Received ; | Revised ; | Accepted

DOI: xxx/xxxx

ARTICLE TYPE

Substantial, Tempered, and Shifted Fractional Derivatives: Three Faces of a Tetrahedron

Manuel D. Ortigueira*[1] | Gabriel Bengochea[2] | J. Tenreiro Machado[3]

[1]CTS-UNINOVA and DEE, NOVA School of Science and Technology of NOVA University of Lisbon , Portugal
[2]Academia de Matemática, Universidad Autónoma de la Ciudad de México, Ciudad de México, México
[3]Dept. of Electrical Engineering, Institute of Engineering of Polytechnic Institute of Porto, Porto, Portugal

Correspondence
*Manuel D. Ortigueira
Email: mdo@fct.unl.pt

Present Address
Campus da FCT, Quinta da Torre, 2829–516 Caparica, Portugal

Summary

The substantial, tempered, and shifted fractional derivatives are introduced in a unified framework. Their properties are studied and, in the light of the strict sense criterion for derivative definitions, they are characterized and assessed. In the scope of the framework, new tempered linear sytems and transfer functions are introduced.

KEYWORDS:
Tempered Fractional Derivative, Substantial Fractional Derivative,Shifted Fractional Derivative, Tempered Fractional System.

1 | INTRODUCTION

1.1 | On the weighted derivatives

The Fractional Calculus (FC) is fertile in variants of differential operators that gave rise to several different formulations and operators. Nonetheless, in many applications the distinct approaches give the same results. The same happens with modified fractional derivative (FD) definitions where a weight function is introduced. This state of affairs led to several formulations with different designations lead to identical operators. The most interesting are Substantial (SD), Tempered (TD) and Shifted (ShD) fractional operators [1,2,3,4,5,6,7,8].

In this paper, we review briefly the historical evolutions and describe the main characteristics of these operators that are in fact the three lateral faces of a thetrahedron having as fourth face (base) the Fractional Calculus. We propose a very general formulation, under the name *Tempered Fractional Calculus,* that includes and generalises these three formulations. Therefore, the classic (non tempered FC) emerges as particular case. However, we show that the three derivatives can be expressed in terms of the of the FD which leads to question of using the designation “derivative”. We will study this problem vis-a-vis the criteria proposed before [9].

After formalising the properties of these derivatives, we can define new systems. We will consider the particular, but very important case, of the linear systems.

The paper outlines as follows. In section 2 we present short histories and descriptions of the SD, TD, and ShD. The new unified formulation is presented in section 3. The different versions of the derivatives are introduced, the reason for the use of the designation TFD, their stability, and frequeny domain representations. The tempered fractional linear systems are introduced in section 4. Finally we present some conclusions (section 5).

[0]**Abbreviations:** FT, Fourier transform; LT, Laplace transform FD, Fractional derivative; GL, Grünwald-Letnikov; L, Liouville; RL, Riemann-Liouville ; TF, Transfer function TFD, Tempered Fractional Derivative; **MSC 2010 classification:** Primary 26A33; Secondary 34A08, 35R11

1.2 | Asumptions

We assume that

- We work on $\mathbb{R}$. Nonetheless, this is not a limitation. If the function at hand is defined on any sub-interval in $\mathbb{R}$, we can extend the definition of the function to the whole real line with null values.
- We use the two-sided Laplace transform (LT):

$$F(s) = \mathcal{L}\left[f(t)\right] = \int_{\mathbb{R}} f(t)e^{-st}\mathrm{d}t, \tag{1}$$

  where $f(t)$ is any function defined on $\mathbb{R}$ and $F(s)$ is its transform, provided that it has a non empty region of convergence (ROC). Sufficient conditions for the existence of the LT can be found in [10,11]
- The Fourier transform (FT), $\mathcal{F}\left[f(t)\right]$, is obtained from the LT through the substitution $s = i\omega$, with $\omega \in \mathbb{R}$ and $i = \sqrt{-1}$
- The functions and distributions have Laplace and/or Fourier transforms
- Current properties of the Dirac delta distribution, $\delta(\cdot)$, and its derivatives, $\delta'(\cdot), \delta''(\cdot) \cdots$, will be used
- The standard convolution operation (denoted by the symbol $*$) will be adopted

$$f(t) * g(t) = \int_{\mathbb{R}} f(\tau)g(t-\tau)\mathrm{d}\tau. \tag{2}$$

- The order of any fractional derivative, α, is any real number. We will not consider the complex order, since it gives non Hermitian derivatives.
- The multi-valued expressions s^α and $(-s)^\alpha$ will be used. To obtain functions from them we will fix for branch-cut lines the negative real half axis for s^α and the positive real half axis for $(-s)^\alpha$; for both the first Riemann surface is chosen.
- The Heaviside unit step will be represented by $\varepsilon(t)$ and the signum function by $sgn(t)$. These functions are related since it is straightforward that $sgn(t) = 2\varepsilon(t) - 1$.
- We define the "floor" of a real number α as the integer $N = \lfloor \alpha \rfloor$ verifying $N \leq \alpha < N + 1$.

2 | ORIGINS AND EVOLUTIONS

2.1 | Substantial derivative

The SD, also called material, total, particle, convective, and some other names, was first proposed in continuum mechanics and fluid dynamics [12] and dates back into the eighteenth century. The SD is useful as a bridge between Lagrangian and Eulerian descriptions for deformations and movements [12]. The SD, denoted as $\dfrac{Df}{Dt}$, is defined as

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}, \tag{3}$$

where $f(x, y, z, t)$ can be a scalar or vectorial function, $u = \frac{dx}{dt}$, $v = \frac{dy}{dt}$, and $w = \frac{dz}{dt}$. The partial derivation $\frac{\partial}{\partial t}$ is called the local derivative and the three terms concerning the space variables are called the convective derivative, [13]. It is easy to see that the SD is nothing more than a total derivative with respect to time. A physical meaning of SD is the rate of change of a quantity as experimented by an observer that is moving along with the flow. The SD has several applications in science. For example, it is considered in the material equations and the energy, mass and momentum conservation equations in case of moving observer [14]. Also it appears in the Navier-Stokes equations which are applied to the unsteady, three dimensional flow of any fluid, compressible or incompressible, viscous or inviscid [13]. In aerodynamics, the SD is the time rate of change of density of the given fluid element as it moves through space [15]. the relation (3) can be rewritten in a more useful form for generalization, namely as:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f, \tag{4}$$

where $\mathbf{v}$ is the flow velocity. If we set $\lambda = \mathbf{v} \cdot \nabla f$ and $n \in \mathbb{N}$, then we obtain easily the n^{th}-order derivative

$$D^{n}_{t,\lambda} = \underbrace{\left(\frac{d}{dt} + \lambda\right)\left(\frac{d}{dt} + \lambda\right) \cdots \left(\frac{d}{dt} + \lambda\right)}_{n-times}, \tag{5}$$

