The solution of the inner and outer Dirichlet problems based on the discrete double-layer potential

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Abstract

Discrete double-layer potentials are powerful tools to describe the solution of discrete Dirichlet problems for the Laplace equation. In discrete potential theory there are already theorems known, regarding the existence and uniqueness of the solution of Dirichlet problems. The results on the solution of these finite difference Dirichlet problems are based on the single-layer potential. In this paper we extend the theory by using double-layer potentials.

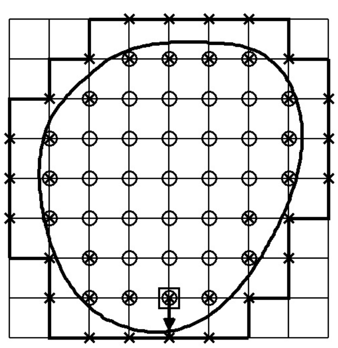
1 Introduction

The final goal is to solve discrete boundary value problems in case of a bounded domain as well as for the complement of such a bounded domain, having in mind the idea of considering transmission problems or coupled problems. This requires an abstract and exact definition of "interior" and of boundary points of a discrete domain. The geometric idea of inner points or of a boundary point must be adjusted to the definition of the finite difference operator and of the corresponding discrete potentials. The geometrically simpler case of a half space or two complementary half spaces was recently studied. Based on fundamental work on discrete potential operators and their boundary operator equations by Ryabenkii [1] first ideas for the here considered problems were already studied in [2], which differs especially in the approach with the double-layer potential in the inner and outer domain. In this context, we come to a second interesting point: In the continuous case, the solvability of boundary integral equations is based on the jump relations of the potentials and on their values on the boundary. When working on a uniform lattice of step size h, however, no limit considerations can be performed. In the following, for this reason, the directly resulting boundary equation system from the approach with the double-layer potential has to be examined for its solvability with a suitable additional summand. For coupled problems, the boundary layers are chosen such that the boundary points on the outer boundary layer of the outer problem forms a subset of the boundary points from the outer boundary layer of the inner problem. This subset is a proper subset in the case of existing concave corners, otherwise the sets coincide.

2 The discrete domain and the boundary layers

2.1 Basics for bounded domains

Let $\mathbb{R}^2$ be the 2-dimensional Euclidean space. By $\mathbb{R}^2_h = \{lh = (l_1h, l_2h) : \; l_1, l_2 \in \mathbb{Z}\}$ we define a uniform lattice of the step size $h > 0$. We consider a bounded, simply connected domain $G \subset \mathbb{R}^2$ with the piecewise smooth boundary Γ. Further we use the notations $M = \{l = (l_1, l_2) : \; l_1, l_2 \in \mathbb{Z}, (l_1h, l_2h) \in (G \cap \mathbb{R}^2_h)\}$ and $K = \{(0,0),\ (1,0),\ (-1,0),\ (0,1),\ (0,-1)\}$. If l runs through the set M, then N is the union of all five-point star sets $N_l = \{l + k : \; k \in K\}$. At all points $r = (r_1, r_2) \in N$ let $K_r = \{k \in K : \; r + k \notin M\}$. The domain G corresponds to the discrete domain $G_h = \{lh = (l_1h, l_2h) : l = (l_1, l_2) \in M\}$ with the two discrete boundary layers $\gamma_h = \{rh : \; r \in N$ with $K_r \neq \emptyset\}$. All points rh with $k = (0,0) \in K_r$ belong to the outer boundary layer $\gamma_h^- \subset \gamma_h$, while the inner boundary layer γ_h^+ consists of the points $rh \in \gamma_h \setminus \gamma_h^-$. The following figure should clarify the introduced notations.



