Convergence and rate of Convergence of of a system of exponential form difference equations

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Abstract

We study the persistence , boundedness and unboundedness , existence and uniqueness of positive equilibrium point, local and global asymptotic stability, and rate of convergence of following system of exponential form difference equations:

$$x_{n+1} = \alpha_1 + \beta_1 y_n + \gamma_1 y_{n-1} e^{-y_n},\ y_{n+1} = \alpha_2 + \beta_2 x_n + \gamma_2 x_{n-1} e^{-x_n}\ n = 0, 1, \cdots,$$

where initial values $x_{-1},\ y_{-1}, x_0,\ y_0$ and parameters $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2,\ \gamma_2$ are positive real numbers.

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1 Introduction

Difference equations or systems of difference equations play a vital role in the development of different sciences ranging from life to decision sciences (see [1]- [16] and references cited therein). One of the most important types of difference equation is the exponential form difference equation .These types have many applications in our life . For instance , El-Metwally et al [4] studied the qualitative behavior of the following population model

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}$$

Papaschinopoulos et al.[14] investigated the asympotitic bahavior of the positive solutions of the following systems of difference equations

$$x_{n+1} = \alpha_1 + \beta_1 y_{n-1} e^{-x_n}, \quad y_{n+1} = \alpha_2 + \beta_2 x_{n-1} e^{-y_n}$$

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A. Q.Khan et al. [11] investigated the qualitative behavior of the positive solution of following system of difference equation

$$x_{n+1} = \frac{\alpha e^{-y_n} + \beta e^{-y_{n-1}}}{\gamma + \alpha x_n + \beta x_{n-1}}, \quad y_{n+1} = \frac{\alpha_1 e^{-x_n} + \beta_1 e^{-x_{n-1}}}{\gamma_1 + \alpha_1 y_n + \beta_1 y_{n-1}}$$

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where initial values x_{-1}, x_0, y_{-1}, y_0 and the parameters $\alpha, \alpha_1, \beta, \beta_1, \gamma, \gamma_1$ are non-negative real numbers.

Motivated by the above studies, our aim in this paper is to investigate the boundedness ,unboundedness and persistence, existence and uniqueness of positive equilibrium point, local and global asymptotic stability, and rate of convergence of following system of difference equations:

$$x_{n+1} = \alpha_1 + \beta_1 y_n + \gamma_1 y_{n-1} e^{-y_n}, \ y_{n+1} = \alpha_2 + \beta_2 x_n + \gamma_2 x_{n-1} e^{-x_n} \ n = 0, 1, \cdots, \tag{1}$$

where initial values $x_{-1}, \ y_{-1}, x_0, \ y_0$ and parameters $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \ \gamma_2$ are positive real numbers. Finally, some numerical examples are given to verify our theoretical results.

2 Preliminaries

Let $I, \ J$ be some intervals of real numbers and let $f : I^2 \times J^2 \to I, \ g : I^2 \times J^2 \to J$ be continuously differentiable functions. Then for every initial conditions $(x_i, y_i) \in I \times J$,for $i \in \{-1, 0\}$, the system of difference equations

$$\begin{aligned} x_{n+1} &= f(x_n, x_{n-1}, y_n, y_{n-1}), \\ y_{n+1} &= g(x_n, x_{n-1}, y_n, y_{n-1}), \ n = 0, 1, \cdots, \end{aligned} \tag{2}$$

has a unique solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$. A point $(\bar{x}, \bar{y}) \in I \times J$ is called an equilibrium point of (2) if

$$\begin{aligned} \bar{x} &= f(\bar{x}, \bar{x}, \bar{y}, \bar{y}) \\ \bar{y} &= g(\bar{x}, \bar{x}, \bar{y}, \bar{y}). \end{aligned}$$

Definition 1. *Assume that $(\bar{x}, \bar{y})$ is an equilibrium point of system (2). Then one has the following*

(i) $(\bar{x}, \bar{y})$ is said to be stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every initial condition (x_i, y_i), $i \in \{-1, 0\}$, $\| \sum_{i=-1}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \| < \delta$ implies $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon$ for all $n > 0$, where $\|.\|$ is the usual Euclidian norm in $\mathbb{R}^2$.

(ii) $(\bar{x}, \bar{y})$ is said to be unstable if it is not stable.