where λ is assumed to be independent of t. In the fractional case, the fractional SD of order α, $n-1 < \alpha < n$, was defined similarly to the classic Riemann-Liouville derivative [16]

$$D^{\alpha}_{t,\lambda} f(t) = D^{n}_{t,\lambda}\left[I^{n-\alpha}_{t,\lambda} f\right](t), \tag{6}$$

with

$$I^{n-\alpha}_{t,\lambda} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} f(\xi)(t-\xi)^{n-\alpha-1} e^{-\lambda(t-\xi)} d\xi, \tag{7}$$

where $\alpha > 0$. This operator plays an important role when studying the anomalous sub-diffusion process with coupling probability density functions [17]. The deterministic equations are derived from the continuous time random walk with coupling probability density function [18]. Carmi et al. [19], applied the fractional SD to obtain the forward and backward fractional Feynman-Kac equations for a functional of continuous-time random walks in a binding potential. The problem was later solved by Chen and Deng [20,21]. Friedrich at al. [22] proposed a fractional equation of the Kramers-Fokker-Planck type, involving the fractional SD for representing important nonlocal couplings in time and space.

2.2 | Tempered Derivative

The notion of "tempered" derivative can be referred back to the seventies in the 20th century. In fact, a TD appeared for the first time in the book by Antosik et al. [23], but using the Gaussian function for tempering and working in the space of the "rapidly decreasing functions in zero (RDZ)" [24,8]. A function f is RDZ iff $|f(x)| \leq M_r |x|^r$, for $|x| \leq 1$. With the help of this concept, the fractional TD, using an exponential as tempering function, was proposed in 1983 by S. Pilipović [6]. In a first step, the tempered integral was introduced with some degree of generality for $\alpha \geq 0$ by

$$I^{\alpha}_{a} f(x) = \exp(-a(x)) \int_{0}^{x} \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \exp(a(t)) f(t) \mathrm{d}t. \tag{8}$$

To introduce the corresponding derivative it was assumed that f is the l-order derivative, $l \in \mathbb{N}$, of a continuous RDZ function in such a way that

$$D^{\alpha}_{a} f(x) = D^{p+l}_{a} I^{p-\alpha}_{a} f(x), \quad p \in \mathbb{Z} \ni 0 \leq p-1 < \alpha \leq p \tag{9}$$

and it was noted that $D^{\alpha}_{a} D^{\beta}_{a} f(x) = D^{\alpha+\beta}_{a} f(x)$, for any $\alpha, \beta \in \mathbb{R}$. This is basically the formulation that would be introduced later in the field of stochastic processes in an independent way [25]. In diffusion processes, the concept of "truncated Leévy flight" was introduced in order to guarantee that the underlying process had finite variance [26]. However, Koponen [27] showed that such goal could be achieved with some advantages by using the "tempered Lévy flight". Similar results were obtained in financial mathematics by Madan et al. [28], Carran et al and Barnerdoff-Nielson et al. with the concepts of "variance gamma process" [29,30] and "normal modified stable processes" [1,31]. These approaches consisted basically in modifying the probability density function by introducing one or two exponentials. J. Rosiński [32,7] published an interesting work on unification and generalization of the above notions. Cartea and del-Castillo-Negrete developed the tempered fractional diffusion equation that governs the probability densities of tempered Lévy flights [33,34]. Working in the context of the Fourier transform they introduced tempered versions of the Liouville derivative (see [16,35]) and of a regularised Caputo derivative. In the last 15 years several other papers on the subject were published, [36,37], being important to refer the Meerschaert's works [38,39,5,40,41].
There are two slightly different approaches based on:

1. The Liouville (regularised) and Grünwald-Letnikov derivatives [41]

2. The Riemann-Liouville and Caputo derivatives [42,37,43]

2.2.1 | Tempered Liouville and Grünwald-Letnikov derivatives

Let $\lambda > 0$ and consider $f(x)$ a function with FT.

Definition 1. We define the positive tempered fractional integral of $f(x)$ by[5,40]

$$I_{+}^{\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} f(u)(x-u)^{\alpha-1} e^{-\lambda(x-u)} \mathrm{d}u \tag{10}$$

and the negative tempered fractional integral by

$$I_{-}^{\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} f(u)(u-x)^{\alpha-1} e^{-\lambda(u-x)} \mathrm{d}u. \tag{11}$$

With $\lambda = 0$, these expressions recover the Liouville integrals[16,44,45]. The corresponding FT are given by $(i\omega+\lambda)^{-\alpha}$ and $(-i\omega+\lambda)^{-\alpha}$, respectively

The corresponding derivatives are obtained using the procedure currently used to define the Riemann-Liouville derivative. For the positive tempered we have

$$D_{+}^{\alpha,\lambda} f(x) = D_{+}^{n,\lambda} I_{+}^{n-\alpha,\lambda} f(x), \tag{12}$$

where $D_{+}^{n,\lambda} = e^{-\lambda x} \left[e^{\lambda x} f(x) \right]^{(n)}$ is the positive integer order TD.

Definition 2. The GL is the fractional incremental ratio version of the Liouville derivative. As above we define the positive and negative tempered GL FD by[41]

$${}^{GL}D_{\pm}^{\alpha,\lambda} f(x) = \lim_{h\to 0^{+}} h^{-\alpha} \sum_{k=0}^{\infty} e^{-\lambda n h} \frac{(-\alpha)_k}{k!} f(x \mp kh), \tag{13}$$

where $f(x),\quad x \in \mathbb{R}$, is any function, and $(a)_k = a(a+1)(a+2)\cdots(a+k-1),\ (a)_0 = 1$ represents the Pochhammer symbol for the raising factorial.

The positive and negative TFD (10) to (12) can be combined to get tempered versions of the Riesz potential and derivative[46].

2.2.2 | Tempered Riemann-Liouville and Caputo derivatives

Let $[a,b] \subset \mathbb{R}$ be a finite interval. Denote $L([a,b])$ as the integrable space which includes the Lebesgue measurable functions on $[a,b]$, i.e.,

$$L([a,b]) = \left\{ y : \|y\|_{L([a,b])} = \int_{a}^{b} |y(x)| \mathrm{dt} < \infty \right\}.$$

Let $AC[a,b]$ be the space of real functions $y(x)$ that are absolutely continuous on $[a,b]$. For $n \in \mathbb{Z}^{+}$, we denote $AC^{m}[a,b]$ as the space of real-valued functions $y(t)$ with continuous derivatives up to order $m-1$ on $[a,b]$, such that $\frac{\mathrm{d}^{m-1}}{\mathrm{dx}^{m-1}} y(x) \in AC[a,b]$.