× . . . point on γ_h^-
⊗ . . . point on γ_h^+
○ . . . point from G_h
$\boxtimes$ . . . special point $rh \in \gamma_h^+$
↓ . . . reference to $k \in K_r$

Figure 1

2.2 Basics for the complement of bounded domains

Similar to subsection 2.1 we define $M^a = \{l = (l_1, l_2) : \; l_1, l_2 \in \mathbb{Z}, (l_1h, l_2h) \in \mathbb{R}^2_h \setminus (G_h \cup \gamma_h^-)\}$ and $K_r^a = \{k \in K : \; r + k \notin M^a\}$. The discrete outer domain is denoted by $G_h^a = \{lh = (l_1h, l_2h) : \; l = (l_1, l_2) \in M^a\}$ with the two boundary layers $\gamma_h^a = \{rh : \; r \in N^a$ with $K_r^a \neq \emptyset\}$. Concerning the union of all five-point star environments, l passes through the set M^a. All points with $k = (0,0) \in K_r^a$ belong to the outer boundary layer $\gamma_h^{a-} \subset \gamma_h^a$, while the inner boundary layer γ_h^{a+} consists of the points $rh \in \gamma_h^a \setminus \gamma_h^{a-}$. We remark that $\gamma_h^{a-} \subset \gamma_h^-$. In many cases, even $\gamma_h^{a-} = \gamma_h^-$. We only have a subset if inside corners are to be considered as in an L-shaped domain. In this case we have the following problem: If we look at the mesh points in the outer domain, then we can not reach the inner corner by adding the five-point star environments.

3 Solution of the inner Dirichlet problem with the double-layer potential

Existence and uniqueness theorems for discrete Dirichlet and Neumann problems and the solvability conditions for the Neumann problems are proved already in [2] by means of discrete potential theory. Away from applying these theorems to boundary value problems for finite difference operators such results are known much longer (see e.g. [3] and references therein). The main problem here is, that in [2] Dirichlet problems were solved by using single-layer potentials and stability properties were not studied. To prepare such investigations we will adapt the theory in this contribution more to the continuous case, where we solve Dirichlet problems in terms of double-layer potentials. To be more precise we study for convenience the Dirichlet problem for the discrete Laplace operator. The discrete Laplace operator is defined by

$$-\Delta_h u_h(lh) = \sum_{k\in K} a_k u_h(lh-kh) \quad \text{with} \quad a_k = \begin{cases} -1/h^2 & \text{for } k\in K, k\neq(0,0) \\ 4/h^2 & \text{for } k=(0,0) \end{cases}.$$

We start with the solution of the inner Dirichlet problem

$$\begin{aligned} -\Delta_h u_h(lh) &= 0 && \forall\, lh \in G_h \\ u_h(lh) &= \varphi_h(lh) && \forall\, lh\in\gamma_h^- \end{aligned} \tag{1}$$

and denote the discrete fundamental solution of the Laplace operator (see [4],[5] and [2]) by E_h. This fundamental solution has the property

$$-\Delta_h E_h(lh-rh) = \delta_h(lh-rh) = \begin{cases} 1/h^2 & \text{for } l=r \\ 0 & \text{for } l\neq r \end{cases}.$$

Based on the second Green's formula (see [2]) we can prove for discrete harmonic functions $u_h(lh)$

$$u_h(lh) = P_h^E u_A(lh) - P_h^D u_h(lh) \qquad \forall\, lh\in G_h\cup\gamma_h^-$$

with the single-layer potential

$$P_h^E u_A(lh) = \sum_{rh\in\gamma_h^-} u_A(rh)E_h(lh-rh)h, \qquad u_A(rh) = h^{-1}\sum_{k\in K\setminus K_r}(u_h(rh)-u_h((r+k)h))$$

and the double-layer potential

$$\begin{aligned} &P_h^D u_h(lh) \\ &= \begin{cases} \sum\limits_{rh\in\gamma_h^-}\sum\limits_{k\in K\setminus K_r}(E_h(lh-rh)-E_h(lh-(r+k)h))u_h(rh) & lh\in G_h \\ \sum\limits_{rh\in\gamma_h^-}\sum\limits_{k\in K\setminus K_r}(E_h(lh-rh)-E_h(lh-(r+k)h))u_h(rh)-u_h(lh) & lh\in\gamma_h^- \end{cases}. \end{aligned}$$

