Chebyshev spectral method for variable order fuzzy fractional advection diffusion equation by using Mittag-Leffler law

Prashant Pandey[1,2] Sunil Kumar[*3]

[1] *Department of Mathematical Sciences*
Indian Institute of Technology (BHU), Varanasi, 221005, India
[2] *Department of Mathematics*
Goverment M.G.M. P.G. College, Itarsi, 461111, India
[3] *Department of Mathematics*
National Institute of Technology, Jharkhand, India

Abstract

In this work, the Chebyshev spectral technique based on shifted fifth-kind Chebyshev polynomials is utilized to analyzed and to obtained the approximate numerical solution of a class of variable order fuzzy partial differential equations (PDEs) with Mittag-Leffler non-singular kernel and its some particular cases by using the basic properties of fuzzy calculus theory. We analyze a variable order mathematical fuzzy model where the coefficients, unknown functions, initial and boundary conditions are some fuzzy numbers and fuzzy valued functions. The variable order fuzzy operational matrix of shifted fifth-kind Chebyshev polynomials is derived for fuzzy fractional derivatives w.r.to space and time where the fuzzy derivative is taken in ABC sense. The Chebyshev fuzzy operational matrix is applied to concerned non-linear fuzzy space-time fractional variable order reaction-diffusion equations with ABC derivative which reduces into a system of non-linear fuzzy algebraic equations and can be deal by using the method given in literature. To validate the high efficiency and capability of proposed numerical scheme few test examples are reported with computation of the absolute error for the obtained numerical solution.

Key words: Fuzzy Calculus, Fractional Calculus, Variable Order ABC Derivatives, Fuzzy PDEs, Diffusion Equation.

Email addresses: ppunique1133@gmail.com (Prashant Pandey[1,2]), skumar.math@nitjsr.ac.in (Sunil Kumar[*3]).

1. Introduction

Recently the scope of variable order partial differential equations (PDEs) is widely applied in many branch of physical science. The concepts of generalized calculus theory was first introduced by Abel and Liouville. The calculus theory in which the concept of any arbitrary order differentiation and integration is discussed that can be a generalization of classical calculus theory. The ordinary differentiation (integer order) is used to characterize the systems of short term memory, to extend the application of derivative to long term systems the fractional order derivative of constant order are being in used. This generalized calculus theory (fractional calculus) has diverse and widely spreaded in applied mathematical sciences, engineering, fluid mechanics, electromagnetic etc. and increasingly applied to mathematical modeling of several complex physical phenomena viz., fluid flow, viscoelasticity, dynamical systems, control, groundwater contamination, transports of molecules via pores etc. Due to its wide application and feasibility fractional calculus seeks the attention of many researchers, scientists, engineers and applied mathematicians[1,2].
Many definitions of the fractional derivatives and integer order derivatives are known. Besides of several definitions of derivatives, the application of non-local operators has seeks the attentions in many field of physical sciences. Many definitions of the fractional derivatives and the variable order fractional derivatives of any arbitrary order are given viz., Caputo's fractional derivatives, Caputo-Fabrizio derivatives[3], Riemann-Liouville's derivatives for variable order, Yang-Srivastava-Machado derivatives[4]. and Grünwald-Letinkov's derivatives for variable order etc. Atangana and Baleanu are defined the generalized Riemann-Liouville and generalized Caputo derivatives [5,6] so these derivatives are also known as ABC derivatives[7,8]. For the mean square displacement the Caputo-Fabrizio and ABC derivative shows the crossover property where as the Riemann-Liouville fractional order derivative is a scalar invariant derivative.

The diverse application of fractional calculus theory leads us to deal with the fractional differential and partial differential equations[9–11]. To find the the exact solution of fractional PDEs is a very tough task as the exact solution of many fractional PDEs do not exist. To overcome the lack of exact solution many researchers developed various techniques for the approximate analytical solution of fractional PDEs. Few of these numerical techniques can also used in to find the numerical solution of integro-differential equations and integral equations. Many numerical schemes have been developed to find the approximate numerical solution of fractional order PDEs[12–14]. Few of very commonly used numerical schemes are based on differential transform technique [15], homotopy perturbation method [16], Adomain decomposition scheme [17] etc. Some researchers have been developed the operational matrix technique to find the numerical solution of fractional PDEs. These techniques basically based on the Chebyshev wavelets[18], sin wavelets[19], Haar wavelets[20], Legendre wavelets[21] etc. Few operational matrix are developed on polynomials such as Genocchi polynomial[22], Chebyshev polynomial, Laguerre polynomial [23], Luca polynomial, Fibonacci polynomials etc.

Many properties and definitions of variable order integration and differentiation have been given by authors. When we extend the concept of constant fractional order derivatives to a space and time depending fractional derivatives then later arise as a very interesting concept in fractional calculus. This new concept of fractional calculus can be applied to several aspects of mathematical physics, signal and control processing, mechanics etc[24–26]. To find the numerical solution of variable order derivatives is little bit tough task as compared to constant order fractional derivatives because the variable order fractional operator have complex kernels for variable powers. In the article [27], a collocation method based on domain type radial basis function has been applied to a constant and variable order derivative to find an approximate solution. Chen et al. [28], presented a new approach of collocation method based on boundary type radial basis function to find the the numerical solution of fractional order diffusion equation. To find the numerical solution of variable order fractional differential equation a finite difference scheme has been proposed in [29] with stability and convergence analysis of scheme. Moreover there are many numerical scheme have been proposed to find the numerical solution of variable order fractional differentiation viz., B-linear spline technique[30], integro quadratic spline interpolations techniques [31], finite difference method [32], cubic spline technique [33], discretization technique [34], spectral collocation technique [35] etc.

Fractional PDEs can be utilized in modeling many linear and non-linear physical processes. Although there are wide application of fractional PDEs but the accurate mathematical model of many complex physical processes can not be found. Zadeh developed the concept of fuzzy theory to overcome this lackness of fractional PDEs and the application of fuzzy theory with fractional PDEs can be able to mathematical modeling of such complex physical process. Fuzzy analytic theory has a very useful and significant part as fuzzy differential equations (DEs). Fuzzy DEs are very efficient tools which explain the many dynamical process accurately where the nature of dynamical process is uncertain with vague information[36]. The applications of fuzzy fractional differential equations fuzzy fractional PDEs are rapidly spread in last few years because of its wide presence in modeling the several physical industrial processes like mass and heat transfer, bio-mechanics, electromagnetic fields etc. Many researchers have been developed some numerical scheme for the approximate solutions of fractional fuzzy PDEs[37–39]. Although there are many numerical schemes are available but the field of fractional fuzzy PDEs yet to be tackled more accurately and needs some more efficient and valid numerical schemes.