Definition 3. Let $y(x)$ be a piecewise continuous real function on (a,b) and $y(x) \in L([a,b]), \lambda > 0, \alpha > 0$. The tempered Riemann–Liouville integral is defined by

$${}_{a}^{RL}I_{+}^{\alpha,\lambda} y(x) := \frac{1}{\Gamma(\alpha)} \int_{a}^{x} e^{-\lambda(t-\xi)} (x-\xi)^{\alpha-1} y(\xi).\mathrm{d}\xi, \tag{14}$$

In agreement with the classic RL derivative we define the tempered Riemann–Liouville derivative by

$${}_{a}^{RL}D_{+}^{\alpha,\lambda} y(x) := \frac{e^{-\lambda x}}{\Gamma(m-\alpha)} \frac{\mathrm{d}^{m}}{\mathrm{d}x^{m}} \int_{a}^{x} e^{\lambda\xi} (x-\xi)^{m-\alpha-1} y(\xi) \mathrm{d}\xi. \tag{15}$$

where $m-1 < \alpha \le m$ and $m \in \mathbb{Z}^{+}$

The corresponding tempered Caputo derivative can be introduced in a similar way.

Definition 4. We define the tempered Caputo derivative by

$${}_{a}^{C}D_{+}^{\alpha,\lambda} y(x) := \frac{e^{-\lambda x}}{\Gamma(m-\alpha)} \int_{a}^{x} e^{\lambda\xi} (x-\xi)^{m-\alpha-1} \frac{\mathrm{d}^{m}}{\mathrm{d}\xi^{m}} y(\xi) \mathrm{d}\xi, \tag{16}$$

where $m-1 < \alpha \le m$ and $m \in \mathbb{Z}^{+}$

2.3 | Shifted derivative

The ShD was introduced by A. Hanyga[47,4,48] when modelling the wave attenuation in complex viscoporous media. Hanyga[47] formulated the relaxation function

$$R_0(x) = (1/\gamma)^\alpha \mathrm{e}^{-x/\gamma} x^{\alpha-1}/\Gamma(\alpha)$$

and defined implicitely a derivative operator by

$$(1+\tau \mathrm{D})^{\pm\alpha} f(x) = \tau^{-\alpha} \mathrm{e}^{-x/\tau} \mathrm{D}^{\pm\alpha}\left[\mathrm{e}^{x/\tau} f(x)\right].$$

This expression introduces the shifted derivative[47,4,48]

$$(D+\lambda)^\alpha f(x) = \mathrm{e}^{-\lambda x} D^\alpha \left[\mathrm{e}^{\lambda x} f(x)\right], \tag{17}$$

where $\lambda > 0$, $0 < \alpha < 1$, and D^α represents the Caputo fractional derivative[49].

2.4 | Some reflections

We reviewed three different ways to define operators differing from the usual derivatives by the introduction of an exponential weight. These derivatives had distinct origins and may involve slightly different formulations, but that are clearly addressing similar concepts. In fact, we can verify easily that the expressions (6) to (9) constitute essentially the Riemann-Liouville framework that is a particular case in the scheme defined by the Liouville procedure[16] shown in (10) and (12). This means that our work domain is $\mathbb{R}$ and not any of its subsets. Therefore, the so called starting point does not have any role and we will assume in the follow-up that we will work always with functions defined on $\mathbb{R}$.
The relation (17) is interesting by showing that the usual fractional derivatives are the base for the above introduced operators. Therefore, the three lateral faces of the thetrahedron need a support (the base) that is a correct formulation of fractional derivatives. For this endeavour we will follow the strategy proposed in[44,35,50,51,45]. From these considerations we can ask if an operator obtained from a fractional derivative following (17) be considered a derivative. This paper will try to answer to this question.
Hereafter, we fix the nomenclature. The designation "substancial derivative" was introduced in a special physical system where it makes sense and allows an interpretation in terms of moving fluids. A similar situation is verified for the shifted derivative. The term "tempered" was tied to some classic operators in mathematical contexts allowing its connection with different applications with a simple and clear interpretation. Since the bibliography about it is much larger than the first and third, we will adopt the designation "TD" in what follows.

3 | THE TFD AND THEIR INVERSES

3.1 | Elemental derivatives

We find in the literature three standard definitions for the derivatives of order one[35]. With a slight modification we obtain their tempered versions. These are elemental derivatives that will be the starting points for the notion of high level derivatives.
Let $\lambda \in \mathbb{R}$. Such TD are defined as:

Definition 5.

- Forward or causal TD

$$D_{\lambda,f} f(t) = \lim_{h\to 0} \frac{f(t) - e^{-\lambda h} f(t-h)}{h}, \tag{18}$$

that has LT

$$\mathcal{L}\left[D_{\lambda,f} f(t)\right] = \lim_{h\to 0} \frac{1 - e^{-\lambda h} e^{-sh}}{h} F(s) = (s+\lambda)F(s), \tag{19}$$

for any $s \in \mathbb{C}$.

- Backward or anti-causal TD

$$D_{\lambda,b} f(t) = \lim_{h\to 0} \frac{e^{\lambda h} f(t+h) - f(t)}{h}. \tag{20}$$

Its LT is

$$\mathcal{L}\left[D_{\lambda,b} f(t)\right] = \lim_{h\to 0} \frac{e^{\lambda h} e^{sh} - 1}{h} = (s+\lambda)F(s), \tag{21}$$

valid for any $s \in \mathbb{C}$.

Remark 1. Substituting $-h$ for $+h$ interchanges the definitions (18) and (20), meaning that we only have to consider $h > 0$.

- Two-sided or centred TD

$$D_{\lambda,c} f(t) = \lim_{h\to 0} \frac{e^{\lambda\frac{h}{2}} f(t+\frac{h}{2}) - e^{-\lambda\frac{h}{2}} f(t-\frac{h}{2})}{h}. \tag{22}$$

This expression is equivalent to the other two, from the LT point of view.

Remark 2. For $\lambda = 0$ we recover the classic first order derivatives.

Remark 3. As it is seen from (19) and (21),

$$D_{\lambda,f(b)} f(t) = D_{0,f,(b)} f(t) + \lambda f(t) \tag{23}$$

that was the original formula for the SD (4).