(iii) $(\bar{x}, \bar{y})$ is said to be asymptotically stable if there exists $\eta > 0$ such that $\| \sum_{i=-1}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \| < \eta$ and $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

(iv) $(\bar{x}, \bar{y})$ is called global attractor if $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

(v) $(\bar{x}, \bar{y})$ is called asymptotic global attractor if it is a global attractor and stable.

Definition 2. *Let $(\bar{x}, \bar{y})$ be an equilibrium point of the map*

$$F = (f, x_n, g, y_n),$$

where f and g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of (2) about the equilibrium point $(\bar{x}, \bar{y})$ is

$$X_{n+1} = F(X_n) = F_J X_n,$$

where $X_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix}$ *and F_J is the Jacobian matrix of system (2) about the equilibrium point $(\bar{x}, \bar{y})$.*

Lemma 1. *[?] Consider the system $X_{n+1} = F(X_n)$, $n = 0, 1, \cdots$, where $\bar{X}$ is a fixed point of F. If all eigenvalues of the Jacobian matrix J_F about $\bar{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\bar{X}$ is locally asymptotically stable. If any of the eigenvalue has a modulus greater than one, then $\bar{X}$ is unstable.*

The following result gives the rate of convergence of solutions of a system of difference equations

$$X_{n+1} = (A + B(n))\, X_n, \tag{3}$$

where X_n is an m-dimensional vector, $A \in C^{m\times m}$ is a constant matrix, and $B : \mathbb{Z}^+ \to C^{m\times m}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \tag{4}$$

as $n \to \infty$,where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm

$$\|(x, y)\| = \sqrt{x^2 + y^2}.$$

Proposition 1. *(Perron's Theorem)[15] Suppose that condition* (4) *holds. If X_n is a solution of* (3)*, then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} (\|X_n\|)^{1/n} \tag{5}$$

or

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{6}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

3 Main results

In this section, we will prove main results for under consideration system.

3.1 boundededness of solutions

The following theorem shows that every positive solution $\{(x_n, y_n)\}$ of system (1) is bounded .

Theorem 1. *If*

$$1 - B - C - D = 1 - (\beta_1 + \gamma_1 e^{-\alpha_1})(\beta_2 + \gamma_2 e^{-\alpha_2}) > 0 \tag{7}$$

where $B = \beta_1\beta_2$,$C = \beta_1\gamma_2 e^{-\alpha_2} + \beta_2\gamma_1 e^{-\alpha_1}$ and $D = \gamma_1\gamma_2 e^{-(\alpha_1+\alpha_2)}$,then every positive solution $\{(x_n, y_n)\}$ of system (1) is bounded .

Proof. Let $\{(x_n, y_n)\}$ be any positive solution of system (1). From (1), we have

$$x_n \geq \alpha_1 \ , y_n \geq \alpha_2 \ , n = 0, 1, \cdots . \tag{8}$$

From (1) and (8), we have

$$\begin{aligned} x_{n+1} &= \alpha_1 + \beta_1 y_n + \gamma_1 y_{n-1} e^{-y_n} \\ &= \alpha_1 + \beta_1(\alpha_2 + \beta_2 x_{n-1} + \gamma_2 x_{n-2} e^{-x_{n-1}}) + \gamma_1(\alpha_2 + \beta_2 x_{n-2} + \gamma_2 x_{n-3} e^{-x_{n-2}}) e^{-y_n} \\ &\leq A_1 + B x_{n-1} + C x_{n-2} + D x_{n-3} \end{aligned}$$

Similarly ,

$$y_{n+1} \leq A_2 + B y_{n-1} + C y_{n-2} + D y_{n-3}$$

where $A_1 = \alpha_1 + \beta_1\alpha_2 + \gamma_1\alpha_2 e^{-\alpha_2}$,$A_2 = \alpha_2 + \beta_2\alpha_1 + \gamma_2\alpha_1 e^{-\alpha_1}$,$B = \beta_1\beta_2$,$C = \beta_1\gamma_2 e^{-\alpha_2} + \beta_2\gamma_1 e^{-\alpha_1}$ and $D = \gamma_1\gamma_2 e^{-(\alpha_1+\alpha_2)}$

Consider the following non-homogeneous difference equations

$$Z_{n+1} = A_1 + BZ_{n-1} + CZ_{n-2} + DZ_{n-3}, \quad T_{n+1} = A_2 + BT_{n-1} + CT_{n-2} + DT_{n-3}, \quad n = 0, 1, \cdots . \tag{9}$$