In order to obtain a special approch which is only based on the double-layer potential we add to $P_h^D u_h(lh)$ a summand $P_h^{harm} u_h(lh)\ \forall\, lh \in G_h \cup \gamma_h^-$ which is a discrete harmonic function inside G_h. In detail we consider

$$
\begin{aligned}
& P_h^{D+harm} u_h(lh) = P_h^D u_h(lh) + P_h^{harm} u_h(lh) \\
= \ & P_h^D u_h(lh) + \sum_{rh \in \gamma_h^-} \sum_{k \in K:(r+k)h \in \gamma_h^-} \left(E_h(lh-rh) - E_h(lh-(r+k)h)\right) u_h(rh) \\
= \ & \begin{cases} \sum\limits_{rh \in \gamma_h^-} \sum\limits_{k \in K:(r+k)h \in \gamma_h} \left(E_h(lh-rh) - E_h(lh-(r+k)h)\right) u_h(rh) & lh \in G_h \\ \sum\limits_{rh \in \gamma_h^-} \sum\limits_{k \in K:(r+k)h \in \gamma_h} \left(E_h(lh-rh) - E_h(lh-(r+k)h)\right) u_h(rh) - u_h(lh) & lh \in \gamma_h^- \end{cases}
\end{aligned}
$$

with $\gamma_h = \gamma_h^+ \cup \gamma_h^-$.

Theorem 3.1 *If the linear equation system*

$$
\varphi_h(lh) = P_h^{D+harm} u_h(lh) \qquad \forall\, lh \in \gamma_h^- \tag{2}
$$

is solvable then the potential $P_h^{D+harm} u_h(lh)$ represents a solution of the inner Dirichlet problem (1).

Proof: The boundary condition is fulfilled automatically. Because of the property of the discrete fundamental solution, $-\Delta_h P_h^{D+harm} u_h(lh) = 0$, if $lh \in G_h \setminus \gamma_h^+$. For all mesh points $lh \in \gamma_h^+$ results

$$
\begin{aligned}
-\Delta_h P_h^{D+harm} u_h(lh) \ &= \ \sum_{rh \in \gamma_h^-} \sum_{k \in K:(r+k)h \in \gamma_h} \left(-\Delta_h E_h(lh-rh) + \Delta_h E_h(lh-(r+k)h\right) u_h(rh) \\
&\quad + h^{-2} \sum_{k \in K:(l+k)h \in \gamma_h^-} u_h((l+k)h) \\
&= \ -h^{-2} \sum_{k \in K:(l-k)h \in \gamma_h^-} u_h((l-k)h) + h^{-2} \sum_{k \in K:(l+k)h \in \gamma_h^-} u_h((l+k)h) = 0 \quad \blacksquare
\end{aligned}
$$

We now investigate whether the equation system (2) is solvable.

Theorem 3.2 *The equation system (2) is uniquely solvable for any given boundary values $\varphi_h(lh)$.*

Proof: The assertion follows from Fredholm's alternative, if the homogeneous problem has only the trivial solution. Fredholm's alternative reads as follows: If for all $lh \in \gamma_h^-$ the system

$$
\begin{aligned}
0 &= \sum_{rh\in\gamma_h^-}\ \sum_{k\in K:(l+k)h\in\gamma_h} (E_h(lh-rh)-E_h((l+k)h-rh))w_h(rh)-w_h(lh) \qquad (3)\\
&= -h^2\sum_{rh\in\gamma_h^-}\Delta_h E_h(lh-rh)w_h(rh)-w_h(lh)\\
&\quad -\sum_{rh\in\gamma_h^-}\ \sum_{k\in K\setminus K_l^a}(E_h(lh-rh)-E_h((l+k)h-rh)w_h(rh)\\
&= -\sum_{rh\in\gamma_h^-}\ \sum_{k\in K\setminus K_l^a}(E_h(lh-rh)-E_h((l+k)h-rh)w_h(rh) \qquad (4)
\end{aligned}
$$

which is adjoint to the homogeneous system (2) has a non-trivial solution $w_h^*(lh)$, then at least one non-zero solution of the homogeneous system (2) exists. The system (3) can be considered as an approximation of the classical integral equation

$$
0=\int_\Gamma \frac{\partial}{\partial n_y}E(y-x)\cdot w(x)\,d\Gamma_x+f(y)-w(y) \quad \text{with} \quad \Delta f(y)=0
$$

related to the inner problem, while representation (4) is related to the outer problem. Note that the approach chosen here, results directly from the double-layer potential, whereas in the continuous case, jump relations play an important role. In detail, the additional summand $f(y)$ is helpful to create a discrete analogue to the jump relations on the boundary.