It is straightforward to invert the relations (18) and (20), and so we obtain the order 1 anti-derivatives

$$D^{-1}_{\lambda,f} f(t) = \lim_{h\to 0} \sum_{n=0}^{\infty} e^{-n\lambda h} f(t-nh)\cdot h, \tag{24}$$

$$D^{-1}_{\lambda,b} f(t) = -\lim_{h\to 0} \sum_{n=0}^{\infty} e^{n\lambda h} f(t+nh)\cdot h. \tag{25}$$

Using the LT, we have

$$\mathcal{L}\left[D^{-1}_{\lambda,f} f(t)\right] = \lim_{h\to 0} h \sum_{n=0}^{\infty} e^{-n\lambda h} e^{-sh} F(s) = \frac{1}{s+\lambda} F(s), \quad Re(s) > -\lambda, \tag{26}$$

$$\mathcal{L}\left[D^{-1}_{\lambda,b} f(t)\right] = -\lim_{h\to 0} h \sum_{n=0}^{\infty} e^{n\lambda h} e^{sh} F(s) = \frac{1}{s+\lambda} F(s), \quad Re(s) < -\lambda. \tag{27}$$

Remark 4. Note that the regions of convergence of the LT are now half-planes of $\mathbb{C}$. This important fact is tied with causality[52,10].

3.2 | Generalizations for integer orders

Definition 6. The repeated use of the above derivatives, (18) and (20), and anti-derivatives, (24) and (25), leads to closed formulae valid for any integer order, $N \in \mathbb{Z}$, such that[35,45]:

$$D^{N}_{\lambda,f} f(t) = \lim_{h\to 0^+} h^{-N} \sum_{n=0}^{\infty} \frac{(-N)_n}{n!} e^{-n\lambda h} f(t-nh), \tag{28}$$

$$D^{N}_{\lambda,b} f(t) = (-1)^N \lim_{h\to 0^+} h^{-N} \sum_{n=0}^{\infty} \frac{(-N)_n}{n!} e^{n\lambda h} f(t+nh), \tag{29}$$

Expressions (28) and (29) reflect, in a unified way, all integer order derivatives and anti-derivatives. Therefore, we can use only the word derivative independently of having positive or negative order.
The corresponding LT are given by:

$$\mathcal{L}\left[D^{N}_{\lambda,f} f(t)\right] = (s+\lambda)^N F(s), \quad Re(s) > -\lambda, \tag{30}$$

$$\mathcal{L}\left[D^{N}_{\lambda,b} f(t)\right] = (s+\lambda)^N F(s), \quad Re(s) < -\lambda, \tag{31}$$

respectively.
Expressions (30) and (31) tell us that the derivative operator represents a system with transfer function (TF) given by $H(s) = (s+\lambda)^N$. In this perspective, we have a generalization of a well-known property of the two-sided LT $\mathcal{L}\left[D^{n}_{\lambda,f,b} f(t)\right] = (s+\lambda)^n \mathcal{L}\left[f(t)\right]$, $n \in \mathbb{Z}$, that suggests the correspondence

$$(s+\lambda)^n \Longleftrightarrow e^{-\lambda t} \frac{t^{-n-1}}{(-n-1)!} \varepsilon(\pm t) \tag{32}$$

as long as we accept that

$$\frac{t^{-n-1}}{(-n-1)!}\varepsilon(\pm t) = \delta^{(n)}(t) \tag{33}$$

if $n \geq 0$[53,11]. If $N > 0$, then, for the causal definition, relation (32) allows us to write

$$D^{-N}_{\lambda,f} f(t) = \int_0^\infty f(t-\tau)e^{-\lambda\tau}\frac{\tau^{N-1}}{(N-1)!}d\tau \tag{34}$$

and

$$D^{-N}_{\lambda,b} f(t) = (-1)^N \int_0^\infty f(t+\tau)e^{-\lambda\tau}\frac{\tau^{N-1}}{(N-1)!}d\tau, \tag{35}$$

for the anti-causal.

Remark 5. We can show that the inverse LT of $\frac{1}{s^n}$, for $Re(s) > 0$, is given by $\frac{t^{n-1}}{(n-1)!}\varepsilon(t)$, only if $n > 0$. Multiplying $\frac{1}{s^n}$ by $s, s^2, s^3, \cdots$ corresponds to computing the first, second, and successive derivatives of $\frac{t^{n-1}}{(n-1)!}\varepsilon(t)$. These derivatives have to be considered in distributional sense[53] and lead to (33). This result will remain valid even if we have a real power α instead of a natural power n, as we will consider in the next sub-section.

3.3 | Second generalisation: real orders

It is straightforward to extend formulae (19) and (21) to any real order[1].

Definition 7. For $\alpha \in \mathbb{R}$ we can write

$$D^{\alpha}_{\lambda,f} f(t) = \lim_{h\to 0^+} h^{-\alpha}\sum_{n=0}^{\infty}\frac{(-\alpha)_n}{n!}e^{-n\lambda h}f(t-nh), \tag{36}$$

$$D^{\alpha}_{\lambda,b} f(t) = e^{-i\alpha\pi}\lim_{h\to 0^+} h^{-\alpha}\sum_{n=0}^{\infty}\frac{(-\alpha)_n}{n!}e^{n\lambda h}f(t+nh), \tag{37}$$

that have LT

$$\mathcal{L}\left[D^{\alpha}_{\lambda,f} f(t)\right] = (s+\lambda)^{\alpha}F(s), \quad Re(s) > -\lambda, \tag{38}$$

and

$$\mathcal{L}\left[D^{\alpha}_{\lambda,b} f(t)\right] = (s+\lambda)^{\alpha}F(s), \quad Re(s) < -\lambda, \tag{39}$$

respectively, for $(s+\lambda)^{\alpha}, \quad \pm Re(s) > -\lambda$.

If $\alpha < 0$, then the inverse LT of this TF can be obtained from the properties of the LT and of the Gamma function. Here, the remark 5 remains valid and allows us to state that

$$\mathcal{L}^{-1}\left[(s+\lambda)^{\alpha}\right] = \pm e^{-\lambda t}\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}\varepsilon(\pm t) \tag{40}$$

Definition 8. The relation (40) and the convolution property of the LT allow us to introduce the integral version of the TD as

$$D^{\alpha}_{\lambda,f} f(t) = \int_0^\infty f(t-\tau)e^{-\lambda\tau}\frac{\tau^{-\alpha-1}}{\Gamma(-\alpha)}d\tau = e^{-\lambda t}\int_{-\infty}^{t} f(\tau)e^{\lambda\tau}\frac{(t-\tau)^{-\alpha-1}}{\Gamma(-\alpha)}d\tau \tag{41}$$

thar generalises the causal expression (34) to real orders. For the anti-causal case, we obtain the expression:

$$D^{\alpha}_{\lambda,b} f(t) = -\int_t^\infty f(\tau)e^{\lambda(\tau-t)}\frac{\tau^{-\alpha-1}}{\Gamma(-\alpha)}d\tau = e^{-i\alpha\pi}\int_0^\infty f(t+\tau)e^{\lambda\tau}\frac{\tau^{-\alpha-1}}{\Gamma(-\alpha)}d\tau \tag{42}$$

For $\alpha > 0$, the integrals in (41) and (42) may be singular, but they can be regularised as in[54].