We can see that

$$\lim_{n\to\infty} Z_n = \frac{A_1}{1 - B - C - D}, \quad \lim_{n\to\infty} T_n = \frac{A_2}{1 - B - C - D}$$

Therefore ,

$$\lim_{n\to\infty} sup(x_n) < \frac{A_1}{1 - B - C - D}, \quad \lim_{n\to\infty} sup(y_n) < \frac{A_2}{1 - B - C - D}$$

Hence the proof is completed. □

3.2 Existence and uniqueness of positive equilibrium point

The following theorem shows the existence and uniqueness of positive equilibrium point of system (1).

Theorem 2. *The system (1) has a unique positive equilibrium point $(\bar{x}, \bar{y})$ in $\left[\alpha_1, \frac{A_1}{\psi}\right] \times \left[\alpha_2, \frac{A_2}{\psi}\right]$ if the following condition hold :*

$$\gamma_1 < \frac{A_1 - \alpha_1\psi - \alpha_2\beta_1\psi - A_1\beta_1\beta_2 - A_1\gamma_2\beta_1 e^{-A_1/\psi}}{(\alpha_2\psi + \beta_2 A_1 + \gamma_2 A_1 e^{-A_1/\psi}) e^{-\frac{\alpha_2\psi + \beta_2 A_1 + A_1\gamma_2 e^{-A_1/\psi}}{\psi}}}$$

where $\psi = 1 - (\beta_1 + \gamma_1 e^{-\alpha_1})(\beta_2 + \gamma_2 e^{-\alpha_2})$, $A_1 = \alpha_1 + \beta_1\alpha_2 + \gamma_1\alpha_2 e^{-\alpha_2}$,$A_2 = \alpha_2 + \beta_2\alpha_1 + \gamma_2\alpha_1 e^{-\alpha_1}$

Proof. Consider the following system:

$$x = \alpha_1 + \beta_1 y + \gamma_1 y e^{-y}, \ y = \alpha_2 + \beta_2 x + \gamma_2 x e^{-x}. \tag{10}$$

Defining $F(x) = \alpha_1 + \beta_1 f(x) + \gamma_1 f(x) e^{-f(x)} - x$ where $f(x) = \alpha_2 + \beta_2 x + \gamma_2 x e^{-x}$ and $x \in \left[\alpha_1, \frac{A_1}{\psi}\right]$.
It is easy to see that $F(\alpha_1) = (\beta_1 + \gamma_1 e^{-f(\alpha_1)}) f(\alpha_1) > 0$
Also, $F\left(\frac{A_2}{\psi}\right) < 0$ if and only if

$$\gamma_1 < \frac{A_1 - \alpha_1\psi - \alpha_2\beta_1\psi - A_1\beta_1\beta_2 - A_1\gamma_2\beta_1 e^{-A_1/\psi}}{(\alpha_2\psi + \beta_2 A_1 + \gamma_2 A_1 e^{-A_1/\psi}) e^{-\frac{\alpha_2\psi + \beta_2 A_1 + A_1\gamma_2 e^{-A_1/\psi}}{\psi}}}$$

Hence, $F(x)$ has at least one positive solution in the interval $\left[\alpha_1, \frac{A_1}{\psi}\right]$. Furthermore, one has

$$F'(x) = -1 + (\beta_2 + \gamma_2 e^{-x} - \gamma_2 x e^{-x}) \times$$
$$(\beta_1 + \gamma_1 e^{-(\alpha_2+\beta_2 x+\gamma_2 x e^{-x})} - \gamma_2(\alpha_2 + \beta_2 x + \gamma_2 x e^{-x}) e^{-(\alpha_2+\beta_2 x+\gamma_2 x e^{-x})}) < 0$$

Hence, $F(x) = 0$ has a unique positive solution $x \in \left[\alpha_1, \frac{A_1}{\psi}\right]$. $\square$

