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Critical exponent for semi-linear structurally damped wave equation of derivative type

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Abstract

The main purpose of this paper is to study the following semi-linear structurally damped wave equation with nonlinearity of derivative type:

$$u_{tt} - \Delta u + \mu(-\Delta)^{\sigma/2} u_t = |u_t|^p, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x),$$

with $\mu > 0$, $n \geq 1$, $\sigma \in (0,2]$ and $p > 1$. In particular, we are going to prove the non-existence of global weak solutions by using a new test function and suitable sign assumptions on the initial data in both the subcritical case and the critical case.

KEYWORDS:
Structural damping, Derivative type, Fractional Laplacian, Critical exponent

1 | INTRODUCTION

This paper is concerned with the Cauchy problem for semi-linear structurally damped wave equation with the power nonlinearity of derivative type (powers of the first order time-derivatives of solutions as nonlinear terms) as follows:

$$\begin{cases} u_{tt} - \Delta u + \mu(-\Delta)^{\sigma/2} u_t = |u_t|^p, & x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $\mu > 0$, $\sigma \in (0,2]$, $n \geq 1$ and $p > 1$. Here $(-\Delta)^{\sigma/2}$ is the fractional Laplacian defined as in Definition 1 below when $\sigma \in (0,2)$, and when $\sigma = 2$ it is the classical Laplacian.

Our main goal is to investigate the critical exponent for (1). By critical exponent $p_c = p_c(n,\sigma)$ we mean that global (in time) weak solutions cannot exist (it sometimes called blow-up in some cases), under suitable sign assumption on the initial data, in the subcritical case $1 < p < p_c$ and in the critical case $p = p_c$ as well, whereas small data global (in time) solutions exist in the supercritical case $p > p_c$.

Regarding the structurally damped wave equation (1) with the power nonlinearity $|u|^p$, the critical exponent has been investigated by D'Abbicco and Reissig [7], where they proposed to distinguish between "parabolic like models" in the case $\sigma \in (0,1]$, the so-called effective damping, and "hyperbolic like models" in the remaining case $\sigma \in (1,2]$, the so-called non-effective damping according to expected decay estimates (see more in detail [3]). In the effective case $\sigma \in (0,1]$, they proved the existence of global (in time) solutions when

$$p > p_0(n,\sigma) := 1 + \frac{2}{(n-\sigma)_+}$$

for the small initial data and low space dimensions $2 \leq n \leq 4$ by using the energy estimates. Here we denote $(r)_+ := \max\{r, 0\}$ as its positive part for any $r \in \mathbb{R}$. Afterwards, D'Abbicco and Ebert [2] extended their global existence results to higher space dimensions by using $L^r - L^q$ estimates for solutions to the corresponding linear equation. On the other hand, the authors indicated in [7] the non-existence of global (in time) solutions, just when $\sigma = 1$, if the condition

$$p \le p_0(n, 1) = 1 + \frac{2}{n-1}$$

holds by using the standard test function method via the non-negativity of the fundamental solution (see also [6]). In these cited papers, one should recognizes that the assumptions

$$u_0 = 0 \quad \text{and} \quad u_1 \ge 0$$

come to guarantee the non-negativity of the fundamental solution, which cannot be expected for any $\sigma \in (0,1]$. Quite recently, the global non-existence result for any $\sigma \in (0,1]$ has been completed by Dao and Reissig [10] when $p \le p_0(n,\sigma)$ and for all $n \ge 1$ by using a modified test function which deals with sign-changing data condition, namely

$$u_0 = 0 \quad \text{and} \quad u_1 \in L^1 \text{ satisfying } \int_{\mathbb{R}^n} u_1(x)dx > 0.$$

Again, we can see that assuming the first data $u_0 = 0$ is necessary to require. It seems that the previous used approaches do not work so well if we assume L^1 regularity for u_0 with no need of any additional sign condition for u_0. For the non-effective case $\sigma \in (1,2]$, the global existence results were also shown by [7] only for $p > 1 + (1+\sigma)/(n-1)$ with $n \ge 2$, while the blow-up of solutions has been obtained by [10] when $p \le 1 + 2/(n-1)$. Unfortunately, there appears a gap between the two exponents $1 + (1+\sigma)/(n-1)$ and $1 + 2/(n-1)$. This is naturally due to the hyperbolic like structure of the problem which seems not suitable to use the test function method in the proof of blow-up results.