[1]The use complex orders has limite interest[45]

Definition 9. Let $\beta \in \mathbb{R}$ and $\varepsilon(\cdot)$ be the Heaviside unit step. The regularised temperate derivatives are defined by

$$D^{\alpha}_{\lambda,f} f(t) = \int_0^{\infty} \left[f(t-\tau) - \varepsilon(\alpha) \sum_0^N \frac{(-1)^m f^{(m)}(t)}{m!} \tau^m \right] e^{-\lambda\tau} \frac{\tau^{-\alpha-1}}{\Gamma(-\alpha)} d\tau, \tag{43}$$

that generalises the causal expression (41) to real orders and where $N = \lfloor \alpha \rfloor$. For the anti-causal case, we get:

$$D^{\alpha}_{\lambda,b} f(t) = e^{-i\alpha\pi} \int_0^{\infty} \left[f(t+\tau) - \varepsilon(\alpha) \sum_0^N \frac{f^{(m)}(t)}{m!} \tau^m \right] e^{\lambda\tau} \frac{\tau^{-\alpha-1}}{\Gamma(-\alpha)} d\tau. \tag{44}$$

Expressions (43) and (44) generalise the too restrict definitions introduced in[41]

Assume that $f(t) = 0,\ t \leq a$. From (41), we obtain:

$$D^{\alpha}_{\lambda,f} f(t) = e^{-\lambda t} \int_a^t f(\tau) e^{\lambda\tau} \frac{(t-\tau)^{-\alpha-1}}{\Gamma(-\alpha)} d\tau. \tag{45}$$

On the other hand, if $f(t) = 0,\ t \geq b$, then, from (42) we obtain:

$$D^{\alpha}_{\lambda,b} f(t) = e^{-i\alpha\pi} e^{-\lambda t} \int_t^b f(\tau) e^{\lambda\tau} \frac{\tau^{-\alpha-1}}{\Gamma(-\alpha)} d\tau. \tag{46}$$

From these two relations, we obtain a TD similar to the Riemann-Liouville and Caputo derivatives[49]. Other interesting results can be listed:

- From (40), we verify that

$$D^{\beta}_{\lambda,f,b} \left[e^{-\lambda t} \frac{t^{\alpha-1}}{\Gamma(\alpha)} \varepsilon(\pm t) \right] = e^{-\lambda t} \frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \varepsilon(\pm t) \tag{47}$$

- From (43) we conclude that

$$D^{\beta}_{\lambda,f} f(t) = e^{-\lambda t} D^{\beta}_{0,f} \left[e^{\lambda t} f(t) \right] \tag{48}$$

- From (44) we obtain

$$D^{\beta}_{\lambda,b} f(t) = e^{\lambda t} D^{\beta}_{0,f} \left[e^{-\lambda t} f(t) \right] \tag{49}$$

3.4 | Are the TFD fractional derivatives?

3.4.1 | Strict sense criterion

In[9] we proposed two criteria for classifying a given operator as fractional derivative: the *wide sense criterion* (WSC), that is useful in studying derivatives defined on $\mathbb{R}^+$, and the *strict sense criterion* (SSC), that is suitable for functions defined on $\mathbb{R}$. Hereafter, we will use the SSC for analysing the fractional operators we introduced above.

An operator is considered a FD in SSC if it enjoys the properties **P** defined as:

P1 Linearity
The TFD we introduced in the last sub-section is linear.

P2 Identity
The zero order TFD of a function returns the function itself, since $(s+\lambda)^0 = 1$, for any $\lambda, s \in \mathbb{C}$.

P3 Backward compatibility
When the order is integer, the TFD gives the same result as the integer order TD and recovers the ordinary derivative, for $\lambda = 0$.

P4 The index law holds

$$D^{\alpha} D^{\beta} f(t) = D^{\alpha+\beta} f(t) \tag{50}$$

for any α and β, since $(s+\lambda)^{\alpha}(s+\lambda)^{\beta} = (s+\lambda)^{\alpha+\beta}$

P5 The generalised Leibniz rule

$$D^{\alpha}\left[f(t)g(t)\right] = \sum_{i=0}^{\infty}\binom{\alpha}{i} D^{i} f(t) D^{\alpha-i} g(t) \tag{51}$$

must be changed to the tempered case. However, the deduction of the formula for derivative of the product is somewhat laborious. We will consider the forward tempered GL derivative, since the deduction for the backward case is similar. We use the so-called Newton series [55,56]

$$f_{\nabla}^{(N)}(t) = \frac{\sum_{k=0}^{N}(-1)^{k}\binom{N}{k} f(t-kh)}{h^{N}}$$

and invert it to express a delay in terms of a derivative [57]:

$$f(t-nh) = \lim_{h\to 0^{+}} \sum_{k=0}^{n}(-1)^{k}\binom{n}{k} h^{k} f_{\nabla}^{(k)}(t). \tag{52}$$

The operator $f_{\nabla}^{(N)}(t)$ is a discrete-time derivative that converges to the classic integer order derivative when calculating the limit when h tends to 0^{+} [56]. To find the property let us start from (36) to obtain successively:

$$\begin{aligned}
D^{\alpha}\left[f(t)g(t)\right] &= \lim_{h\to 0^{+}} \frac{\sum\limits_{n=0}^{\infty}(-1)^{n}\binom{\alpha}{n} e^{-n\lambda h} f(t-nh)g(t-nh)}{h^{\alpha}} \\
&= \lim_{h\to 0^{+}} \frac{\sum\limits_{n=0}^{\infty}(-1)^{n}\binom{\alpha}{n} e^{-n\lambda h} g(t-nh)\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} h^{k} f^{(k)}(t)}{h^{\alpha}} \\
&= \lim_{h\to 0^{+}} \frac{\sum\limits_{i=0}^{\infty}(-1)^{i} h^{i} f^{(i)}(t)\sum\limits_{k=i}^{\infty}(-1)^{k}\binom{k}{i}\binom{\alpha}{k} e^{-k\lambda h} g(t-kh)}{h^{\alpha}}
\end{aligned}$$

However, we have

$$\binom{k+i}{i}\binom{\alpha}{k+i} = \binom{\alpha}{i}\binom{\alpha-i}{k}$$

that replaced in the above expression gives

$$D^{\alpha}\left[f(t)g(t)\right] = \lim_{h\to 0^{+}} \sum_{i=0}^{\infty}\binom{\alpha}{i} f^{(i)}(t)\left[\frac{\sum\limits_{k=0}^{\infty}(-1)^{k}\binom{\alpha-i}{k} e^{-k\lambda h} g(t-kh-ih)}{h^{\alpha-i}}\right].$$

Finally, we obtain

$$D_{\lambda,f}^{\alpha}\left[f(t)g(t)\right] = \sum_{i=0}^{\infty}\binom{\alpha}{i} f^{(i)}(t) D_{\lambda,f}^{\alpha-i} g(t) \tag{53}$$

that is very close to the generalised Leibniz rule (51) [44]. However, when $\alpha = N \in \mathbb{Z}^{+}$ and $\lambda = 0$ we obtain the classical Leibniz rule. As already mentioned, for the backward case we obtain a similar formula.