In the present article, we have proposed an accurate and efficient numerical technique to solve variable order fractional order fuzzy reaction-diffusion equation with Mittag-Leffler kernel arising in porous media as

$$^{ABC}D_t^{\mu(x,t)}\tilde{\zeta}(x,t) = \tilde{d}\frac{\partial^2\tilde{\zeta}(x,t)}{\partial x^2} + \tilde{\gamma}(\frac{\partial\tilde{\zeta}(x,t)}{\partial x})^a + \tilde{f}(\tilde{\zeta}(x,t)) + \tilde{h}(x,t), \quad (1.1)$$

with initial and boundary conditions

$$\tilde{\zeta}(0,t) = \tilde{h_1}(t), \;\; \tilde{\zeta}(x,0) = \tilde{h_2}(x), \;\; \tilde{\zeta}(1,t) = \tilde{h_3}(t). \quad (1.2)$$

where $0 \leq x, t \leq 1$, $\mu(x,t)$ denotes the fractional variable order of fuzzy derivative. The field variable $\tilde{\zeta}(x,t)$ denotes a fuzzy values function w.r.to the crisp variables x, t. The fractional order derivatives $^{ABC}D_t^{\mu(x,t)}\tilde{\zeta}(x,t)$ is considered w.r.to the Hukuhara derivatives. The constant coefficients viz., $\tilde{d}$ and $\tilde{\gamma}$ denotes some fixed fuzzy numbers. The unknown functions viz., $\tilde{f}(\tilde{\zeta}(x,t)), \tilde{h}(x,t)$ and known functions $\tilde{h}_i' s$ represents some known fuzzy valued functions.

In the research paper, our main purpose is to develop a most powerful and highly efficient scheme viz., shifted fifth-kind Chebyshev fuzzy operational matrix method to find more accurate approximate numerical solution of the considered non-linear fuzzy space-time fractional reaction-diffusion equations with Mittag-Leffler kernel. The Chebyshev spectral collocation scheme is very efficient tool in handling the non-linear fuzzy PDEs over other known schemes. Allahviranloo [40] developed a method for the solution of system of fuzzy algebraic equations. This method is being used to solve the non-linear fuzzy algebraic system of equations during the process of our proposed scheme. This proposed scheme is applied to many non-linear fuzzy fractional mathematical models to show the high convergence and efficiency of the proposed scheme. As per the best analysis and knowledge, the Chebyshev spectral techniques is first of its kind to find the approximate numerical solution of non-linear fuzzy fractional PDEs with Mittag-Leffler kernel.

This paper is organized as: basic concepts and properties of fuzzy set theory, fuzzy fractional derivatives in ABC sense is discussed in section 2. Section 3 contains definitions and fundamental results of shifted fifth-kind Chebyshev polynomials. The approximation of unknown fuzzy valued functions in terms of Chebyshev polynomials is given in section 4. The derivation of novel fuzzy operational matrix for fractional order and proposed algorithm of the present scheme are discussed in section 5 and section 6 respectively. Few test examples are carried out in order to validate the capability and efficiency of the numerical scheme in section 7. The outcomes of the scientific work is given in conclusion section.

2. Preliminaries and Notations

This part of the work contains some basic introduction and properties of fuzzy calculus and fuzzy set theory. The definitions of fractional fuzzy derivatives and integrals are given in ABC sense which can be further used in the manuscript.

2.1 Motivation behind ABC derivatives

Here we are going to provide the main motivations behind the use of ABC fractional derivatives in this manuscript. The main purpose of using the ABC fractional derivative as a basic tool in modeling of the dynamics fluid in porous media is to describe and encounter the crossover behavior of model and non-local and non-singularity of its kernel where as other type of fractional derivatives which do not posses the these important properties are may be not able to accurate analysis and simulation of concerned physical model.

The definitions of general RiemannLiouville fractional derivatives and Caputo fractional derivatives are given by:

$$ {}^{RL}{}_{0}D_t^{\mu}\zeta(t) = \frac{d}{dt}\int_0^t p(t-x)\zeta(x)dx = \frac{d}{dt}p \star \zeta, \tag{2.1} $$

and

$$ {}^{C}{}_{0}D_t^{\mu}\zeta(t) = \int_0^t p(t-x)\frac{d}{dx}\zeta(x)dx = p \star \frac{d}{dt}\zeta. \tag{2.2} $$

There are generally two special types of kernels viz., the kernel which generates the fractional derivatives with power kernel law $p(t-x) = \frac{1}{\Gamma(1-\mu)}(t-x)^{-\mu}$ and second one is the kernel which generates fractional derivatives with properties of Dirac-Delta function with Mittag-Leffler law, this type of fractional derivatives commonly known as AB fractional derivatives[5]. The first type of kernel generally describes the fitting of a large portion of wealth to a small part of population using Pareto distribution where as the second one derivative (ABC derivatives) is corresponds to the Mittag-Leffler distribution. Due to its non-commutative property, the AB fractional derivatives can be applied to chaotic problems, fractal models and in dealing with the phase space in quantum mechanics and brillouin zone in quantum hall effect.

Some of the very important properties of fractional derivatives are:

• It can be used in certain even where the waiting time is independent of elapsed time.
• Due to comparable with Brownian motion, the AB fractional derivatives is stochastic.
• The AB distribution has crossover from Gaussian to non-Gaussian.

• The AB derivatives has asymptotic behavior due to which it match the behavior of power law and can relate the fading memory with the non-singular kernels.

2.2 Fuzzy Set Theory:

The idea of fuzzy sets has been introduced by L. Zadeh[41] in 1965 to tackle the uncertainty arises because of imprecision and vagueness. Consider a nonempty set Z which is called as base a set and every member $z \in Z$ is associated to a membership grade $\zeta(z)$. A nonempty subset of $Z \times [0, 1]$ is considered as a fuzzy subset of Z by L. Zadeh.
The definition of a fuzzy set B is given as: $B \subseteq \{(z, \zeta(z)) : z \in Z\}$ where ζ is a function from Z to $[0, 1]$. The symbol ζ is commonly used for the notation of the fuzzy set B.

Definition 1. (Fuzzy Numbers): Now we are going to provide the definition of a fuzzy number $\tilde{\omega}$. A real valued function $\tilde{\omega}$ from $\mathbb{R}$ to unit interval $[0, 1]$ i.e. $\tilde{\omega} : \mathbb{R} \longrightarrow [0, 1]$ is said to be a fuzzy number if it satisfies the following basic properties:

• The function $\tilde{\omega}$ should be upper semi-continuous.
• The function $\tilde{\omega}$ should satisfy the normality properties i.e., $\exists$ a real number z_0 such that $\tilde{\omega}(z_0) = 1$.
• The convexity property should be satisfied by the function $\tilde{\omega}$ i.e., $\forall k \in [0, 1]$ and $\forall z_1, z_2 \in \mathbb{R}$, we have

$$\tilde{\omega}(kz_1 + (1 - k)z_2) \geq min\{\tilde{\omega}(z_1), \tilde{\omega}(z_2)\}. \tag{2.3}$$

• The closure set of the support of function $\tilde{\omega}$ is a compact set. The support of the function $\tilde{\omega}$ is defined as $supp(\tilde{\omega}) = \{z \in \mathbb{R} : \tilde{\omega}(z) > 0\}$.