Let us come back our interest to consider the structurally damped wave equation with the power nonlinearity of derivative type (1). At present, there do not seem to be so many related manuscripts. D'Abbicco and Ebert [4] proved the global (in time) existence of small data solutions for any

$$p > p_1(n,\sigma) := 1 + \frac{\sigma}{n}$$

in the case of $\sigma \in (0,1)$ and lower space dimensions, as well as for any

$$p > p_1(n,1) = 1 + \frac{1}{n}$$

in the case of $\sigma = 1$ and all $n \ge 1$. For the purpose of looking for the global (in time) existence of small data Sobolev solutions to (1), with $\sigma \in (0,1)$, from suitable function spaces basing on L^q spaces, with $q \in (1,\infty)$, we address the interested readers to the new papers of Dao and Reissig [8]. When $\sigma \in (1,2]$, the only global existence results known up to our knowledge can be found in [9] for any $p > \bar{p}$, where $\bar{p}$ are a suitable exponent, under small initial data in Sobolev space. From these observations, it still keeps an open problem so far to indicate a non-existence result for (1) in all cases $\sigma \in (0,2]$.

For this reason, our main motivation of this paper is to fill this lack. Especially, we would like to face up to dealing with the fractional Laplacian $(-\Delta)^{\sigma/2}$, the well-known nonlocal operators, where σ is supposed to be a fractional number in $(0,2)$. As we can see, this case was not included in [4] since the standard test function method seems difficult to be directly applied to these fractional Laplacian. To overcome this difficulty, the application of a new modified test function developed by Dao and Fino in the recent work [5], and mentioned in [1], comes into play. Moreover, as analyzed above, we want to point out that it is challenging to follow the recent papers ([7, 10]) in terms of the treatment of $u_0 \ne 0$. Hence, the other point worthy of noticing in the present paper is that our method can be applicable effectively to relax the limitation of the assumption for $u_0 = 0$, which plays an important role in the proofs of blow-up results in severval previous literatures (see, for example, [4, 6, 7, 10]).

Notations

- We denote the constant $\bar{\sigma} := \min\{\sigma, 1\}$, where $\sigma \in (0,2]$.
- For later convenience, C and C_i with $i \in \mathbb{Z}$ stand for suitable positive constants.
- For given nonnegative f and g, we write $f \lesssim g$ if $f \le Cg$. We write $f \approx g$ if $g \lesssim f \lesssim g$.

Our main result reads as follows.

Theorem 1 (Blow-up). Let $\sigma \in (0,2]$. We assume that

$$(u_0, u_1) \in \left(L^1(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)\right) \times \left(L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\right)$$

satisfying the following condition:

$$\int_{\mathbb{R}^n} u_1(x)dx > 0. \tag{2}$$

If

$$p \in \left(1, 1 + \frac{\bar{\sigma}}{n}\right], \tag{3}$$

then, there is no global (in time) weak solution to (1).

Remark 1. From the statement of Theorem 1, we want to underline that no need of any additional sign assumption for u_0 brings a new contribution in comparison with the previous studies [4, 6, 7, 10].

Remark 2. By Theorem 1 and Theorem 7 in [4], it is clear that $p_c := 1 + \sigma/n$ is the critical exponent of (1) when $\sigma \in (0, 1]$. It is still an open problem whether $1 + 1/n$ is the critical exponent of (1) when $\sigma \in (1, 2]$.

2 | PRELIMINARIES

In this section, we collect some preliminary knowledge needed in our proofs.

Definition 1. [11] Let $s \in (0, 1)$. Let X be a suitable set of functions defined on $\mathbb{R}^n$. Then, the fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$ is a non-local operator given by

$$(-\Delta)^s : \ v \in X \to (-\Delta)^s v(x) := C_{n,s}\ p.v. \int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x - y|^{n+2s}} dy$$

as long as the right-hand side exists, where $p.v.$ stands for Cauchy's principal value, $C_{n,s} := \dfrac{4^s\Gamma(\frac{n}{2} + s)}{\pi^{\frac{n}{2}}\Gamma(-s)}$ is a normalization constant and Γ denotes the Gamma function.