We start with representation (3) and denote by $\tilde{P}_h^{harm}$ the adjoint operator with

$$
\tilde{P}_h^{harm}w_h^*(lh)=\sum_{rh\in\gamma_h^-}\ \sum_{k\in K:(l+k)h\in\gamma_h^-}(E_h(lh-rh)-E_h((l+k)h-rh))w_h^*(rh)
$$

Based on the Gaussian formula

$$
P_h^D(rh)=\begin{cases}\sum\limits_{lh\in\gamma_h^-}\ \sum\limits_{k\in K\setminus K_l}(E_h(rh-lh)-E_h(rh-(l+k)h))=-1 & rh\in G_h\\ \sum\limits_{lh\in\gamma_h^-}\ \sum\limits_{k\in K\setminus K_l}(E_h(rh-lh)-E_h(rh-(l+k)h))=0 & rh\notin G_h\end{cases}
$$

(see [2]) and the symmetry property $E_h(l_1h,l_2h)=E_h(-l_1h,l_2h)=E_h(l_1h,-l_2h)$ of the discrete fundamental solution we obtain

$$
\begin{aligned}
0 &= \sum_{lh\in\gamma_h^-}(w_h^*(lh)-\tilde{P}_h^{harm}w_h^*(lh))h\\
&\quad -\sum_{lh\in\gamma_h^-}\left(\sum_{rh\in\gamma_h^-}\ \sum_{k\in K\setminus K_l}(E_h(lh-rh)-E_h((l+k)h-rh)\right)w_h^*(rh)h
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{lh\in\gamma_h^-} (w_h^*(lh) - \tilde{P}_h^{harm} w_h^*(lh))h \\
&\quad - \sum_{rh\in\gamma_h^-} \left(\sum_{lh\in\gamma_h^-} \sum_{k\in K\setminus K_l} (E_h(rh-lh) - E_h(rh-(l+k)h)) \right) w_h^*(rh)h \\
&= \sum_{lh\in\gamma_h^-} (w_h^*(lh) - \tilde{P}_h^{harm} w_h^*(lh))h \\
&= \sum_{lh\in\gamma_h^-} w_h^*(lh)h - \sum_{rh\in\gamma_h^-} \left(\sum_{lh\in\gamma_h^-} \sum_{k\in K:(l+k)h\in\gamma_h^-} (E_h(lh-rh) - E_h((l+k)h-rh) \right) w_h^*(rh)h \\
&= \sum_{lh\in\gamma_h^-} w_h^*(lh)h\,.
\end{aligned}
$$

The last relation is true, because all terms inside the big brackets are added as often as substracted.

We consider now the inner Neumann problem

$$
\begin{aligned}
-\Delta_h u_h(lh) &= 0 & \forall\, lh \in G_h \\
u_A(lh) &= \psi_h(lh) & \forall\, lh \in \gamma_h^-
\end{aligned}
$$

In case of $\psi_h(lh) = w_h^*(lh) - \tilde{P}_h^{harm} w_h^*(lh)$ the single-layer potential $P_h^E w_h^*(lh)$ is a solution of this problem for all $lh \in G_h \cup \gamma_h^-$.

Let us expand $P_h^E w_h^*(lh)$ to all $lh \in G_h^a = \mathbb{R}_h^2 \setminus (G_h \cup \gamma_h^-)$. For all $lh \in G_h^a$ the Laplace eqution is fulfilled and from (4) we know that the outer normal derivatives $\psi_h^a(lh)$ are zero $\forall\, lh \in \gamma_h^{a-} \subseteq \gamma_h^-$. Also the trivial solution is a solution of the homogeneous outer Neumann problem. Therefore we conclude $P_h^E w_h^*(lh) = C \quad \forall\, lh \in G_h^a \cup \gamma_h^{a-}$. We remark that the single-layer potential is bounded for $|lh| \to \infty$ if the condition $\sum\limits_{rh\in\gamma_h^-} w_h^*(lh)h = 0$ is satisfied.