Definition 2. (v-Level Set of Fuzzy Numbers): Consider the collection of all fuzzy numbers defined on set of real number $\mathbb{R}$ is denoted by $\mathbb{R}_F$. Let the fuzzy number $\tilde{\omega} \in \mathbb{R}_F$ for some $v \in [0, 1]$ then for every $v \in [0, 1]$ the v-level set of fuzzy number $[\tilde{\omega}_v]$ is defined as

$$[\tilde{\omega}] = \begin{cases} \{z \in \mathbb{R} : \tilde{\omega}(z) \leq v\}, & v \in (0, 1] \\ \text{closure}(\text{supp}(\tilde{\omega})), & v = 0. \end{cases} \tag{2.4}$$

From the above definition we can find that the $v-$level set $[\tilde{\omega}]$ is a closed and bounded set. Let $\tilde{\omega}^-(v)$ and $\tilde{\omega}^+(v)$ are the end points of the v-level fuzzy interval then we can write the v-level fuzzy interval as $[\tilde{\omega}] = [\tilde{\omega}^-(v), \tilde{\omega}^+(v)]$. The following definition of the fuzzy number can be very useful in embedding the set of real number to the set of fuzzy number for all $z, y \in \mathbb{R}$:

$$\tilde{\omega}(y) = \begin{cases} 1, & y = z \\ 0, & y \neq z. \end{cases} \tag{2.5}$$

Definition 3. (Parametric Interval Form): For any fuzzy number $\tilde{\omega} \in \mathbb{R}_F$, the parametric interval form can be given as:

$$\tilde{\omega}[v] = [\tilde{\omega}_l(v), \tilde{\omega}_u(v)], \;\; v \in [0, 1]. \tag{2.6}$$

The above form of the fuzzy number satisfies the following properties:

• For every $v \in [0, 1]$ The functions $\tilde{\omega}_u(v)$ and $\tilde{\omega}_l(v)$ satisfy the inequality $\tilde{\omega}_l(v) \leq \tilde{\omega}_u(v)$.
• The function $\tilde{\omega}_u(v)$ is a non-increasing and left continuous function of v.
• The function $\tilde{\omega}_l(v)$ is a non-decreasing and left continuous function of v.

The arithmetic operations i.e. vector addition and scalar multiplication of any two arbitrary fuzzy numbers $\tilde{\omega}_1(v)$ and $\tilde{\omega}_2(v)$ are defined for $v \in [0, 1]$ as:

$$\begin{aligned} (\tilde{\omega}_1 \oplus \tilde{\omega}_2)[v] &= [\tilde{\omega}_{1_l} + \tilde{\omega}_{2_l}(v), \tilde{\omega}_{1_u} + \tilde{\omega}_{2_u}(v)], \\ (k \odot \tilde{\omega})[v] &= \begin{cases} [k\tilde{\omega}_l(v), k\tilde{\omega}_u(v)], & k \geq 0, \\ [k\tilde{\omega}_u(v), k\tilde{\omega}_l(v)], & k < 0. \end{cases} \end{aligned} \tag{2.7}$$

Definition 4. (gH-difference): Let M and N are two nonempty compact set then there gH-difference (generalized Hukuhara difference) as the compact set P is given as:

$$M \ominus_{gH} N = P \Leftrightarrow \begin{cases} \quad (a) \;\; M = N + P \\ \text{or } (b) \;\; N = M - P. \end{cases} \tag{2.8}$$

Definition 5. (gH-derivatives): Here we will provide the definition of fuzzy derivatives (gH-derivatives) of any arbitrary fuzzy valued function. Consider a point z_0 in (l, m) and a fuzzy valued function ζ such that $\zeta : (l, m) \longrightarrow \mathbb{R}_F$. Then the function ζ is H-differentiable at point z_0 and is equal to a fuzzy number $\zeta^{'}(z_0)$ if it is satisfy the following equations:

(i) **Case 1:** if the H-difference for two fuzzy number $\zeta(z_0 + h \ominus \zeta(z_0))$ and $\zeta(z_0) \ominus \zeta(z_0 - h)$ exists then we have:

$$\zeta^{'}(z_0) = \lim_{h \to 0^+} \frac{\zeta(z_0 + h \ominus \zeta(z_0))}{h} = \lim_{h \to 0^+} \frac{\zeta(z_0) \ominus \zeta(z_0 - h)}{h}. \tag{2.9}$$

This definition of differentiation is called as 1-differentiation of function ζ at (l, m).

(ii) **Case 2:** if the H-difference for two fuzzy number $\zeta(z_0 \ominus \zeta(z_0+h))$ and $\zeta(z_0-h)\ominus\zeta(z_0)$ exists then we have:

$$\zeta^{'}(z_0) = \lim_{h\to 0^+} \frac{\zeta(z_0 \ominus \zeta(z_0+h))}{-h} = \lim_{h\to 0^+} \frac{\zeta(z_0-h)\ominus\zeta(z_0)}{-h}. \tag{2.10}$$

This definition of differentiation is called as 2-differentiation of function ζ at (l,m).
The gH-derivative can also be given in same manner as:

$$\zeta^{'}(z_0) = \lim_{h\to 0} \frac{\zeta(z_0+h \ominus_{gH} \zeta(z_0))}{h}. \tag{2.11}$$

In order to establish the fractional derivatives in Caputo and Riemann-Liouville we are going to provide the definition of Lebesgue integration of any function $\zeta^{'}(t)$ in parametric fuzzy interval form as:

$$[\int_0^t \zeta^{'}(z)dz]_v = \int_0^t [\zeta^{'}(z)]_v dz = \begin{cases} [\int_0^t \zeta_{-}^{'}(z;v)dz, \int_0^t \zeta_{+}^{'}(z;v)dz], \text{for case-1}, \\ [\int_0^t \zeta_{-}^{'}(z;v)dz, \int_0^t \zeta_{+}^{'}(z;v)dz], \text{for case-2}. \end{cases} \tag{2.12}$$

2.3 Fuzzy Fractional Derivatives

Here we are going to present fuzzy fuzzy fractional derivatives of a fuzzy differential function $\zeta(t)$. The fuzzy fractional derivatives are the generalization of classical fractional differentiation in crisp sense.

Definition 6. (Caputo fractional g-derivatives:) The fractional g-derivatives of any fuzzy valued measurable continuous function $\zeta(t)$ of any arbitrary fractional order in Caputo sense at point t is given as:

$$^{g}D^{\mu}_{a^+}\zeta(t) = \lim_{h\to 0} \frac{\lambda(t+h)\ominus_g \lambda(t)}{h}, \tag{2.13}$$

where the function λ is given by

$$\lambda(t) = \frac{1}{\Gamma(1-\mu)} \int_a^t (t-\rho)^{-\mu}\zeta(\rho)d\rho. \tag{2.14}$$

Let the function $\zeta(t)$ is absolutely continuous fuzzy valued function then Caputo fractional fuzzy derivatives is defined for both previously cases as:

$$\begin{aligned} [^{C}_{a^+}D^{i,\mu}_t\zeta(t)] &= [^{C}_{a^+}D^{i,\mu}_t\zeta_-(t;v), {}^{C}_{a^+}D^{i,\mu}_t\zeta_+(t;v)] \quad \text{for case-1}, \\ [^{C}_{a^+}D^{ii,\mu}_t\zeta(t)] &= [^{C}_{a^+}D^{ii,\mu}_t\zeta_+(t;v), {}^{C}_{a^+}D^{ii,\mu}_t\zeta_-(t;v)] \quad \text{for case-2}. \end{aligned} \tag{2.15}$$

where $[{}^{C}_{a^+}D_t^{i,\mu}\zeta_-(t;v), {}^{C}_{a^+}D_t^{i,\mu}\zeta_+(t;v)], [{}^{C}_{a^+}D_t^{ii,\mu}\zeta_-(t;v)$ and ${}^{C}_{a^+}D_t^{ii,\mu}\zeta_+(t;v)]$ are given by the following equations:

$$\begin{aligned}
{}^{C}_{a^+}D_t^{i,\mu}\zeta_-(t;v) &= \frac{1}{\Gamma(1-\mu)}\int_a^t (t-\rho)^{-\mu}\zeta_-'(\rho)d\rho,\\
{}^{C}_{a^+}D_t^{i,\mu}\zeta_+(t;v) &= \frac{1}{\Gamma(1-\mu)}\int_a^t (t-\rho)^{-\mu}\zeta_+'(\rho)d\rho,\\
{}^{C}_{a^+}D_t^{ii,\mu}\zeta_+(t;v) &= \frac{1}{\Gamma(1-\mu)}\int_a^t (t-\rho)^{-\mu}\zeta_+'(\rho)d\rho,\\
{}^{C}_{a^+}D_t^{ii,\mu}\zeta_-(t;v) &= \frac{1}{\Gamma(1-\mu)}\int_a^t (t-\rho)^{-\mu}\zeta_-'(\rho)d\rho.
\end{aligned} \tag{2.16}$$