Lemma 1. [5, Lemma 2.3] Let $\langle x\rangle := (1+(|x|-1)^4)^{1/4}$ for all $x \in \mathbb{R}^n$. Let $s \in (0, 1]$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ be a function defined by

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ \langle x\rangle^{-n-2s} & \text{if } |x| \ge 1. \end{cases} \tag{4}$$

Then, $\phi \in C^2(\mathbb{R}^n)$ and the following estimate holds:

$$|(-\Delta)^s\phi(x)| \lesssim \phi(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{5}$$

Lemma 2. [10, Lemma 2.4] Let $s \in (0, 1)$. Let ψ be a smooth function satisfying $\partial_x^2\psi \in L^\infty(\mathbb{R}^n)$. For any $R > 0$, let ψ_R be a function defined by

$$\psi_R(x) := \psi(x/R) \quad \text{for all } x \in \mathbb{R}^n.$$

Then, $(-\Delta)^s(\psi_R)$ satisfies the following scaling properties:

$$(-\Delta)^s(\psi_R)(x) = R^{-2s}((-\Delta)^s\psi)(x/R) \quad \text{for all } x \in \mathbb{R}^n.$$

Using Lemmas 1 and 2, or directly from [5, Lemma 2.5], we may conclude easily the following result.

Lemma 3. Let $s \in (0, 1]$, $R > 0$ and $p > 1$. Then, the following estimate holds

$$\int_{\mathbb{R}^n} (\phi_R(x))^{-\frac{1}{p-1}} \left|(-\Delta)^s\phi_R(x)\right|^{\frac{p}{p-1}} dx \lesssim R^{-\frac{2sp}{p-1}+n},$$

where $\phi_R(x) := \phi(x/R)$ and ϕ is given in (4).

3 | PROOF OF THE MAIN RESULT

Before starting our proof, we define weak solutions for (1).

Definition 2. Let $T > 0$, $p > 1$, and $(u_0, u_1) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. A function u is said to be a global weak solution to (1) if

$$u \in L^1_{loc}\big((0,\infty), L^2(\mathbb{R}^n)\big)$$

satisfying

$$u_t \in L^p_{loc}\big((0,\infty), L^{2p}(\mathbb{R}^n)\big) \cap L^1_{loc}\big((0,\infty), L^2(\mathbb{R}^n)\big),$$

and the following formulation holds

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} |u_t|^p \varphi(t,x)\, dx\, dt + \int_{\mathbb{R}^n} u_1(x)\varphi(0,x)\, dx \\
&\quad = -\int_0^\infty \int_{\mathbb{R}^n} u_t(t,x)\varphi_t(t,x)\, dx\, dt \\
&\quad\quad - \int_0^\infty \int_{\mathbb{R}^n} u(t,x)\, \Delta\varphi(t,x)\, dx\, dt \\
&\quad\quad + \mu \int_0^\infty \int_{\mathbb{R}^n} u_t(t,x)\, (-\Delta)^{\sigma/2}\varphi(t,x)\, dx\, dt
\end{aligned}$$

for any test function $\varphi \in \mathcal{C}\big([0,\infty); H^2(\mathbb{R}^n)\big) \cap \mathcal{C}^1\big([0,\infty); L^2(\mathbb{R}^n)\big)$ such that its support in time is compact.

Proof of Theorem 1. First, we introduce the function $\phi = \phi(x)$ as defined in (4) with $s = \sigma/2$ and the function $\eta = \eta(t)$ having the following properties:

$$\begin{aligned}
&1. \quad \eta \in \mathcal{C}_0^\infty([0,\infty)) \text{ and } \eta(t) = \begin{cases} 1 & \text{if} \quad 0 \le t \le 1/2, \\ \text{decreasing} & \text{if} \quad 1/2 \le t \le 1, \\ 0 & \text{if} \quad t \ge 1, \end{cases} \\
&2. \quad \eta^{-\frac{1}{p}}(t)\, |\eta'(t)| \le C \qquad \text{for any } t \in [1/2, 1].
\end{aligned} \tag{6}$$