3.3 Local and global stability

Theorem 3. *The unique positive equilibrium point $(\bar{x}, \bar{y})$ in $[\alpha_1, \psi_1] \times [\alpha_2, \psi_2]$ of system (1) is locally asymptotically stable if*

$$(-\beta_2+\gamma_2\psi_1 e^{-\alpha_1})(-\beta_2+\gamma_2\psi_2 e^{-\alpha_2})+\beta_1\gamma_2 e^{-\alpha_1}+\beta_2\gamma_1 e^{-\alpha_2}+\gamma_1\gamma_2 e^{-(\alpha_1+\alpha_2)}[1+e^{\alpha_2}+\psi_2] < 1. \quad (11)$$

where $\psi_1 = \frac{A_1}{1-B-C-D}$, $\psi_2 = \frac{A_2}{1-B-C-D}$, $A_1 = \alpha_1+\beta_1\alpha_2+\gamma_1\alpha_2 e^{-\alpha_2}$,$A_2 = \alpha_2+\beta_2\alpha_1+\gamma_2\alpha_1 e^{-\alpha_1}$,$B = \beta_1\beta_2$,$C = \beta_1\gamma_2 e^{-\alpha_2} + \beta_2\gamma_1 e^{-\alpha_1}$ *and* $D = \gamma_1\gamma_2 e^{-(\alpha_1+\alpha_2)}$

Proof. The linearized system of (1) about the equilibrium point $(\bar{x}, \bar{y})$ is given by

$$\Phi_{n+1} = E\Phi_n,$$

where $\Phi_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix}$, $E = \begin{pmatrix} 0 & \beta_1 - \gamma_1\bar{y}e^{-\bar{y}} & 0 & \gamma_1 e^{-\bar{y}} \\ \beta_2 - \gamma_2\bar{x}e^{-\bar{x}} & 0 & \gamma_2 e^{-\bar{x}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

The characteristic polynomial of Jacobain matix E about $(\bar{x}, \bar{y})$ is given by

$$P(\lambda) = \lambda^4 + a\lambda^2 + b\lambda + c \quad (12)$$

where

$$a = (-\beta_1 + \gamma_1\bar{y}e^{-\bar{y}})(\beta_2 - \gamma_2\bar{x}e^{-\bar{x}}),$$
$$b = -\beta_2\gamma_1 e^{-\bar{y}} + \gamma_1\gamma_2\bar{x}e^{-\bar{x}-\bar{y}} - \beta_1\gamma_2 e^{-\bar{x}} + \gamma_1\gamma_2\bar{y}e^{-\bar{x}-\bar{y}}$$
$$c = -\gamma_1\gamma_2\bar{y}e^{-\bar{x}-\bar{y}}.$$

Let $\phi(\lambda) = \lambda^4$ and $\psi(\lambda) = a\lambda^2 + b\lambda + c$.
Assume that condition(11) is satisfied and $|\lambda| = 1$. Thus one has

$$|\psi(\lambda)| < (-\beta_2+\gamma_2\psi_1 e^{-\alpha_1})(-\beta_2+\gamma_2\psi_2 e^{-\alpha_2})+\beta_1\gamma_2 e^{-\alpha_1}+\beta_2\gamma_1 e^{-\alpha_2}+\gamma_1\gamma_2 e^{-(\alpha_1+\alpha_2)}[1+e^{\alpha_2}+\psi_2] < 1.$$

where $\psi_1 = \frac{A_1}{1-B-C-D}$, $\psi_2 = \frac{A_2}{1-B-C-D}$. Then, by Rouche's theorem $\phi(\lambda)$ and $\phi(\lambda) + \psi(\lambda)$ have the same number of zeros in an open unit disc $|\lambda| < 1$. Hence all the roots of (12) satisfies $|\lambda| < 1$.

This implies , by lemma(1),that the unique equilibrium point $(\bar{x}, \bar{y}) \in [\alpha_1, \psi_1] \times [\alpha_2, \psi_2]$ of system (1) is locally asymptotically stable. $\square$

Theorem 4. *The unique equilibrium point $(\bar{x}, \bar{y}) \in [\alpha_1, \psi_1] \times [\alpha_2, \psi_2]$ of system (1) is global attractor.*

Proof. Let $\{(x_n, y_n)\}$ be an arbitrary solution of system (1) and let $\lim_{n\to\infty} \sup x_n = S_1 < \infty$, $\lim_{n\to\infty} \inf x_n = I_1 > 0$, , $\lim_{n\to\infty} \sup y_n = S_2 < \infty$, $\lim_{n\to\infty} \inf y_n = I_2 > 0$ where $I_i, S_i \in (0, \infty)$, $i = 1, 2$. Then from system (1), we have

$$S_1 \leq \alpha_1 + \beta_1 S_2 + \gamma_1 S_2 e^{-I_2}, \quad I_1 \geq \alpha_1 + \beta_1 I_2 + \gamma_1 I_2 e^{-S_2}. \tag{13}$$