Definition 7. (Variable order fuzzy ABC derivative) Here we are going to define variable order fractional fuzzy ABC derivatives of an absolutely continuous fuzzy valued function $\zeta(t)$. Then variable order fractional fuzzy ABC derivatives of order $\mu(x,t)$ is defined for both previously cases as:

$$\begin{aligned}
[{}^{ABC}D_t^{i,\mu(x,t)}\zeta(t)] &= [{}^{ABC}D_t^{i,\mu(x,t)}\zeta_-(t;v), {}^{ABC}D_t^{i,\mu(x,t)}\zeta_+(t;v)] \quad \text{for case-1},\\
[{}^{ABC}D_t^{ii,\mu(x,t)}\zeta(t)] &= [{}^{ABC}D_t^{ii,\mu(x,t)}\zeta_+(t;v), {}^{ABC}D_t^{ii,\mu(x,t)}\zeta_-(t;v)] \quad \text{for case-2}.
\end{aligned} \tag{2.17}$$

where $[{}^{ABC}D_t^{i,\mu(x,t)}\zeta_-(t;v), {}^{ABC}D_t^{i,\mu(x,t)}\zeta_+(t;v)], [{}^{ABC}D_t^{ii,\mu(x,t)}\zeta_-(t;v)$ and ${}^{ABC}D_t^{ii,\mu(x,t)}\zeta_+(t;v)]$ are given by the following equations:

$$\begin{aligned}
{}^{ABC}D_t^{i,\mu(x,t)}\zeta_-(t;v) &= \frac{M(\mu(x,t))}{1-\mu(x,t)}\int_0^t E_{\mu(x,t)}[\frac{-\mu(x,t)}{1-\mu(x,t)}(t-\rho)^{\mu(x,t)}]\zeta_-'(x,\rho)d\rho,\\
{}^{ABC}D_t^{i,\mu(x,t)}\zeta_+(t;v) &= \frac{M(\mu(x,t))}{1-\mu(x,t)}\int_0^t E_{\mu(x,t)}[\frac{-\mu(x,t)}{1-\mu(x,t)}(t-\rho)^{\mu(x,t)}]\zeta_+'(x,\rho)d\rho,\\
{}^{ABC}D_t^{ii,\mu(x,t)}\zeta_+(t;v) &= \frac{M(\mu(x,t))}{1-\mu(x,t)}\int_0^t E_{\mu(x,t)}[\frac{-\mu(x,t)}{1-\mu(x,t)}(t-\rho)^{\mu(x,t)}]\zeta_+'(x,\rho)d\rho,\\
{}^{ABC}D_t^{ii,\mu(x,t)}\zeta_-(t;v) &= \frac{M(\mu(x,t))}{1-\mu(x,t)}\int_0^t E_{\mu(x,t)}[\frac{-\mu(x,t)}{1-\mu(x,t)}(t-\rho)^{\mu(x,t)}]\zeta_-'(x,\rho)d\rho.
\end{aligned} \tag{2.18}$$

In the above definitions $0 < \mu(x,t) < 1$.

3. Basic Properties and Definitions of the Fifth-kind Chebyshev Polynomials

Now a days the Chebyshev polynomials are world wide useful in various area of applied and engineering mathematics[42,43]. The l^{th} degree fifth-kind

Chebyshev polynomials are defined in the interval [-1,1] as:

$$\omega_l(x) = \frac{1}{\sqrt{\delta_l}}\bar{\Upsilon}_l^{-3,2,-1,1}(x), -1 \leq x \leq 1; \tag{3.1}$$

In the definition of fifth-kind Chebyshev polynomials $\bar{\Upsilon}_l^{-3,2,-1,1}(x)$ is given by following general formula

$$\bar{\Upsilon}_l^{f,g,h,i}(x) = (\prod_{s=0}^{\lfloor \frac{l}{2} \rfloor -1} \frac{(2s+(-1)^{l+1}+2)i+g}{(2s+(-1)^{l+1}+2\lfloor \frac{l}{2} \rfloor)h+f})\Upsilon_l^{f,g,h,i}(x), \tag{3.2}$$

and

$$\Upsilon_l^{f,g,h,i}(x) = \sum_{u=0}^{\lfloor \frac{l}{2} \rfloor}(\binom{\lfloor \frac{l}{2} \rfloor}{u}(\prod_{s=0}^{\lfloor \frac{l}{2} \rfloor -u-1} \frac{(2s+(-1)^{l+1}+2\lfloor \frac{l}{2} \rfloor)h+f}{(2s+(-1)^{l+1}+2)i+g})x^{l-2u}). \tag{3.3}$$

The other constant δ_l in the equation (3.1) is given by the following expression:

$$\delta_l = \begin{cases} \frac{\pi(l+2)}{l2^{2l+1}}, & \text{when l is odd,} \\ \frac{\pi}{2^{2l+1}}, & \text{when l is even.} \end{cases} \tag{3.4}$$

The collection of Chebyshev polynomials $\omega_l(x)$ form an orthonormal set over the interval [-1,1] i.e.

$$\int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}\omega_k(x)\omega_l(x) = \begin{cases} 0, & \text{when } k \neq l, \\ 1, & \text{when } k = l. \end{cases} \tag{3.5}$$

Now the l^{th} degree shifted fifth-kind Chebyshev polynomials are defined in the interval [0,1] as:

$$\Omega_l(x) = \omega_l(2x-1). \tag{3.6}$$

The collection of shifted fifth-kind Chebyshev polynomials $\Omega_l(x)$ form an orthonormal set over the interval [0,1] i.e.

$$\int_{0}^{1} \frac{(2x-1)^2}{\sqrt{x-x^2}}\Omega_k(x)\Omega_l(x) = \begin{cases} 0, & \text{when } k \neq l, \\ 1, & \text{when } k = l. \end{cases} \tag{3.7}$$

Moreover the shifted fifth-kind Chebyshev polynomials are bounded over the unit interval [0,1] i.e.,

$$\mid \Omega_l(x) \mid \leq \sqrt{2/\pi}(l+2), \forall t \in [0,1] \text{ and } l \geq 0. \tag{3.8}$$

We can rewrite $\Omega_l(x)$ in series form as

$$\Omega_l(x) = \sum_{p=0}^{l} \lambda_{p,l}x^p, \tag{3.9}$$

where $\lambda_{p,l}$ is given by

$$\lambda_{p,l} = \frac{2^{2p+3/2}}{\sqrt{\pi}(2p)!} \begin{cases} \frac{1}{\sqrt{l(l+2)}} \sum_{\epsilon=\lfloor \frac{p}{2} \rfloor}^{\frac{l-1}{2}} \frac{(-1)^{\frac{l+1}{2}+\epsilon-p}+(2\epsilon+1)^2(2\epsilon+p)!}{(2\epsilon-p+1)!}, & \text{when l is odd ,} \\ 2\sum_{\epsilon=\lfloor \frac{p+1}{2} \rfloor}^{\frac{l}{2}} \frac{(-1)^{\frac{l}{2}+\epsilon-p}\epsilon\theta_\epsilon(2\epsilon+p-1)!}{(2\epsilon-1)!}, & \text{when l is even ,} \end{cases} \tag{3.10}$$

and θ_ϵ is;

$$\theta_\epsilon = \begin{cases} 1, & \epsilon > 0, \\ 1/2, & \epsilon = 0. \end{cases} \tag{3.11}$$