For the existence of such function, see e.g. [12, Chapter 1]. Let R be a large parameter in $[0,\infty)$. We define the following test function:

$$\varphi_R(t,x) := \eta_R(t)\phi_R(x),$$

where

$$\eta_R(t) := \eta\big(R^{-\bar{\sigma}} t\big) \quad \text{and} \quad \phi_R(x) := \phi\big(R^{-1}K^{-1}x\big)$$

for some $K \ge 1$ which will be fixed later. Moreover, we introduce the function

$$\Psi_R(t) = \int_t^\infty \eta_R(\tau) d\tau \quad \text{for all } t \ge 0.$$

Because of $\text{supp}\eta_R \subset \big[0, R^{\bar{\sigma}}\big]$, it follows $\text{supp}\Psi_R \subset \big[0, R^{\bar{\sigma}}\big]$. Here we also notice that the relation $\Psi'_R(t) = -\eta_R(t)$ holds. We define the functionals

$$\begin{aligned}
I_R &:= \int_0^\infty \int_{\mathbb{R}^n} |u_t(t,x)|^p \varphi_R(t,x)\, dxdt \\
&= \int_0^{R^{\bar{\sigma}}} \int_{\mathbb{R}^n} |u_t(t,x)|^p \varphi_R(t,x)\, dxdt,
\end{aligned}$$

and

$$I_{R,t} := \int_{R^{\bar\sigma}/2}^{R^{\bar\sigma}} \int_{\mathbb{R}^n} |u_t(t,x)|^p \varphi_R(t,x)\,dxdt,$$

$$I_{R,x} := \int_0^{R^{\bar\sigma}} \int_{|x|\geq RK} |u_t(t,x)|^p \varphi_R(t,x)\,dxdt.$$

Let us assume that $u = u(t,x)$ is a global weak solution to (1) according to Definition 2, then

$$\begin{aligned} I_R + \int_{\mathbb{R}^n} u_1(x)\phi_R(x)\,dx = &- \int_{R^{\bar\sigma}/2}^{R^{\bar\sigma}} \int_{\mathbb{R}^n} u_t(t,x)\eta'_R(t)\phi_R(x)\,dx\,dt \\ &+ \int_0^{R^{\bar\sigma}} \int_{|x|\geq RK} u(t,x)\,\Psi'_R(t)\Delta\phi_R(x)\,dx\,dt \\ &+ \mu \int_0^{R^{\bar\sigma}} \int_{\mathbb{R}^n} u_t(t,x)\,\eta_R(t)(-\Delta)^{\sigma/2}\phi_R(x)\,dx\,dt. \end{aligned}$$

Using integrating by parts, we conclude that

$$\begin{aligned} I_R &+ \int_{\mathbb{R}^n} u_1(x)\phi_R(x)\,dx + \int_{\mathbb{R}^n} u_0(x)\Psi_R(0)\Delta\phi_R(x)\,dx \\ &= - \int_{R^{\bar\sigma}/2}^{R^{\bar\sigma}} \int_{\mathbb{R}^n} u_t(t,x)\eta'_R(t)\phi_R(x)\,dx\,dt \\ &\quad - \int_0^{R^{\bar\sigma}} \int_{|x|\geq RK} u_t(t,x)\,\Psi_R(t)\Delta\phi_R(x)\,dx\,dt \\ &\quad + \mu \int_0^{R^{\bar\sigma}} \int_{\mathbb{R}^n} u_t(t,x)\,\eta_R(t)(-\Delta)^{\sigma/2}\phi_R(x)\,dx\,dt \\ &=: -J_1 - J_2 + J_3. \end{aligned} \tag{7}$$

Applying Hölder's inequality with $\frac{1}{p} + \frac{1}{p'} = 1$ we may estimate J_1 as follows:

$$\begin{aligned} |J_1| &\leq \int_{R^{\bar\sigma}/2}^{R^{\bar\sigma}} \int_{\mathbb{R}^n} |u_t(t,x)| \left|\eta'_R(t)\right| \phi_R(x)\,dx\,dt \\ &\lesssim \Big(\int_{R^{\bar\sigma}/2}^{R^{\bar\sigma}} \int_{\mathbb{R}^n} \left|u_t(t,x)\varphi_R^{\frac{1}{p}}(t,x)\right|^p dx\,dt \Big)^{\frac{1}{p}} \Big(\int_{R^{\bar\sigma}/2}^{R^{\bar\sigma}} \int_{\mathbb{R}^n} \left|\varphi_R^{-\frac{1}{p}}(t,x)\eta'_R(t)\phi_R(x)\right|^{p'} dx\,dt \Big)^{\frac{1}{p'}} \\ &\lesssim I_{R,t}^{\frac{1}{p}} \Big(\int_{R^{\bar\sigma}/2}^{R^{\bar\sigma}} \int_{\mathbb{R}^n} \eta_R^{-\frac{p'}{p}}(t) \left|\eta'_R(t)\right|^{p'} \phi_R(x)\,dx\,dt \Big)^{\frac{1}{p'}}. \end{aligned}$$