We conclude that *the TFD verifies the SSC and therefore can be considered a derivative.*

3.4.2 | The perpective of Bode diagrams

Bode diagrams are useful tools for the analysis and design of linear systems [52,10], since they provide a direct insight into models adopted in engineering and natural systems.

Definition 10. From formulae (38) and (39) we define two spectra:

1. *Amplitude* spectrum

$$A(\omega) = \left|\lambda^{2} + \omega^{2}\right|^{\frac{\alpha}{2}} \tag{54}$$

2. *Phase* spectrum

$$\Phi(\omega) = \alpha \arctan\left(\frac{\omega}{\lambda}\right) \tag{55}$$

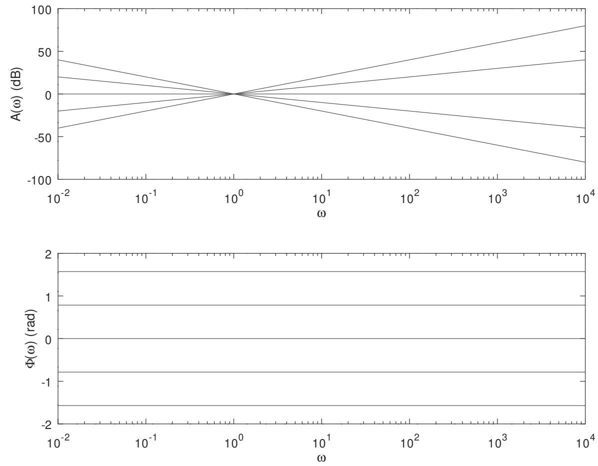

FIGURE 1 Bode plots for $\alpha = \{-1, -0.5, 0.5, 1\}$ with $\lambda = 0$, corresponding to the amplitude and phase spectra, given in (54) and (55).

For real-valued functions, the amplitude and the phase are even and odd functions, respectively[52,10]. Due to this reason, we only need to represent log plots for positive frequencies that are called *Bode diagrams*. The amplitude spectrum, $A(\omega)$, is usual expressed in deciBell (dB). Then, it results

$$A(\omega)|_{dB} = 10\alpha \log\left(\lambda^2 + \omega^2\right). \tag{56}$$

In[50] we proposed a criterion based on Bode diagrams. According to those ideas, a fractional derivative is an operator having a frequency response such that the amplitude and phase spectra are oblique and horizontal straight lines, respectively. This concept is illustrated in figure 1 that represents the Bode diagrams for the Liouville and GL derivatives[50] for orders $\alpha = \{-1, -0.5, 0.5, 1\}$. This case corresponds to have $\lambda = 0$ in (54) and (55), since the function $\arctan\left(\frac{\omega}{\lambda}\right)$ degenerates into the signum function $\frac{\pi}{2} sgn(\omega)$.
Similarly, in figure 2 we represent the amplitude and phase spectra corresponding to a TFD with orders $\alpha = \{-1, -0.5, 0.5, 1\}$ and $\lambda = 0.25$. As we can see, the diagrams are no longer straight lines. Therefore and according to[50] the TFD should not be considered as fractional derivatives. However, we can see that for higher frequences, namely for frequencies greater than a decade above $|\lambda|$, we obtain half straight lines with slopes equal to the fractional derivatives ($\lambda = 0$) with the same order. Therefore, we will keep the designation: Tempered Fractional Derivative.

If we compare figures 1 and 2 , then we conclude that the TFD acts as a filter that reduces the effect of the low frequences. This characteristic corresponds to a decrease of the amplitudes of the corresponding impulse responses as the time grows up. Indeed, the asymptotic polynomial decrease in the time response is transformed in an exponential decrease, reducing the memory captured by the derivatives and systems adopting this formulation. This characteristic can be seen clearly for negative derivative orders as it is illustated in figure 3 .

3.5 | On the stability of the TFD

The TFD introduced in 3.3 resulted from the usual FD[35,50] by inserting an exponential in the impulse responses. Therefore, we can guarantee stability by a suitable choice of the exponent. If the derivative is causal the exponent must have negative real part, while if it is anti-causal the real part of the exponent must be positive. For the above forward and backward TFD, we conclude immediately that, for the same λ, if the forward is stable, then the backward is unstable and vice-versa. If we decide to use always stable derivatives, then we can redefine them as follows:

1. Set $\lambda \in \mathbb{R}_0^+$
2. Make the substitution $-\lambda$ for λ in the backward derivatives.

In agreement with this convention, we obtain the derivatives in table 1 .

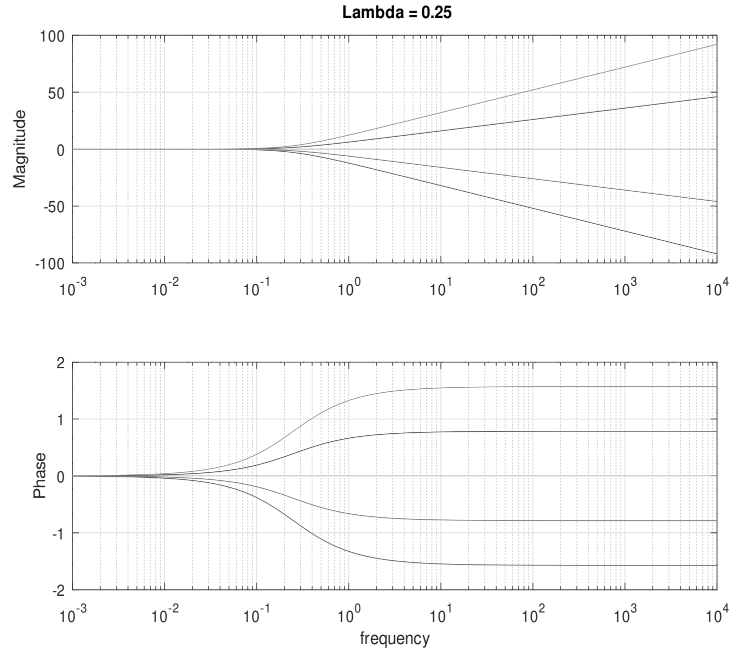

FIGURE 2 Bode diagrams of the TFD for $\alpha = \{-1, -0.5, 0.5, 1\}$ and $\lambda = 0.25$