The matrix form shifted fifth-kind Chebyshev polynomials can be written as

$$\vartheta(x) = M.P_l(x), \tag{3.12}$$

where $\vartheta(x) = [\Omega_0(x), \Omega_1(x), ..., \Omega_l(x)]^T$, $P_l(x) = [1, x, x^2, ..., x^l]^T$ and M is a lower triangular matrix as:

$$M = \begin{bmatrix} \lambda_{0,0} & 0 & 0 & \dots & 0 \\ \lambda_{1,0} & \lambda_{1,1} & 0 & \dots & 0 \\ \lambda_{2,0} & \lambda_{2,1} & \lambda_{2,2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{l,0} & \lambda_{l,1} & \lambda_{l,2} & \dots & \lambda_{l,l} \end{bmatrix}. \tag{3.13}$$

From the structure of matrix M it is clear that M is an invertible matrix i.e., $|M| \neq 0$.

4. Approximation of the function $\zeta(x, t)$

Since the set of shifted fifth-kind Chebyshev polynomials forms a complete basis in the Hilbert space $L^2[0, 1]$ therefore each function $\zeta(x) \in L^2[0, 1]$ can be expressed in terms of this polynomials as:

$$\zeta(x) \simeq \zeta_r(x) = \sum_{g=0}^{r} b_g \Omega_g(x) = B^T.\vartheta(x), \tag{4.1}$$

where the unknown constants matrix $B^T = [b_g]$ is known as shifted fifth-kind Chebyshev coefficients.
Similarly a function of two variable $\zeta(x, t) \in L^2[0, 1]$ can be expressed in terms of Chebyshev polynomials as:

$$\zeta(x, t) \simeq \zeta_r(x, t) = \sum_{g=0}^{r} \sum_{h=0}^{r} b_{g,h} \Omega_g(x) \Omega_h(t) = \vartheta(x)^T.B.\vartheta(t), \tag{4.2}$$

where the unknown constants matrix $B = [b_{g,h}]$ is known as shifted fifth-kind Chebyshev coefficients. These coefficients can be determine by using the initial and boundary conditions.

4.1 Approximation of fuzzy valued function $\tilde{\zeta}(x,t)$

Here we use the shifted fifth-kind Chebyshev polynomial in order to approximate a fuzzy valued measurable and continuous function $\tilde{\zeta}(t)$. The approximation of the fuzzy valued function in terms of shifted fifth-kind Chebyshev polynomials is given as:

$$\tilde{\zeta}(x) \simeq \tilde{\zeta}_r(x) = \sum_{g=0}^{r} \tilde{b}_g \odot \Omega_g(x) = \tilde{B}^T \odot \vartheta(x), \tag{4.3}$$

where the unknown constants matrix $\tilde{B}^T = [\tilde{b}_g]$ is known as Chebyshev coefficients.
Similarly a function of two variable $\tilde{\zeta}(x,t) \in L^2[0,1]$ can be expressed in terms of Chebyshev polynomials as:

$$\tilde{\zeta}(x,t) \simeq \tilde{\zeta}_r(x,t) = \sum_{g=0}^{r}\sum_{h=0}^{r} \tilde{b}_{g,h} \odot \Omega_g(x) \odot \Omega_h(t) = \vartheta(x)^T \odot \tilde{B} \odot \vartheta(t), \tag{4.4}$$

where the unknown constants matrix $\tilde{B} = [\tilde{b}_{g,h}]$ is known as Chebyshev coefficients. Here also the summation is taken in accordance with fuzzy algebraic addition $\oplus$ and $\odot$ denotes the fuzzy scalar multiplication. The unknown fuzzy coefficients matrix $\tilde{B} = [\tilde{b}_{g,h}]$ can be calculated later where all the operation will be taken as fuzzy set algebra.

5. Operational Matrix of ABC Derivative for Variable Order

In this section of the manuscript we are going to derive the operational matrix for the variable order derivative. From the equation (3.12), we can write the derivative of vector $\vartheta(t)$ as

$$^{ABC}D_t^{\mu(x,t)}\vartheta(t) = {}^{ABC}D_t^{\mu(x,t)} M.P_r(t) = M.{}^{ABC}D_t^{\mu(x,t)} \begin{bmatrix} 1 \\ t \\ \vdots \\ t^r \end{bmatrix}. \tag{5.1}$$

Or,

$$\begin{aligned} {}^{ABC}D_t^{\mu(x,t)}\vartheta(t) =& M.[0, \frac{M(\mu(x,t))}{1-\mu(x,t)}\sum_{s=0}^{\infty}\frac{(-\mu(x,t))^s t^{\mu(x,t)s+1}}{(1-\mu(x,t))^s\Gamma(\mu(x,t)s+2)}, \cdots, \\ & \frac{M(\mu(x,t))}{1-\mu(x,t)}\sum_{s=0}^{\infty}\frac{(-\mu(x,t))^s\Gamma(r+1)t^{\mu(x,t)s+r}}{(1-\mu(x,t))^s\Gamma(\mu(x,t)s+r+1)}]^T. \end{aligned} \tag{5.2}$$

After simplification above equation can be written as:

$$ {}^{ABC}D_t^{\mu(x,t)}\vartheta(t) = M.\Xi.P_r(t), \tag{5.3}$$

where Ξ is given by the expression:

$$\Xi = [a_{pq}]_{(r+1)\times(r+1)} = \begin{cases} 0, & \text{elsewhere}, \\ \frac{M(\mu(x,t))}{1-\mu(x,t)}\sum_{s=0}^{\infty}\frac{(-\mu(x,t))^s\Gamma(q+1)t^{\mu(x,t)s}}{(1-\mu(x,t))^s\Gamma(\mu(x,t)s+q+1)}, & \text{when } p=q\geq g. \end{cases} \tag{5.4}$$

In view of equation (3.12) and(5.3), we can write

$$ {}^{ABC}D_t^{\mu(x,t)}\vartheta(t) = M.\Xi.M^{-1}.\vartheta(t), \tag{5.5}$$

where $M.\Xi.M^{-1}$ is operational matrix of ABC derivative for variable fractional order w.r.to time. Similarly we can find the operational matrix for variable order ABC derivative w.r.to x. Now collocating the our concerned model (1.1) with give initial and boundary conditions (1.2), we find a system of non-linear algebraic equation. After solving this non-linear algebraic equation, we can get the arbitrary constant matrix $\tilde{B}$ in the approximation (4.4).