Finally we consider the inner domain $\hat{G}_h = \mathbb{R}_h^2 \setminus (G_h^a \cup \gamma_h^{a-})$ with $\hat{\gamma}_h^- \subseteq \gamma_h^{a-}$. For all $lh \in \hat{\gamma}_h^- \subseteq \gamma_h^{a-}$ we have $P_h^E w_h^*(lh) = C$. Since the inner Dirichlet problem has an unique solution we obtain $P_h^E w_h^*(lh) = C \quad \forall\, lh \in \hat{G}_h \cup \hat{\gamma}_h^-$. Consequently, for all $lh \in \gamma_h^- \setminus \hat{\gamma}_h^-$ (inside corners of $G_h \cup \gamma_h^-$) we get $0 = -\Delta_h P_h^E w_h^*(lh) = h^{-1} w_h^*(lh)$. This means $w_h^*(lh) = 0$. In case $lh \in \gamma_h^- \cap \hat{\gamma}_h^-$ the inner normal derivatives and all derivatives of $P_h^E w_h^*(lh)$ along the boundary γ_h^- are zero. Therefore from (3) the identity $w_h^*(lh) = 0$ follows and the adjoint system has only the trivial solution ∎

4 Solution of the outer Dirichlet problem with the double-layer potential

We consider an outer discrete domain G_h^a with the boundary layers $\gamma_h^a = \gamma_h^{a+} \cup \gamma_h^{a-}$. In order to find a solution of the outer Dirichlet problem

$$\begin{aligned} -\Delta_h u_h(lh) &= 0 && \forall\, lh \in G_h^a \\ u_h(lh) &= \varphi_h^a(lh) && \forall\, lh \in \gamma_h^{a-}, \end{aligned} \tag{5}$$

based on the double-layer potential, we use the approach

$$P_{h,a} u_h(lh) = P_{h,a}^{D+harm} u_h(lh) + \sum_{rh \in \gamma_h^{a-}} u_h(rh) h \tag{6}$$

with

$$\begin{aligned} & P_{h,a}^{D+harm} u_h(lh) \\ = \;& P_h^{D,a} u_h(lh) + \sum_{rh \in \gamma_h^{a-}} \sum_{k \in K:(r+k)h \in \gamma_h^{a-}} \left(E_h(lh-rh) - E_h(lh-(r+k)h)\right) u_h(rh) \\ = \;& \begin{cases} \sum\limits_{rh \in \gamma_h^{a-}} \sum\limits_{k \in K:(r+k)h \in \gamma_h^{a}} \left(E_h(lh-rh) - E_h(lh-(r+k)h)\right) u_h(rh) & lh \in G_h^a \\ \sum\limits_{rh \in \gamma_h^{a-}} \sum\limits_{k \in K:(r+k)h \in \gamma_h^{a}} \left(E_h(lh-rh) - E_h(lh-(r+k)h)\right) u_h(rh) - u_h(lh) & lh \in \gamma_h^{a-}. \end{cases} \end{aligned} \tag{7}$$

Theorem 4.1 *If the linear equation system*

$$\varphi_h^a(lh) = P_{h,a} u_h(lh) \qquad \forall\, lh \in \gamma_h^{a-} \tag{8}$$

is solvable then the potential $P_{h,a} u_h(lh)$ represents a solution of the Dirichlet problem (5).

Proof: The boundary condition is obviously fulfilled. Based on the property of the discrete fundamental solution we find $-\Delta_h P_{h,a} u_h(lh) = 0$ for all $lh \in G_h^a \setminus \gamma_h^{a+}$. In the mesh points $lh \in \gamma_h^{a+}$ we obtain

$$\begin{aligned} -\Delta_h P_{h,a} u_h(lh) &= \sum_{rh \in \gamma_h^{a-}} \sum_{k \in K:(r+k)h \in \gamma_h^{a}} (-\Delta_h E_h(lh-rh) + \Delta_h E_h(lh-(r+k)h) u_h(rh) \\ &\quad + h^{-2} \sum_{k \in K:(l+k)h \in \gamma_h^{a-}} u_h((l+k)h) - \Delta_h \sum_{rh \in \gamma_h^{a-}} u_h(rh)\, h \\ &= -h^{-2} \sum_{k \in K:(l-k)h \in \gamma_h^{a-}} u_h((l-k)h) + h^{-2} \sum_{k \in K:(l+k)h \in \gamma_h^{a-}} u_h((l+k)h) = 0 \quad \blacksquare \end{aligned}$$

The following property is important for further consideration:

Lemma 4.1 *At infinity we have the behavior* $P_{h,a}^{D+harm} u_h(lh) \to 0$ *for* $|lh| \to \infty$.

