International Journal of Partial Differential Equationsl.* 2014;2014:6. Fld.3996.
8. Chalub FACC, Souza MO. The SIR epidemic model from a PDE point of view. *Mathematical and Computer Modelling.* 2011;53(7–8):1568–1574.
9. Elkadry A. Transmission Rate in Partial Differential Equation in Epidemic Models. *Master Thesis, Marshall University.* 2013;:52.
10. Samarskii AA. *The theory of difference schemes.* Moscow, Russia: CRC Press; 2001.
11. Ashyralyev A, Sobolevskii PE. *New difference schemes for partial differential equations.* Verlag, Basel: Birkhäuser; 2004. Volume 148.

12. Kreui SG. *Linear differential equations in Banach space*. Nauka, Moscow: American Mathematical Soc.; 2011.

13. Ashyralyev A. Fractional spaces generated by the positive differential and difference operators in a Banach space. *Mathematical Methods in Engineering.* 2007;:13–22.

14. Ashyralyev A, Sarsenbi AM. Well-posedness of an elliptic equation with involution. *Electronic Journal of Differential Equations.* 2015;284:1–8.

15. Sobolevskii PE. *Difference methods for the approximate solution of differential equations*. Russia: Voronezh State University Press, Voronezh; 1975.

16. Ashyralyev A, Agirseven D. Bounded Solutions of Semilinear Time Delay Hyperbolic Differential and Difference Equations. *Mathematics.* 2019;7(12):1163.

How to cite this article: Williams K., B. Hoskins, R. Lee, G. Masato, and T. Woollings (2016), A regime analysis of Atlantic winter jet variability applied to evaluate HadGEM3-GC2, *Q.J.R. Meteorol. Soc.*, *2017;00:1–6*.