6. Proposed Algorithm

In this section of the article we investigate the concerned fuzzy model under the environment of fuzzy calculus theory. Our variable order fractional fuzzy advection-diffusion equation is

$$ {}^{ABC}D_t^{\mu(x,t)}\tilde{\zeta}(x,t) = \tilde{d}\frac{\partial^2\tilde{\zeta}(x,t)}{\partial x^2} + \tilde{\gamma}(\frac{\partial\tilde{\zeta}(x,t)}{\partial x})^a + \tilde{f}(\tilde{\zeta}(x,t)) + \tilde{h}(x,t), \tag{6.1}$$

with initial and boundary conditions

$$\tilde{\zeta}(0,t) = \tilde{h_1}(t),\ \ \tilde{\zeta}(x,0) = \tilde{h_2}(x),\ \ \tilde{\zeta}(1,t) = \tilde{h_3}(t). \tag{6.2}$$

After the fuzzyfication above equation can be written for $v \in [0,1]$ as

$$\begin{aligned}
&[\tilde{\zeta}(x,t)]^v = [\zeta_-(x,t,v), \zeta_+(x,t,v)],\\
&[{}^{ABC}D_t^{\mu(x,t)}\tilde{\zeta}(x,t)]^v = [{}^{ABC}D_t^{\mu(x,t)}\zeta_-(x,t,v), {}^{ABC}D_t^{\mu(x,t)}\zeta_+(x,t,v)],\\
&[\frac{\partial^2\tilde{\zeta}(x,t)}{\partial x^2}]^v = [\frac{\partial^2\zeta_-(x,t,v)}{\partial x^2}, \frac{\partial^2\zeta_+(x,t,v)}{\partial x^2}],\\
&[\frac{\partial\tilde{\zeta}(x,t)}{\partial x}]^v = [\frac{\partial\zeta_-(x,t,v)}{\partial x}, \frac{\partial\zeta_+(x,t,v)}{\partial x}],\\
&[\tilde{h}(x,t)]^v = [h_-(x,t,v), h_+(x,t,v)].
\end{aligned} \tag{6.3}$$

Now we can rewrite concerned model in upper and lower approximations as

$$\begin{aligned}
{}^{ABC}D_t^{\mu(x,t)}\zeta_+(x,t,v) = {}& d_+\frac{\partial^2\zeta_+(x,t,v)}{\partial x^2} + \gamma_+(\frac{\partial\zeta_+(x,t,v)}{\partial x})^a\\
&+ f_+(\zeta_+(x,t,v)) + h_+(x,t,v),
\end{aligned} \tag{6.4}$$

with initial and boundary conditions as

$$\zeta_+(0,t) = h_{1_+}(x),\ \ \zeta_+(x,0) = h_{2_+}(t),\ \ \zeta_+(1,t) = h_{3_+}(t). \tag{6.5}$$

and

$$\begin{aligned}
{}^{ABC}D_t^{\mu(x,t)}\zeta_-(x,t,v) = {}& d_-\frac{\partial^2\zeta_-(x,t,v)}{\partial x^2} + \gamma_-(\frac{\partial\zeta_-(x,t,v)}{\partial x})^a\\
&+ f_-(\zeta_-(x,t,v)) + h_-(x,t,v),
\end{aligned} \tag{6.6}$$

with initial and boundary conditions as

$$\zeta_-(0,t) = h_{1_-}(x),\ \ \zeta_-(x,0) = h_{2_-}(t),\ \ \zeta_-(1,t) = h_{3_-}(t). \tag{6.7}$$

On solving above equation we can find the numerical solution of this non-linear variable order fuzzy PDE for both upper and lower approximations.

7. Test Examples

This section of the article is mainly devoted find the approximate numerical solution of few test examples. Here the fuzzy fractional variable order operational matrix method based on Chebyshev spectral collocation method has been applied to some linear and non-linear variable order fuzzy fractional reaction-diffusion equation with Mittag-Leffler kernel and demonstration of absolute errors. In order to show the capability and efficiency of the present numerical scheme for fuzzy fractional PDEs the absolute error is reported for test examples.

Example 1: Consider the following non-linear variable order fuzzy advection-diffusion equation with ABC derivative for some particular values of constant coefficients in the concerned model (1.1) as

$$ {}^{ABC}D_t^{\mu(x,t)}\tilde{\zeta}(x,t) = \frac{\partial^2\tilde{\zeta}(x,t)}{\partial x^2} - \frac{\partial\tilde{\zeta}(x,t)}{\partial x} + \tilde{h}(x,t), \tag{7.1} $$

The exact solution for this non-linear variable order fractional fuzzy PDE is $\tilde{\zeta}(x,t;v) = \tilde{\omega}(v)e^{-x^3t}x^{\mu(x,t)}$, where $\tilde{\omega}(v) = [v-1, 1-v]$. The initial and boundary conditions can be obtain from the exact solution. The plot of exact solution and numerical solution is shown through the Fig. 1 for variable fractional order $\mu(x,t) = 1.25 + 0.35sin(3\pi xt)$ at $t = 0.5$, which ensures the high accuracy and efficiency of the proposed numerical scheme. The behavior of field variable $\tilde{\zeta}(x,t)$ w.r.to the crisp value v can be seen from the Fig. 2 for different value of the variable fractional order $\mu(x,t)$ at $x = t = 0.5$ for the order of approximation $r = 6$.

Figure 1 (a-b)
Figure 2 (a-b)

The absolute error is also computed to show the high accuracy of proposed numerical scheme for both upper and lower approximations. The variation of absolute error can be seen from the Table 1 for different values of x at $t = v = 0.5$ for the order of approximation $r = 7$.

Table 1

Example 2: Consider the following non-linear variable order fuzzy advection-diffusion equation with ABC derivative for some particular values of constant coefficients in the concerned model (1.1) as

$$ {}^{ABC}D_t^{\mu(x,t)}\tilde{\zeta}(x,t) = \frac{\partial^2\tilde{\zeta}(x,t)}{\partial x^2} - \frac{\partial\tilde{\zeta}(x,t)}{\partial x} + \tilde{\zeta}(x,t)(1-\tilde{\zeta}(x,t)) + \tilde{h}(x,t), \tag{7.2} $$

The exact solution for this non-linear variable order fractional fuzzy PDE is $\tilde{\zeta}(x,t;v) = \tilde{\omega}(v)(xt)^{\mu(x,t)}$, where $\tilde{\omega}(v) = [0.9 + 0.1v, 1.1 - 0.1v]$. The initial and boundary conditions can be obtain from the exact solution. The plot of exact solution and numerical solution is shown through the Fig. 3 for variable fractional order $\mu(x,t) = 1.25 + 0.35sin(3\pi xt)$ at $t = 0.5$, which ensures the high accuracy and efficiency of the proposed numerical scheme. The behavior of field variable $\tilde{\zeta}(x,t)$ w.r.to the crisp value v can be seen from the Fig. 4 for different value of the variable fractional order $\mu(x,t)$ at $x = t = 0.5$ for the order of approximation $r = 6$.

Figure 3 (a-b)
Figure 4 (a-b)

The absolute error is also computed to show the high accuracy of proposed numerical scheme for both upper and lower approximations. The variation of absolute error can be seen from the Table 2 for different values of x at $t = v = 0.5$ for the order of approximation $r = 7$.

Table 2

Example 3: Consider the following non-linear variable order fuzzy advection-diffusion equation with ABC derivative for some particular values of constant coefficients in the concerned model (1.1) as

$$ {}^{ABC}D_t^{\mu(x,t)}\tilde{\zeta}(x,t) = \frac{1}{2}\frac{\partial^2\tilde{\zeta}(x,t)}{\partial x^2} - \frac{\partial\tilde{\zeta}(x,t)}{\partial x} + \tilde{\zeta}(x,t)(1-\tilde{\zeta}^2(x,t)) + \tilde{h}(x,t), \quad (7.3) $$

The exact solution for this non-linear variable order fractional fuzzy PDE is $\tilde{\zeta}(x,t;v) = \tilde{\omega}(v)cos(t)e^{-5x}x^{\mu(x,t)}$, where $\tilde{\omega}(v) = [0.75 + 0.25v, 1.25 - 0.25v]$. The initial and boundary conditions can be obtain from the exact solution. The plot of exact solution and numerical solution is shown through the Fig. 5 for variable fractional order $\mu(x,t) = 1.25 + 0.35sin(3\pi xt)$ at $t = 0.5$, which ensures the high accuracy and efficiency of the proposed numerical scheme. The behavior of field variable $\tilde{\zeta}(x,t)$ w.r.to the crisp value v can be seen from the Fig. 6 for different value of the variable fractional order $\mu(x,t)$ at $x = t = 0.5$ for the order of approximation $r = 6$.