By the change of variables $\tilde{t} := R^{-\bar{\sigma}}t$ and $\tilde{x} := R^{-1}K^{-1}x$, a straight-forward calculation gives

$$|J_1| \lesssim I_{R,t}^{\frac{1}{p}}\, R^{-\bar{\sigma}+\frac{n+\bar{\sigma}}{p'}}\, K^{\frac{n}{p'}} \Big(\int_{\mathbb{R}^n} \langle \tilde{x} \rangle^{-n-\sigma}\, d\tilde{x} \Big)^{\frac{1}{p'}}$$
$$\lesssim I_{R,t}^{\frac{1}{p}}\, R^{-\bar{\sigma}+\frac{n+\bar{\sigma}}{p'}}\, K^{\frac{n}{p'}}. \tag{8}$$

Here we used $\eta'_R(t) = R^{-\bar{\sigma}}\eta'(\tilde{t})$ and the assumption (6). Now let us turn to estimate J_2 and J_3. Applying Hölder's inequality again as we estimated J_1 leads to

$$|J_2| \le I_{R,x}^{\frac{1}{p}} \Big(\int_0^{R^{\bar{\sigma}}} \int_{|x|\ge RK} \Psi_R^{p'}(t)\eta_R^{-\frac{p'}{p}}(t)\phi_R^{-\frac{p'}{p}}(x) \left|\Delta\phi_R(x)\right|^{p'} dx\, dt \Big)^{\frac{1}{p'}},$$

and

$$|J_3| \le I_R^{\frac{1}{p}} \Big(\int_0^{R^{\bar{\sigma}}} \int_{\mathbb{R}^n} \eta_R(t)\phi_R^{-\frac{p'}{p}}(x) \left|(-\Delta)^{\sigma/2}\phi_R(x)\right|^{p'} dx\, dt \Big)^{\frac{1}{p'}}.$$

In order to control J_2, we derive the following estimate:

$$\begin{aligned}
\eta_R^{-\frac{p'}{p}}(t)\Psi_R^{p'}(t) &= \eta_R^{-\frac{p'}{p}}(t)\Big(\int_t^{\infty} \eta_R(\tau)d\tau \Big)^{p'} \\
&= \eta_R^{-\frac{p'}{p}}(t)\Big(\int_t^{R^{\bar{\sigma}}} \eta_R(\tau)d\tau \Big)^{p'} \\
&\le \eta_R^{-\frac{p'}{p}}(t)\eta_R(t)(R^{\bar{\sigma}} - t)^{p'} \\
&\le R^{\bar{\sigma}p'}\eta_R(t) \\
&\le R^{\bar{\sigma}p'},
\end{aligned}$$

where we have used the fact that η_R is a non-increasing function satisfying $\eta_R \le 1$. Then, carrying out the change of variables $\tilde{t} := R^{-\bar{\sigma}}t$, $\tilde{x} := R^{-1}K^{-1}x$ and Lemma 3 with $s = 1$ we arrive at

$$|J_2| \lesssim I_{R,x}^{\frac{1}{p}}\, R^{-2+\bar{\sigma}+\frac{n+\bar{\sigma}}{p'}}\, K^{-2+\frac{n}{p'}}. \tag{9}$$

Next carrying out again the change of variables $\tilde{t} := R^{-\bar{\sigma}}t$ and $\tilde{x} := R^{-1}K^{-1}x$ and employing Lemma 2, then Lemma 3, with $s = \sigma/2$, we can proceed J_3 as follows:

$$|J_3| \lesssim I_R^{\frac{1}{p}}\, R^{-\sigma+\frac{n+\bar{\sigma}}{p'}}\, K^{-\sigma+\frac{n}{p'}}. \tag{10}$$

Combining the estimates from (7) to (10) we may arrive at

$$\begin{aligned}
&I_R + \int_{\mathbb{R}^n} u_1(x)\phi_R(x)\, dx \\
&\le C_0\Big(I_{R,t}^{\frac{1}{p}} R^{-\bar{\sigma}+\frac{n+\bar{\sigma}}{p'}} K^{\frac{n}{p'}} + I_{R,x}^{\frac{1}{p}} R^{-2+\bar{\sigma}+\frac{n+\bar{\sigma}}{p'}} K^{-2+\frac{n}{p'}} + I_R^{\frac{1}{p}} R^{-\sigma+\frac{n+\bar{\sigma}}{p'}} K^{-\sigma+\frac{n}{p'}} \Big) + \int_{\mathbb{R}^n} |u_0(x)|\Psi_R(0)|\Delta\phi_R(x)|\, dx.
\end{aligned}$$