TABLE 1 Stable TFD with $\lambda \geq 0$

Derivative	$D^{\alpha}_{\lambda,f} f(t) =$	LT
Forward GL	$\lim\limits_{h\to 0^+} h^{\alpha} \sum\limits_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} e^{-n\lambda h} f(t-nh)$	$(s+\lambda)^{\alpha}$
Backward GL	$e^{-i\alpha\pi} \lim\limits_{h\to 0^+} h^{\alpha} \sum\limits_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} e^{-n\lambda h} f(t+nh)$	$(s-\lambda)^{\alpha}$
Reg. forw. L	$\int_0^{\infty} \left[f(t-\tau) - \varepsilon(\alpha) \sum_0^N \frac{(-1)^m f^{(m)}(t)}{m!} \tau^m \right] e^{-\lambda\tau} \frac{\tau^{-\alpha-1}}{\Gamma(-\alpha)} d\tau$	$(s+\lambda)^{\alpha}$
Reg. back. L	$e^{-i\alpha\pi} \int_0^{\infty} \left[f(t+\tau) - \varepsilon(\alpha) \sum_0^N \frac{f^{(m)}(t)}{m!} \tau^m \right] e^{-\lambda\tau} \frac{\tau^{-\alpha-1}}{\Gamma(-\alpha)} d\tau$	$(s-\lambda)^{\alpha}$

4 | ON THE TEMPERED LINEAR SYSTEMS

With the derivatives defined in 3.3 and the final form stated in table 1 , we can introduce formally the notion of linear systems.

Definition 11. Let $x(t)$ and $y(t)$ be two functions assumed almost everywhere continuous, with bounded variation, and of exponential order. Therefore, they have LT with a non empty regions of convergence. We define tempered fractional linear system with input $x(t)$ and output $y(t)$ through the following differential equation

$$\sum_{k=0}^{N} a_k D^{\alpha_k}_{\lambda_k,f} y(t) = \sum_{k=0}^{M} b_k D^{\beta_k}_{\gamma_k,f} x(t), \tag{57}$$

where $t \in \mathbb{R}$, $a_k,\ k = 0, 1, \cdots, N$, and $b_k,\ k = 0, 1, \cdots, M$, are real constant coefficients. The parameters α_k and β_k are the derivative orders that, without loss of generality, we assume to form strictly increasing sequences of positive real numbers. The exponential coefficients $\lambda_k \in \mathbb{R}^+_0,\ k = 0, 1, \cdots, N$, and $\gamma_k \in \mathbb{R}^+_0,\ k = 0, 1, \cdots, M$, are chosen in agreement with table reftable1. The formulation stated in (57) is very general i the sense that we can use forward, backward or both derivatives. However, for most pratical applications where we deal with causal systems and therefore the use of the forward GL or L derivatives (table 1) is more appropriate.

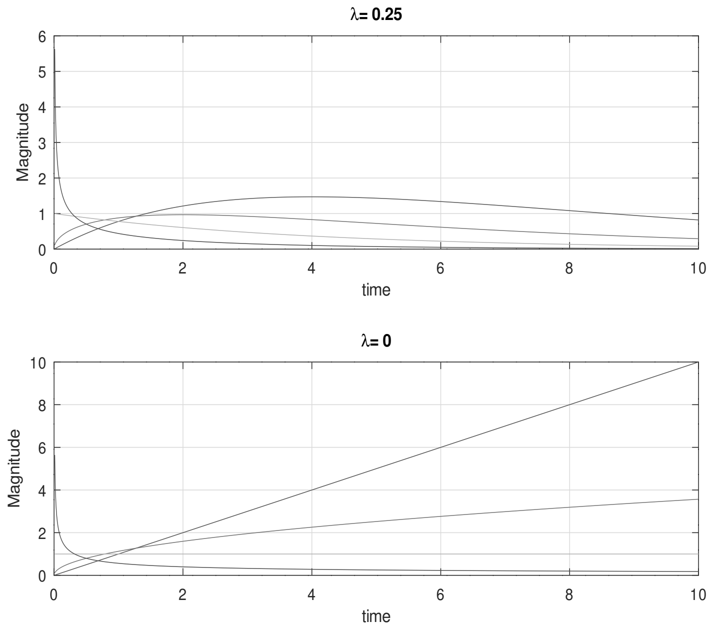

FIGURE 3 Impulse responses of the forward TFD for $\alpha = \{-2 - 1.5 - 1 - 0.5\}$ and $\lambda = \{0.25\}$

According to table 1 , the TF corresponding to the differential equation (57) is given by:

$$H(s) = \frac{\sum_{k=0}^{M} b_k (s+\gamma_k)^{k\alpha}}{\sum_{k=0}^{N} a_k (s+\lambda_k)^{k\alpha}}. \tag{58}$$

Nonetheless, the general form of this TF is not easy to manipulate. In the so-called commensurate case we can write $\alpha_k = \beta_k = k\alpha,\ k \in \mathbb{N}_0$. However, this case is only manageable if $\lambda_k = \gamma_k = \lambda_0, k = 1, 2, 3, \cdots$, that can be written as

$$\sum_{k=0}^{N} a_k D_{\lambda_0}^{k\alpha} y(t) = \sum_{k=0}^{M} b_k D_{\lambda_0}^{k\alpha} x(t). \tag{59}$$

The corresponding TF is:

$$H(s) = \frac{\sum_{k=0}^{M} b_k (s+\lambda_0)^{k\alpha}}{\sum_{k=0}^{N} a_k (s+\lambda_0)^{k\alpha}}, \tag{60}$$

or, equivalently

$$H(s) = K_0 \frac{\prod_{k=1}^{M} \left[(s+\lambda_0)^{\alpha} - z_k\right]}{\prod_{k=1}^{N} \left[(s+\lambda_0)^{\alpha} - p_k\right]} \tag{61}$$

where $p_k, z_k,\ k = 1, 2, \cdots$ are the pseudo-poles and -zeroes[2] [45] and K_0 is a constant. The shift property of the LT allows us to conclude that, if $h_0(t)$ is the impulse response (i.e., the inverse LT of $H_0(s)$), of the non tempered system ($\lambda_0 = 0$), the impulse response corresponding to (60) is

$$h(t) = e^{-\lambda_0 t} h_0(t). \tag{62}$$

[2] Roots of $\sum_{k=0}^{N} a_k z^k$ and $\sum_{k=0}^{M} b_k z^k$, respectively.

Equations (57) or (58) show that we have other possibilities of generating other models. For example, if α_k are constant equal to 1, then we obtain

$$H(s) = \frac{\sum_{k=0}^{M} b_k(s+\gamma_k)}{\sum_{k=0}^{N} a_k(s+\lambda_k)}$$

that is merely a system with one pole and one zero.