And

$$S_2 \leq \alpha_2 + \beta_2 S_1 + \gamma_2 S_1 e^{-I_1}, \quad I_2 \geq \alpha_2 + \beta_2 I_1 + \gamma_2 I_1 e^{-S_1}. \tag{14}$$

From (13), we have

$$I_1 S_1 \leq \alpha_1 I_1 + \beta_1 I_1 S_2 + \gamma_1 I_1 S_2 e^{-I_2}, \quad I_1 S_1 \geq \alpha_1 S_1 + \beta_1 I_2 S_1 + \gamma_1 I_2 S_1 e^{-S_2}. \tag{15}$$

From (14), we get

$$I_2 S_2 \leq \alpha_2 I_2 + \beta_2 I_2 S_1 + \gamma_2 I_2 S_1 e^{-I_2}, \quad I_2 S_2 \geq \alpha_2 S_2 + \beta_2 I_1 S_2 + \gamma_2 I_1 S_2 e^{-S_1}. \tag{16}$$

From (15) and (16), we get

$$\alpha_1 S_1 + \beta_1 I_2 S_1 + \gamma_1 I_2 S_1 e^{-S_2} \leq \alpha_1 I_1 + \beta_1 I_1 S_2 + \gamma_1 I_1 S_2 e^{-I_2}, \tag{17}$$

$$\alpha_2 S_2 + \beta_2 I_1 S_2 + \gamma_2 I_1 S_2 e^{-S_1} \leq \alpha_2 I_2 + \beta_2 I_2 S_1 + \gamma_1 I_2 S_1 e^{-I_1} \tag{18}$$

From (17) and (18) we have $S_1 \leq I_1$ and $S_2 \leq I_2$. Hence $S_1 = I_1$ and $S_2 = I_2$. Hence, the unique equilibrium point $(\bar{x}, \bar{y})$ of system (1) is a global attractor. □

Theorem 5. *If*

$$(-\beta_2+\gamma_2\psi_1 e^{-\alpha_1})(-\beta_2+\gamma_2\psi_2 e^{-\alpha_2})+\beta_1\gamma_2 e^{-\alpha_1}+\beta_2\gamma_1 e^{-\alpha_2}+\gamma_1\gamma_2 e^{-(\alpha_1+\alpha_2)}[1+e^{\alpha_2}+\psi_2] < 1. \tag{19}$$

where $\psi_1 = \frac{A_1}{1-B-C-D}$, $\psi_2 = \frac{A_2}{1-B-C-D}$, $A_1 = \alpha_1+\beta_1\alpha_2+\gamma_1\alpha_2 e^{-\alpha_2}$,$A_2 = \alpha_2+\beta_2\alpha_1+\gamma_2\alpha_1 e^{-\alpha_1}$,$B = \beta_1\beta_2$,$C = \beta_1\gamma_2 e^{-\alpha_2}+\beta_2\gamma_1 e^{-\alpha_1}$ *and* $D = \gamma_1\gamma_2 e^{-(\alpha_1+\alpha_2)}$ *,then the unique positive equilibrium point* $(\bar{x}, \bar{y}) \in [\alpha_1, \psi_1] \times [\alpha_2, \psi_2]$ *of system (1) is globally asymptotically stable*

Proof. The proof comes directly from theorem(3) , theorem(4) and definition (1) . □

3.4 Rate of convergence

In this section, we will determine the rate of convergence of a solution that converges to the unique positive equilibrium point of system (1).

Let $\{(x_n, y_n)\}$ be any solution of system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$, and $\lim_{n\to\infty} y_n = \bar{y}$, where $\bar{x} \in [\alpha_1, \psi_1]$ and $\bar{y} \in [\alpha_2, \psi_2]$.