Proof: According to a theorem proved by Thomée in [6], for finite differences of the discrete fundamental solution of the Laplace operator we obtain

$$\begin{aligned} & |P_{h,a}^{D+harm} u_h(lh)| \\ \leq & \max_{rh \in \gamma_h^{a-}} |u_h(rh)| \sum_{rh \in \gamma_h^{a-}} \left| h^{-1} \sum_{k \in K:(r+k)h \in \gamma_h^a} (E_h(lh-rh) - E_h(lh-(r+k)h)) \right| h \\ \leq & \max_{rh \in \gamma_h^{a-}} |u_h(rh)| \cdot C_1 \max_{rh \in \gamma_h^{a-}} (|lh-rh|+h)^{-1} \sum_{rh \in \gamma_h^{a-}} h, \end{aligned}$$

where the constant C_1 is independent of h. Since the boundary γ_h^{a-} consists only of a finite number of mesh points, the assertion of the lemma follows ■

We now come to the uniqueness theorem for the solution of the boundary equation system:

Theorem 4.2 *The equation system (8) is uniquely solvable for any given boundary values* $\varphi_h^a(lh)$.

Proof: We consider the homogeneous equation and assume that any solution $u_h^*(llh)$ exists. By Theorem 4.1 the potential $P_{h,a} u_h^*(lh)$ is a discrete harmonic function in G_h^a. Since the trivial solution is also a solution of the homogeneous outer Dirichlet problem, we have $P_{h,a} u_h^*(lh) = 0 \quad \forall\, lh \in G_h^a \cup \gamma_h^{a-}$ because of the uniqueness theorem of the outer Dirichlet problem. From Lemma 4.1 we know that $P_{h,a}^{D+harm} u_h^*(lh)$ tends to zero for $|lh| \to \infty$. Consequently, we obtain

$$\sum_{rh \in \gamma_h^{a-}} u_h(rh) h = 0 \,. \tag{9}$$

This condition simplifies approach (6). In the following we will show that the homogeneous system (8) with the simplified approach (6) has only $u_h^*(lh) = 1$ as linear independent solution. Therefore we obtain $u_h^*(lh) = C$ in the general case and from (9) $u_h^*(lh) = 0$ follows.

For all $lh \in \gamma_h^{a-}$ we write the simplified homogeneous equation system in the form

$$\begin{aligned} 0 &= \sum_{rh \in \gamma_h^{a-}} \sum_{k \in K} (E_h(lh-rh) - E_h(lh-(r+k)h))\, u_h^*(rh) - u_h^*(lh) \\ &\quad - \sum_{rh \in \gamma_h^{a-}} \sum_{k \in K \setminus K_r} (E_h(lh-rh) - E_h(lh-(r+k)h))\, u_h^*(rh) \\ &= -\sum_{rh \in \gamma_h^{a-}} \sum_{k \in K \setminus K_r} (E_h(lh-rh) - E_h(lh-(r+k)h))\, u_h^*(rh), \end{aligned}$$

where $K \setminus K_r = \{k \in K : (r+k)h \in G_h\}$ and $G_h = \mathbb{R}^2_h \setminus (G^a_h \cup \gamma^{a-}_h)$. We remark that $\gamma^-_h \subset \gamma^{a-}_h$. Points $rh \in \gamma^{a-}_h \setminus \gamma^-_h$ are outside corners with respect to the domain G_h. In this outside corners $\{k \in K : (r+k)h \in G_h\} = \emptyset$, such that

$$
\begin{aligned}
0 &= -\sum_{rh\in\gamma^{a-}_h} \sum_{k\in K\setminus K_r} (E_h(lh-rh) - E_h(lh-(r+k)h))\, u^*_h(rh) \\
&= -\sum_{rh\in\gamma^-_h} \sum_{k\in K\setminus K_r} (E_h(lh-rh) - E_h(lh-(r+k)h))\, u^*_h(rh)\,.
\end{aligned}
$$

Because of the Gaussian formula (see section 3) this equation has for all $lh \in \gamma^{a-}_h$ and $rh \in \gamma^-_h$ the nontrivial solution $u^*_h(rh) = 1$. We show that this is the only linear independent solution by proving that also the adjoint equation has only one linear independent solution.