Figure 5 (a-b)
Figure 6 (a-b)

The absolute error is also computed to show the high accuracy of proposed numerical scheme for both upper and lower approximations. The variation of absolute error can be seen from the Table 3 for different values of x at $t = v = 0.5$ for the order of approximation $r = 7$.

Table 3

The above three numerical examples shows a high accuracy of the proposed numerical scheme for finding the numerical solution of variable order non-linear fractional fuzzy partial differential equations with Mittag-Leffler kernel. The variation of the solute profile $\tilde{\zeta}(x,t)$ w.r.to the crisp value is shown from the Fig. 2, Fig. 4 and Fig. 6 for the different variable fractional order. The advancement of solute variable for fractional order system can be easily seen from these figures and from figures Fig. 1, Fig. 3 and Fig. 5. From these figures it can be seen that the numerical solution of the given test examples satisfies the double parametric form of fuzzy number system by achieving the convex triangular fuzzy number shape for $v \in [0,1]$.

8. Conclusion

In this paper the shifted fifth-kind Chebyshev polynomials is utilized for development of a numerical technique to find the approximate analytical solution of non-linear variable order fractional fuzzy partial differential equations with ABC derivative. The theory of fuzzy calculus has been discussed and we approximated the fuzzy valued function in terms of Chebyshev polynomials. The variable order fuzzy operational matrix is developed and with the help of this matrix we analyze the space-time fractional non-linear fractional variable order fuzzy reaction-advection-diffusion model with ABC derivative for the first time. The numerical solution which is very close to exact solution is obtained after the fuzzyfication concerned model with proper crisp points. The feasibility and efficiency of proposed numerical scheme is shown from few test examples by comparing the computed numerical solution with the existing exact solution. The proposed method can be utilized to investigate the behavior of system of fractional fuzzy PDEs.

References

[1] S. Kumar, R. Kumar, C. Cattani, B. Samet, Chaotic behaviour of fractional predator-prey dynamical system, Chaos, Solitons & Fractals 135 (2020) 109811.

[2] S. Kumar, R. Kumar, R. P. Agarwal, B. Samet, A study of fractional lotka-volterra population model using haar wavelet and adams-bashforth-moulton methods, Mathematical Methods in the Applied Sciences 43 (8) (2020) 5564–5578.

[3] M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, Progr. Fract. Differ. Appl 1 (2) (2015) 1–13.

[4] A.-M. Yang, Y. Han, J. Li, W.-X. Liu, On steady heat flow problem involving yang-srivastava-machado fractional derivative without singular kernel, Thermal Science 20 (suppl. 3) (2016) 717–721.

[5] A. Atangana, B. S. T. Alkahtani, New model of groundwater flowing within a confine aquifer: application of caputo-fabrizio derivative, Arabian Journal of Geosciences 9 (1) (2016) 8.

[6] A. Alshabanat, M. Jleli, S. Kumar, B. Samet, Generalization of caputo-fabrizio fractional derivative and applications to electrical circuits, FrP 8 (2020) 64.

[7] D. Baleanu, M. Jleli, S. Kumar, B. Samet, A fractional derivative with two singular kernels and application to a heat conduction problem, Advances in Difference Equations 2020 (1) (2020) 1–19.

[8] P. Veeresha, D. Prakasha, S. Kumar, A fractional model for propagation of classical optical solitons by using nonsingular derivative, Mathematical Methods in the Applied Sciences.

[9] E. F. D. Goufo, S. Kumar, S. Mugisha, Similarities in a fifth-order evolution equation with and with no singular kernel, Chaos, Solitons & Fractals 130 (2020) 109467.

[10] B. Ghanbari, S. Kumar, R. Kumar, A study of behaviour for immune and tumor cells in immunogenetic tumour model with non-singular fractional derivative, Chaos, Solitons & Fractals 133 (2020) 109619.

[11] V. Morales-Delgado, J. Gómez-Aguilar, S. Kumar, M. Taneco-Hernández, Analytical solutions of the keller-segel chemotaxis model involving fractional operators without singular kernel, The European Physical Journal Plus 133 (5) (2018) 200.

[12] S. Kumar, P. Pandey, A legendre spectral finite difference method for the solution of non-linear space-time fractional burgers–huxley and reaction-diffusion equation with atangana–baleanu derivative, Chaos, Solitons & Fractals 130 (2020) 109402.

[13] S. Kumar, P. Pandey, Quasi wavelet numerical approach of non-linear reaction diffusion and integro reaction-diffusion equation with atangana–baleanu time fractional derivative, Chaos, Solitons & Fractals 130 (2020) 109456.

[14] P. Pandey, S. Kumar, J. Gómez-Aguilar, Numerical solution of the time fractional reaction-advection-diffusion equation in porous media, Journal of Applied and Computational Mechanics 7 (2) (2021) x–xx.

[15] A. Arikoglu, I. Ozkol, Solution of fractional differential equations by using differential transform method, Chaos, Solitons & Fractals 34 (5) (2007) 1473–1481.

[16] S. Johnston, H. Jafari, S. Moshokoa, V. Ariyan, D. Baleanu, Laplace homotopy perturbation method for burgers equation with space-and time-fractional order, Open Physics 14 (1) (2016) 247–252.

[17] S. Das, S. Rajeev, Solution of fractional diffusion equation with a moving boundary condition by variational iteration method and adomian decomposition method, Zeitschrift für Naturforschung A 65 (10) (2010) 793–799.

[18] E. Babolian, F. Fattahzadeh, Numerical computation method in solving integral equations by using chebyshev wavelet operational matrix of integration, Applied Mathematics and Computation 188 (1) (2007) 1016–1022.

[19] M. T. Kajani, M. Ghasemi, E. Babolian, Comparison between the homotopy perturbation method and the sine–cosine wavelet method for solving linear integro-differential equations, Computers & Mathematics with Applications 54 (7-8) (2007) 1162–1168.

[20] Y. Li, W. Zhao, Haar wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations, Applied Mathematics and Computation 216 (8) (2010) 2276–2285.

[21] A. Bhrawy, E. Doha, S. Ezz-Eldien, M. Abdelkawy, A numerical technique based on the shifted legendre polynomials for solving the time-fractional coupled kdv equations, Calcolo 53 (1) (2016) 1–17.

[22] S. Kumar, A. Kumar, S. Momani, M. Aldhaifallah, K. S. Nisar, Numerical solutions of nonlinear fractional model arising in the appearance of the stripe patterns in two-dimensional systems, Advances in Difference Equations 2019 (1) (2019) 413.

[23] P. Pandey, S. Kumar, S. Das, Approximate analytical solution of coupled fractional order reaction-advection-diffusion equations, The European Physical Journal Plus 134 (7) (2019) 364.