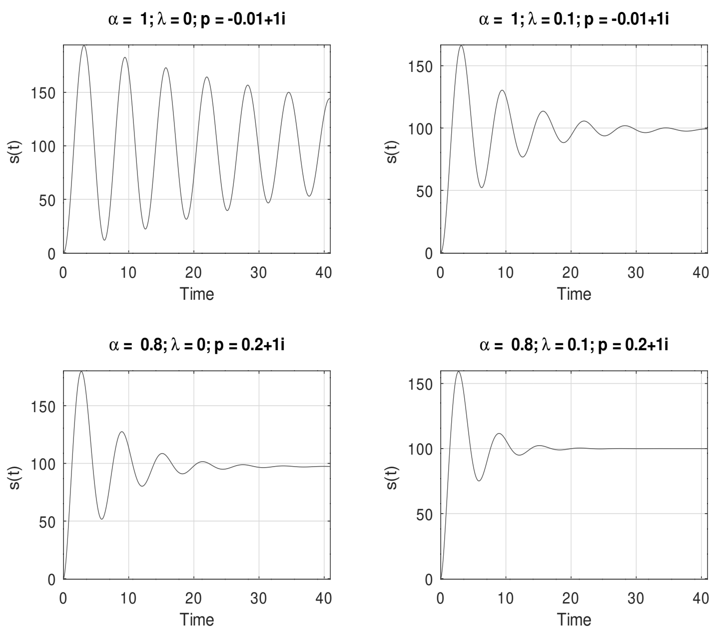

FIGURE 4 Step responses of a system with TF (63) for $\alpha = \{0, 0.8\}$ and $\lambda = \{0, 0.1\}$ and $p = \{-0.01 + i, 0.2 + i\}$

Example 4.1 (A two pseudo-pole system)**.**

Consider a simple system with TF

$$H(s) = \frac{1}{((s+\lambda)^\alpha - p)\,((s+\lambda)^\alpha - p^*)}, \tag{63}$$

where p is a pseudo-pole[45] and p^* denotes its conjugated. In figure 4 we illustrate the behaviour of the step responses of this system for two values of the order, α, and of the coefficient of the exponential, λ. We observe the decreasing of the settling time, while the overshoot reamins almost unchanged.
The numerical computations were carried by the use of the bilinear transformation and the fast Fourier transform[58]. The step response was obtained by integrating the impulse response.

Systems falling in the formulation (61) are usually called *implicit systems* and our approach opens the possibility of computing their impulse responses. In sub-section, 3.3 we introduced the TFD as linear systems with TF

$$G(s) = (s+\lambda)^{\pm\alpha}\,,\ \alpha, \lambda \in \mathbb{R}^+, \tag{64}$$

having impulse response given by (40), so that $g(t) = e^{-\lambda t}\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}\varepsilon(t)$, and in 3.4.2 we illustrated its behaviour in the frequency domain. Combinations of these elemental systems originate (61)[59]. An interesting example with the form (64) is the Cole–Davidson model for dielectric relaxation modelling:

$$\mathbf{Z}(j\omega) = \frac{1}{(1+j\omega\tau)^\beta}$$

where $0 < \beta \leq 1$ and τ is a time constant[60].

Example 4.2. 1. Fractional lead/lag compensator

The fractional lead compensator, used in Control to increase the phase of a system around a chosen frequency, is defined by the TF[61,45]

$$C(s) = \left(\frac{\tau s + a}{s + a}\right)^{\alpha}, \ \alpha, a \in \mathbb{R}^{+}, \ \tau > 1. \tag{65}$$

The fractional lag compensator is used in Control to increase the static gain of a plant. It is defined by the TF

$$C(s) = \left(\frac{s + a}{s + \frac{a}{\tau}}\right)^{\alpha}, \ \alpha, a \in \mathbb{R}^{+}, \ \tau > 1. \tag{66}$$

Attending to the similarity between (65) and (66), we are going to compute the impulse response merely for the TF in (65). We can write

$$C(s) = \tau^{\alpha}\left(\frac{s + \frac{a}{\tau}}{s + a}\right)^{\alpha} = \tau\left(s + \frac{a}{\tau}\right)^{-\alpha}(s + a)^{\alpha}.$$

The inverse LT gives

$$c(t) = \tau^{\alpha}\left[e^{-\frac{at}{\tau}}\frac{t^{\alpha-1}}{\Gamma(\alpha)}\varepsilon(t)\right] * \left[e^{-at}\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}\varepsilon(t)\right] \tag{67}$$

that can be written as

$$c(t) = \frac{\tau^{\alpha}e^{-at}}{\Gamma(\alpha)\Gamma(-\alpha)}\int_0^t u^{\alpha-1}(t-u)^{-\alpha-1}e^{-au\left(\frac{1}{\tau}-1\right)}\mathrm{d}u\ \varepsilon(t). \tag{68}$$

2. Fractional PID
   It is not worth mentioning the importance of the fractional proportional - integral - derivative (PID) controller. In implicit form, it is defined by its TF which is given by[61,45]

$$C(s) = K_p\left(1 + \frac{1}{T_i s}\right)^{\nu}\left(1 + T_d s\right)^{\mu}, \ K_p, T_i, T_d, \nu, \mu > 0, \tag{69}$$

where K_p is the proportional gain, T_i is the integral time, and T_d the derivative time. We can rewrite (69) in a slightly different form more useful to be expressed in terms of (64):

$$C(s) = K_p T_d s^{-\nu}\left(s + \frac{1}{T_i}\right)^{\nu}\left(s + \frac{1}{T_d}\right)^{\mu}. \tag{70}$$

The corresponding impulse response, $c(t)$, is given by

$$c(t) = K_p T_d\left[\frac{t^{\nu-1}}{\Gamma(\nu)}\varepsilon(t)\right] * \left[e^{-\frac{t}{T_i}}\frac{t^{\nu-1}}{\Gamma(\nu)}\varepsilon(t)\right] * \left[e^{-\frac{t}{T_d}}\frac{t^{\mu-1}}{\Gamma(\mu)}\varepsilon(t)\right]. \tag{71}$$

5 | CONCLUSIONS

This paper presented a brief historic review of the substantial, tempered, and shifted FD. A unified framework for these derivatives was discussed for the case of one-sided derivatives. Future work will address the case of two-sided derivatives. The conformity of these operators as studied in the perspective of a criterion for fractional derivatives. Moreover, the concept of tempered fractional linear systems was introduced as a direct consequence of the study.

Financial disclosure

This work was partially funded by Portuguese National Funds through the FCT – Foundation for Science and Technology within the scope of the CTS Research Unit - Center of Technology and Systems / UNINOVA / FCT / NOVA, under the reference UIDB / 00066/2020

Conflict of interest

The author declare no potential conflict of interests.

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How to cite this article: Ortigueira, M. D., Bengochea, G. and Machado, J. T., (2020), Substantial, Tempered, and Shifted Fractional Derivatives: Three Faces of a Tetrahedron, *Mathematical Methods in the Applied Sciences*, *2020;00:1–6*.