Let us assume that the adjoint equation

$$
0 = \sum_{rh\in\gamma^-_h} \sum_{k\in K\setminus K_l} (E_h(lh-rh) - E_h((l+k)h-rh))\, w^*_h(rh) \tag{10}
$$

has the nontrivial solution $w^*_h(rh)$ for $rh \in \gamma^-_h$. In the outside corners $lh \in \gamma^{a-}_h \setminus \gamma^-_h$ the set $K \setminus K_l$ is empty. Therefore we concentrate on $lh \in \gamma^-_h$. Obviously, the single-layer potential $P^E_h w^*_h(lh)$ is in all points $lh \in G_h \cup \gamma^-_h$ a solution of the homogeneous inner Neumann problem. On the other side, this problem has the trivial solution, such that from the uniqueness theorem the relation $P^E_h w^*_h(lh) = C \quad \forall\, lh \in G_h \cup \gamma^-_h$ follows. We consider now the potential $P^E_h w^*_h(lh)$ in the outer domain $\hat{G}^a_h = \mathbb{R}^2_h \setminus (G_h \cup \gamma^-_h)$ with $\hat{\gamma}^{a-}_h = \gamma^-_h$ and prove that $\sum\limits_{rh\in\gamma^-_h} w^*_h(rh)h \neq 0$. Let us assume that the oposit property is true. Then we can show analog to [2] that $P^E_h w^*_h(lh)$ is bounded for $|lh| \to \infty$. Since the single-layer potential is constant in all mesh points $rh \in \hat{\gamma}^{a-}_h = \gamma^-_h$ it follows from the uniqueness theorem of the outer Dirichlet problem that $P^E_h w^*_h(lh) = C \quad \forall\, lh \in \hat{G}^a_h \cup \hat{\gamma}^{a-}_h$ and therefore $P^E_h w^*_h(lh) = C$ in each mesh point $lh \in \mathbb{R}^2_h$. In addition we have in all mesh points $lh \in \gamma^-_h$ $0 = -\Delta_h P^E_h w^*_h(lh) = w^*_h(lh)h^{-1}$. But we consider here a nontrivial solution $w^*_h(lh)$, so that the assumption was wrong and we have $\sum\limits_{rh\in\gamma^-_h} w^*_h(rh)h = C_1 \neq 0$. In the following we prove that the homogeneous system (10) has no nontrivial solution, which is linear independent from $w^*_h(lh)$. Let $w^{**}_h(lh)$ be a second nontrivial solution of (10). From the above considerations we can follow that $\sum\limits_{rh\in\gamma^-_h} w^{**}_h(rh)h = C_2 \neq 0$. Furthermore, $w^{***}_h = w^*_h C_2 - w^{**}_h C_1$ is a solution of the homogeneous system (10). For this solution we have $\sum\limits_{rh\in\gamma^-_h} w^{***}_h(rh)h = 0$ and therefore $w^{***}_h(rh) = 0 \quad \forall\, rh \in \gamma^-_h$. Consequently, $w^{**}_h = \frac{C_2}{C_1} w^*_h$ and the system (10) has only one linear independent solution as well as the homogeneous system (8) with the simplified approch related to (9) ∎

Remark: Based on the 4th Fredholm theorem the inhomogeneous equation system (8) is solvable for only this boundary values $\varphi^a_h(lh)$ which are orthogonal to all solutions $w^*_h(lh)$

of the homogeneous adjoint equation system. Therefore the condition

$$\sum_{rh\in\gamma_h^-} \varphi_h^a(rh)\, w_h^*(rh)h = 0$$

is a necessary and sufficient solvability condition. If this equation is fulfilled, then a solution of the outer Dirichlet problem exists, which can be described with a double-layer potential and which has at infinity the behavior $|lh|^{-1}$. If the equation is not fulfilled, then there exists no solution which can be described with an approach based on the double-layer potential.

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