In order to find the error terms, one has from system (1)

$$\begin{aligned} x_{n+1} - \bar{x} &= \alpha_1 + \beta_1 y_n + \gamma_1 y_{n-1} e^{-y_n} - \{\alpha_1 + \beta_1\bar{y} + \gamma_1\bar{y}e^{-\bar{y}}\} \\ &= \beta_1(y_n - \bar{y}) + \gamma_1 e^{-y_n}(y_{n-1} - \bar{y}) + \gamma_1\bar{y}(e^{-y_n} - e^{-\bar{y}}) \end{aligned}$$

$$= \beta_1(y_n - \bar{y}) + \gamma_1 e^{-y_n}(y_{n-1} - \bar{y}) + \gamma_1 \bar{y} e^{-\bar{y}}(-(y_n - \bar{y}) + O((y_n - \bar{y})^2))$$

$$\approx (\beta_1 - \gamma_1 \bar{y} e^{-\bar{y}})(y_n - \bar{y}) + \gamma_1 e^{-y_n}(y_{n-1} - \bar{y})$$

Similarly ,

$$y_{n+1} - \bar{y} \approx (\beta_2 - \gamma_2 \bar{x} e^{-\bar{x}})(x_n - \bar{x}) + \gamma_2 e^{-x_n}(x_{n-1} - \bar{x})$$

Set $e_n^1 = x_n - \bar{x}$ and $e_n^2 = y_n - \bar{y}$, one has

$$e_{n+1}^1 = A_n e_n^1 + B_n e_n^2 + C_n e_{n-1}^1 + D_n e_{n-1}^2,$$

$$e_{n+1}^2 = E_n e_n^1 + F_n e_n^2 + G_n e_{n-1}^1 + H_n e_{n-1}^2,$$

where

$$A_n = 0,\ B_n = \beta_1 - \gamma_1 \bar{y} e^{-\bar{y}},\ C_n = 0,\ D_n = \gamma_1 e^{-y_n}.$$

$$E_n = \beta_2 - \gamma_2 \bar{x} e^{-\bar{x}},\ F_n = 0,\ G_n = \gamma_2 e^{-x_n},\ H_n = 0.$$

Taking the limits, we obtain

$$\lim_{n\to\infty} A_n = 0,\ \lim_{n\to\infty} B_n = \beta_1 - \gamma_1 \bar{y} e^{-\bar{y}}, \lim_{n\to\infty} C_n = 0,\ \lim_{n\to\infty} D_n = \gamma_1 e^{-\bar{y}}.$$

$$\lim_{n\to\infty} E_n = \beta_2 - \gamma_2 \bar{x} e^{-\bar{x}},\ \lim_{n\to\infty} F_n = 0, \lim_{n\to\infty} G_n = \gamma_2 e^{-\bar{x}},\ \lim_{n\to\infty} H_n = 0.$$

Hence, the limiting system of error terms at $(\bar{x}, \bar{y})$ can be written as

$$E_{n+1} = K E_n, \tag{20}$$

where $E_n = \begin{pmatrix} e_{n+1}^1 \\ e_{n+1}^2 \\ e_n^1 \\ e_n^2 \end{pmatrix}$, $K = \begin{pmatrix} 0 & \beta_1 - \gamma_1 \bar{y} e^{-\bar{y}} & 0 & \gamma_1 e^{-\bar{y}} \\ \beta_2 - \gamma_2 \bar{x} e^{-\bar{x}} & 0 & \gamma_2 e^{-\bar{x}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$,

which is similar to the linearized system of (1) about the equilibrium point $(\bar{x}, \bar{y})$. Using proposition (1), one has following result.

Theorem 6. *Assume that $\{(x_n, y_n)\}$ be a positive solution of system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$, and $\lim_{n\to\infty} y_n = \bar{y}$, where $(\bar{x}, \bar{y}) \in [\alpha_1, \psi_1] \times [\alpha_2, \psi_2]$. Then, the error vector E_n of every solution of (1) satisfies both of the following asymptotic relations*

$$\lim_{n\to\infty} (\|E_n\|)^{\frac{1}{n}} = |\lambda_{1,2,3,4} F_J(\bar{x}, \bar{y})|,\ \lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda_{1,2,3,4} F_J(\bar{x}, \bar{y})|,$$

where $\lambda_{1,2,3,4} F_J(\bar{x}, \bar{y})$ are the characteristic roots of the Jacobian matrix $F_J(\bar{x}, \bar{y})$ about $(\bar{x}, \bar{y})$.

3.5 Existence of unbounded solutions

Our goal in this section is to study the existence of the unbounded solutions of system (1) .