Moreover, it is clear that $\Psi_R(0) \le R^{\bar{\sigma}}$. By the change of variables, using Lemma 1 we can easily check that

$$|\Delta\phi_R(x)| \le R^{-2}\phi_R(x).$$

Therefore, this implies that

$$\begin{aligned}
&I_R + \int_{\mathbb{R}^n} u_1(x)\phi_R(x)\, dx \\
&\le C_0\Big(I_{R,t}^{\frac{1}{p}} R^{-\bar{\sigma}+\frac{n+\bar{\sigma}}{p'}} K^{\frac{n}{p'}} + I_{R,x}^{\frac{1}{p}} R^{-2+\bar{\sigma}+\frac{n+\bar{\sigma}}{p'}} K^{-2+\frac{n}{p'}} + I_R^{\frac{1}{p}} R^{-\sigma+\frac{n+\bar{\sigma}}{p'}} K^{-\sigma+\frac{n}{p'}} \Big) + R^{\bar{\sigma}-2} \int_{\mathbb{R}^n} |u_0(x)|\phi_R(x)\, dx.
\end{aligned} \tag{11}$$

Because of the assumption (2), there exists a sufficiently large constant $R_1 > 0$ such that it holds

$$\int_{\mathbb{R}^n} u_1(x)\phi_R(x)\,dx > 0 \tag{12}$$

for all $R > R_1$. Since $u_0 \in L^1$, it implies immediately that

$$R^{\bar{\sigma}-2}\int_{\mathbb{R}^n} |u_0(x)|\phi_R(x)\,dx \to 0 \quad \text{as } R \to \infty.$$

Hence, from (12) there exists there exists a sufficiently large constant $R_2 > 0$ such that

$$R^{\bar{\sigma}-2}\int_{\mathbb{R}^n} |u_0(x)|\phi_R(x)\,dx < \frac{1}{2}\int_{\mathbb{R}^n} u_1(x)\phi_R(x)\,dx$$

for all $R > R_2$. Now we choose $R_0 := \max\{R_1,\ R_2\}$. Then, from (11) we have

$$\begin{aligned} &I_R + \frac{1}{2}\int_{\mathbb{R}^n} u_1(x)\phi_R(x)\,dx \\ &\le C_0\Big(I_{R,t}^{\frac{1}{p}} R^{-\bar{\sigma}+\frac{n+\bar{\sigma}}{p'}} K^{\frac{n}{p'}} + I_{R,x}^{\frac{1}{p}} R^{-2+\bar{\sigma}+\frac{n+\bar{\sigma}}{p'}} K^{-2+\frac{n}{p'}} + I_R^{\frac{1}{p}} R^{-\sigma+\frac{n+\bar{\sigma}}{p'}} K^{-\sigma+\frac{n}{p'}}\Big) \end{aligned} \tag{13}$$

for all $R > R_0$. By choosing $K = 1$ and noticing the relations $I_{R,t} \le I_R$ and $I_{R,x} \le I_R$ we may arrive, particularly, at

$$I_R + \frac{1}{2}\int_{\mathbb{R}^n} u_1(x)\phi_R(x)\,dx \le C_0\, I_R^{\frac{1}{p}}\, R^{-\bar{\sigma}+\frac{n+\bar{\sigma}}{p'}} \tag{14}$$

for all $R > R_0$. Thanks to the following ε-Young's inequality:

$$ab \le \varepsilon a^p + C(\varepsilon) b^{p'} \quad \text{for all } a, b > 0 \text{ and for any } \varepsilon > 0,$$

we conclude

$$C_0\, I_R^{\frac{1}{p}}\, R^{-\bar{\sigma}+\frac{n+\bar{\sigma}}{p'}} \le \frac{1}{2} I_R + C_1\, R^{-\bar{\sigma}p'+n+\bar{\sigma}}.$$