[24] R. Ganji, H. Jafari, A numerical approach for multi-variable orders differential equations using jacobi polynomials, International Journal of Applied and Computational Mathematics 5 (2) (2019) 34.

[25] D. Valério, J. Sá da Costa, Variable order fractional controllers, Asian Journal of Control 15 (3) (2013) 648–657.

[26] M. D. Ortigueira, D. Valério, J. T. Machado, Variable order fractional systems, Communications in Nonlinear Science and Numerical Simulation 71 (2019) 231–243.

[27] Z.-J. Fu, W. Chen, L. Ling, Method of approximate particular solutions for constant-and variable-order fractional diffusion models, Engineering Analysis with Boundary Elements 57 (2015) 37–46.

[28] Z.-J. Fu, W. Chen, H.-T. Yang, Boundary particle method for laplace transformed time fractional diffusion equations, Journal of Computational Physics 235 (2013) 52–66.

[29] R. Lin, F. Liu, V. Anh, I. Turner, Stability and convergence of a new explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation, Applied Mathematics and Computation 212 (2) (2009) 435–445.

[30] J. A. T. Machado, B. P. Moghaddam, A robust algorithm for nonlinear variable-order fractional control systems with delay, International Journal of Nonlinear Sciences and Numerical Simulation 19 (3-4) (2018) 231–238.

[31] F. K. Keshi, B. P. Moghaddam, A. Aghili, A numerical approach for solving a class of variable-order fractional functional integral equations, Computational and Applied Mathematics 37 (4) (2018) 4821–4834.

[32] B. P. Moghaddam, J. A. T. Machado, Extended algorithms for approximating variable order fractional derivatives with applications, Journal of Scientific Computing 71 (3) (2017) 1351–1374.

[33] B. P. Moghaddam, J. A. T. Machado, A computational approach for the solution of a class of variable-order fractional integro-differential equations with weakly singular kernels, Fractional Calculus and Applied Analysis 20 (4) (2017) 1023–1042.

[34] M. Hajipour, A. Jajarmi, D. Baleanu, H. Sun, On an accurate discretization of a variable-order fractional reaction-diffusion equation, Communications in Nonlinear Science and Numerical Simulation 69 (2019) 119–133.

[35] M. Zayernouri, G. E. Karniadakis, Fractional spectral collocation methods for linear and nonlinear variable order fpdes, Journal of Computational Physics 293 (2015) 312–338.

[36] X. Guo, D. Shang, X. Lu, Fuzzy approximate solutions of second-order fuzzy linear boundary value problems, Boundary Value Problems 2013 (1) (2013) 212.

[37] T. Allahviranloo, N. Taheri, An analytic approximation to the solution of fuzzy heat equation by adomian decomposition method, Int. J. Contemp. Math. Sciences 4 (3) (2009) 105–114.

[38] A. Jameel, N. Anakira, A. Alomari, I. Hashim, S. Momani, A new approximation method for solving fuzzy heat equations, Journal of Computational and Theoretical Nanoscience 13 (11) (2016) 7825–7832.

[39] M. Stepnicka, R. Valasek, Numerical solution of partial differential equations with help of fuzzy transform, in: The 14th IEEE International Conference on Fuzzy Systems, 2005. FUZZ'05., IEEE, 2005, pp. 1104–1109.

[40] T. Allahviranloo, M. A. Kermani, Solution of a fuzzy system of linear equation, Applied Mathematics and Computation 175 (1) (2006) 519–531.

[41] L. A. Zadeh, Fuzzy sets, Information and control 8 (3) (1965) 338–353.

[42] W. Abd-Elhameed, Y. Youssri, Fifth-kind orthonormal chebyshev polynomial solutions for fractional differential equations, Computational and Applied Mathematics 37 (3) (2018) 2897–2921.

[43] R. Ganji, H. Jafari, D. Baleanu, A new approach for solving multi variable orders differential equations with mittag–leffler kernel, Chaos, Solitons & Fractals 130 (2020) 109405.

Table 1
Absolute error for upper and lower approximations of $\tilde{\zeta}(x, t; v)$

	$\mu(x,t) = 1.25 + 0.35sin(3\pi xt)$		$\mu(x,t) = 1.45 + 0.35sin(2\pi xt)$	
x	Lower case error	Upper case error	Lower case error	Upper case error
0.2	2.473×10^{-04}	6.585×10^{-05}	2.445×10^{-04}	1.204×10^{-04}
0.4	2.394×10^{-05}	9.654×10^{-05}	4.478×10^{-06}	7.361×10^{-05}
0.6	1.758×10^{-07}	4.689×10^{-06}	2.033×10^{-06}	1.941×10^{-06}
0.8	3.594×10^{-07}	4.390×10^{-07}	1.300×10^{-07}	6.439×10^{-06}
1	2.310×10^{-07}	4.347×10^{-07}	4.304×10^{-07}	5.495×10^{-07}

Table 2
Absolute error for upper and lower approximations of $\tilde{\zeta}(x, t; v)$

	$\mu(x,t) = 1.65 + 0.35sin(\pi xt)$		$\mu(x,t) = 1.45 + 0.35sin(2\pi xt)$	
x	Lower case error	Upper case error	Lower case error	Upper case error
0.2	3.403×10^{-04}	7.595×10^{-04}	3.504×10^{-04}	2.009×10^{-05}
0.4	5.438×10^{-04}	1.483×10^{-05}	2.963×10^{-05}	2.475×10^{-05}
0.6	4.012×10^{-05}	9.784×10^{-05}	1.308×10^{-06}	5.843×10^{-05}
0.8	1.384×10^{-05}	3.574×10^{-05}	1.043×10^{-06}	7.574×10^{-05}
1	9.840×10^{-06}	1.491×10^{-06}	5.675×10^{-06}	8.839×10^{-07}

Table 3
Absolute error for upper and lower approximations of $\tilde{\zeta}(x, t; v)$

	$\mu(x,t) = 1.65 + 0.35sin(\pi xt)$		$\mu(x,t) = 1.45 + 0.35sin(2\pi xt)$	
x	Lower case error	Upper case error	Lower case error	Upper case error
0.2	4.358×10^{-05}	3.474×10^{-05}	1.394×10^{-05}	2.483×10^{-04}
0.4	3.480×10^{-05}	7.659×10^{-05}	6.549×10^{-05}	5.584×10^{-05}
0.6	3.630×10^{-06}	1.382×10^{-06}	6.489×10^{-06}	6.785×10^{-06}
0.8	1.394×10^{-07}	3.484×10^{-06}	5.474×10^{-07}	7.592×10^{-06}
1	8.695×10^{-07}	1.403×10^{-07}	8.605×10^{-07}	1.495×10^{-06}

Figure Captions

1-Plot of $\tilde{\zeta}(x,t;\vartheta)$ for exact and numerical cases vs. x, v at $t = 0.5$.

2-Plot of $\tilde{\zeta}(x,t;\vartheta)$ for exact and numerical cases vs. v at $x = t = 0.5$.

3-Plot of $\tilde{\zeta}(x,t;\vartheta)$ for exact and numerical cases vs. x, v at $t = 0.5$.

4-Plot of $\tilde{\zeta}(x,t;\vartheta)$ for exact and numerical cases vs. v at $x = t = 0.5$.

5-Plot of $\tilde{\zeta}(x,t;\vartheta)$ for exact and numerical cases vs. x, v at $t = 0.5$.

6-Plot of $\tilde{\zeta}(x,t;\vartheta)$ for exact and numerical cases vs. v at $x = t = 0.5$.