Theorem 7. *Suppose that the following statements are true :*

(i) $0 < \beta_1 + \gamma_1 e^{-\alpha_2} < 1$, $\quad 0 < \beta_2 + \gamma_2 e^{-\alpha_1} < 1$.

(ii) $\alpha_1(1-\beta_2) > \alpha_2(1-\beta_1-\gamma_1 e^{-\alpha_2})$

(iii) $\alpha_2(1-\beta_1) > \alpha_1(1-\beta_2-\gamma_2 e^{-\alpha_1})$

Then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$$

Proof. Consider a solution $\{(x_n, y_n)\}$ of system (1) with intial values x_0, x_{-1}, y_0, y_{-1} such that $x_{-1} < m$, $x_0 < m$, $y_{-1} < m$,$y_0 < m$,$x_0 > M$ and $y_0 > M$ with

$$m = max\{\frac{\alpha_1}{1-\beta_1-\gamma_1 e^{-\alpha_2}}, \frac{\alpha_2}{1-\beta_2-\gamma_2 e^{-\alpha_1}}\},$$
$$M = max\{ln\frac{\gamma_1 m}{m-\alpha_1-\beta_1 m}, ln\frac{\gamma_2 m}{m-\alpha_2-\beta_2 m}\},$$

Then ,

$$x_1 = \alpha_1 + \beta_1 y_0 + \gamma_1 y_{-1} e^{-y_0} < \alpha_1 + \beta_1 m + \gamma_1 m e^{-M} = m$$

Similarly ,

$$y_1 < \alpha_2 + \beta_2 m + \gamma_2 m e^{-M} = m$$

Also,

$$x_2 = \alpha_1 + \beta_1 y_1 + \gamma_1 y_0 e^{y_1} < \alpha_1 + \beta_1 m + \gamma_1 m e^{-\alpha_2} = m$$

Similarly ,

$$y_2 < \alpha_2 + \beta_2 m + \gamma_2 m e^{-\alpha_1} = m$$

Therefore , by induction , we have

$$x_n < m, \quad y_n < m$$

Now

$$x_{n+1} = \alpha_1 + \beta_1 y_n + \gamma_1 y_{n-1} e^{-y_n} > \alpha_1 + \beta_1 y_n + \gamma_1 y_{n-1} e^{-m}$$

,

$$y_{n+1} = \alpha_2 + \beta_2 x_n + \gamma_2 x_{n-1} e^{-x_n} > \alpha_2 + \beta_2 x_n + \gamma_2 x_{n-1} e^{-m}$$

Consider the following linear difference equations :

$$U_{n+1} = a_1 + a_2 U_n + a_3 U_{n-1} \tag{21}$$

$$V_{n+1} = b_1 + b_2 V_n + b_3 V_{n-1} \tag{22}$$

The solution of equation (21)

$$U_n = \frac{a_1}{1-a_2-a_3} + r_1 2^{-n}\{a_2 - \sqrt{a_2^2+4a_3}\}^n + r_2 2^{-n}\{a_2 + \sqrt{a_2^2+4a_3}\}^n$$

The solution of equation (22)

$$V_n = \frac{b_1}{1-b_2-b_3} + r_3 2^{-n}\{b_2 - \sqrt{b_2^2+4b_3}\}^n + r_4 2^{-n}\{b_2 + \sqrt{b_2^2+4b_3}\}^n$$

□

We note that $U_n, V_n \to \infty$ as $n \to \infty$. So $x_n, y_n \to \infty$ as $n \to \infty$.

4 Conclusions

In this paper, we study the behavior of following system of difference equations:

$$x_{n+1} = \alpha_1 + \beta_1 y_n + \gamma_1 y_{n-1} e^{-y_n},\ y_{n+1} = \alpha_2 + \beta_2 x_n + \gamma_2 x_{n-1} e^{-x_n}\ n = 0, 1, \cdots ,$$

where initial values $x_{-1},\ y_{-1}, x_0,\ y_0$ and parameters $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2,\ \gamma_2$ are positive real numbers. We proved that every positive solution $\{(x_n, y_n)\}$ of this system is bounded and persists. Furthermore, existence and uniqueness of positive equilibrium, local and global stability, and rate of convergence of positive solutions which converges to its unique positive point and unboundedness are demonstrated. Finally, some numerical examples are given to verify theoretical results.

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