Consequently, from (14) we derive

$$\frac{1}{2} I_R + \frac{1}{2}\int_{\mathbb{R}^n} u_1(x)\phi_R(x)\,dx \le C_1\, R^{-\bar{\sigma}p'+n+\bar{\sigma}},$$

which follows that

$$I_R \le 2C_1\, R^{-\bar{\sigma}p'+n+\bar{\sigma}}, \tag{15}$$

$$\int_{\mathbb{R}^n} u_1(x)\phi_R(x)\,dx \le 2C_1\, R^{-\bar{\sigma}p'+n+\bar{\sigma}}, \tag{16}$$

for all $R > R_0$. It is clear that the assumption (3) is equivalent to

$$-\bar{\sigma}p' + n + \bar{\sigma} \le 0.$$

For this reason, let us now separate our considerations into two cases as follows.

Case 1: $-\bar{\sigma}p' + n + \bar{\sigma} < 0$, i.e. the subcritical case. Letting $R \to \infty$ in (16) we infer a contradiction to (2).

Case 2: $-\bar{\sigma}p' + n + \bar{\sigma} = 0$, i.e. the critical case. Then, we can see from (15) that $I_R \le 2C_1$ for all $R > R_0$. Using Beppo Levi's theorem on monotone convergence, on the one hand, we derive

$$\int_0^\infty \int_{\mathbb{R}^n} |u_t(t,x)|^p\,dx\,dt = \lim_{R\to\infty} \int_0^{R^{\bar{\sigma}}} \int_{\mathbb{R}^n} |u_t(t,x)|^p\,\varphi_R(t,x)\,dx\,dt = \lim_{R\to\infty} I_R \le 2C_1,$$

that is, $u_t \in L^p((0,\infty)\times\mathbb{R}^n)$. By the absolute continuity of the Lebesgue integral, it follows that

$$I_{R,t} \to 0 \quad \text{and} \quad I_{R,x} \to 0 \quad \text{as } R \to \infty.$$

On the other hand, using again the fact that $p = 1 + \bar{\sigma}/n$ we obtain from (13) the following estimate:

$$I_R + \frac{1}{2}\int_{\mathbb{R}^n} u_1(x)\phi_R(x)\,dx \le C_0\Big(I_{R,t}^{\frac{1}{p}} K^{\frac{n}{p'}} + I_{R,x}^{\frac{1}{p}} R^{-2+2\bar{\sigma}} K^{-2+\frac{n}{p'}} + I_R^{\frac{1}{p}} R^{-\sigma+\bar{\sigma}} K^{-\sigma+\frac{n}{p'}}\Big) \tag{17}$$

for all $K \ge 1$ and all $R > R_0$.

- If $\sigma \in (0,1]$, then $\bar{\sigma} = \sigma$. As a consequence, from (17) we have

$$I_R + \frac{1}{2}\int_{\mathbb{R}^n} u_1(x)\phi_R(x)\,dx \le C_0\Big(I_{R,t}^{\frac{1}{p}} K^{\frac{n}{p'}} + I_{R,x}^{\frac{1}{p}} R^{-2(1-\sigma)} K^{-2+\frac{n}{p'}} + I_R^{\frac{1}{p}} K^{-\sigma+\frac{n}{p'}}\Big) \tag{18}$$

  for all $K \ge 1$ and all $R > R_0$. Letting $R \to \infty$ in (18) we get

$$\int_{\mathbb{R}^n} u_1(x)dx \lesssim K^{-\sigma+\frac{n}{p'}} \quad \text{for all } K \ge 1. \tag{19}$$

  Due to $p = 1 + \bar{\sigma}/n = 1 + \sigma/n$, it is clear that $-\sigma + n/p' = -\sigma/p' < 0$. Therefore, we can fix a sufficiently large constant $K \ge 1$ in (19) to gain a contradiction to (2).

- If $\sigma \in (1,2]$, then $\bar{\sigma} = 1$. As a result, choosing $K = 1$ we may conclude from (17) that

$$\int_{\mathbb{R}^n} u_1(x)\phi_R(x)\,dx \le 2C_0\Big(I_{R,t}^{\frac{1}{p}} + I_{R,x}^{\frac{1}{p}} + I_R^{\frac{1}{p}} R^{1-\sigma}\Big) \tag{20}$$

  for all $R > R_0$. Since $\sigma > 1$, letting $R \to \infty$ in (20) we obtain a contradiction to (2) again.

Summarizing, the proof Theorem 1 is completed. □

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