On Conformable Double Sumudu Transform

Suliman Alfaqeih, Gizel BAKICIERLER, Emine Misirli
Ege University, Faculty of Science
Department of Mathematics, 35100
Bornova – İzmir – TURKEY
alfaqeihsuliman@gmail.com
gizelbakicierler@gmail.com
emine.misirli@ege.edu.tr

Abstract

In the current work, we have introduced a new deformation and generalization of the Sumudu transform called the Conformable Double Sumudu Transform (CDST), which can be applied to solve conformable fractional partial differential equations (CFPDEs). Further, we proposed some fundamental properties of the (CDST). To show the efficiency, applicability, high accuracy, and simplicity of the proposed transform, we apply this new transform to solve a wide class of linear fractional partial differential equations in the sense of Conformable fractional derivative (CFD).

Keywords

Conformable fractional derivative (CFD), Partial differential equation (PDE), Telegraph equation, Mittag-Leffler function, Sumudu transform (ST), Laplace transforms (LT)

1 Introduction

In current years, fractional partial differential equations (FPDEs) have found to be very important for modeling many applications in real life sciences and engineering such as fluid dynamics, mathematical biology, electrical circuits, optics, quantum mechanics, etc. [9, 16, 12]. In literature, many definitions of fractional derivatives and integrals have been stated such as Rizez, Weyl, Riemann-Liouville, Caputo, Hadamard and so on. These types of fractional derivatives have many unusual properties such as not all obey chain rule, product, and quotient rule of two functions, these important properties lead to some flaws in applications in physics and engineering. In 2014, Khalil et. al [8] defined a new type of fractional derivative called the Conformable fractional derivative (CFD) which satisfies the classical properties of the known derivative. During the last few years, authors have been developing many analytical and numerical methods to obtain the solutions of conformable fractional partial differential equations, such as tanh method [19], reliable methods [10], exponential rational function method [7], Kudryashov method [11], simplest equation method (SEM) [3], single conformable Laplace transform method (CLT) [6], conformable double Laplace transform (CDLT) [15]. Sumudu transform (ST), which derived from the classical Fourier transform, (ST) was first introduced by Watugala in 1993 [5] and has been implemented to obtain the solution of many problems in real life science and engineering. (ST) proved to be an efficient method for solving physical problems because of its unit and scale preserving properties. For more about (ST) see [13, 4, 17, 1]. Recently, in 2019, the single Conformable Sumudu transform (CST) was introduced by Kamyar Hosseini [20] and coupled this transform with invariant subspace method (ISM) to solve nonlinear conformable fractional dispersive equation of the fifth order. Due to the some advantage of (ST) over the (LT), we introduce the (CDST) to solve linear fractional differential equations in the conformable fractional derivative sense.

2 Preliminaries

In this section, we present some basic notations about the conformable fractional derivative (CFD).

[2] The (CFD) of a function $\chi : (0, \infty) \to \mathbb{R}$ of order ν is given by:

$$D_x^{\nu} \chi \left(\frac{x^{\nu}}{\nu} \right) = \lim_{\varepsilon \to 0} \frac{\chi \left(\frac{x^{\nu}}{\nu} + \varepsilon x^{1-\nu} \right) - \chi \left(\frac{x^{\nu}}{\nu} \right)}{\varepsilon}, \frac{x^{\nu}}{\nu} > 0, 0 < \nu \leq 1.$$

[18] The (CFPD) of a function $\varphi \left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta} \right) : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ of order ν is defined by:

$$D_x^{\nu} \varphi \left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta} \right) = \lim_{\vartheta \to 0} \frac{\varphi \left(\frac{x^{\nu}}{\nu} + \vartheta x^{1-\nu}, \frac{\tau^{\beta}}{\beta} \right) - \varphi \left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta} \right)}{\vartheta}, \frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta} > 0.$$

[18] The (CFPD) of a function $\varphi \left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta} \right) : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ of order β is defined by:

$$D_{\tau}^{\beta} \varphi \left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta} \right) = \lim_{\gamma \to 0} \frac{\varphi \left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta} + \gamma \tau^{1-\beta}, \right) - \varphi \left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta} \right)}{\gamma}, \ 0 < \nu, \beta \leqslant 1, \ \frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta} > 0.$$

Let $0 < \nu, \beta \leq 1$ and $\varphi (x, \tau)$ be ν and β- differentiable at a point x, t > 0, then

$$D_x^{\nu} \varphi (x, \tau) = x^{1-\nu} \frac{\partial \varphi (x, \tau)}{\partial x},$$
$$D_{\tau}^{\beta} \varphi (x, \tau) = \tau^{1-\beta} \frac{\partial \varphi (x, \tau)}{\partial \tau}.$$

2.1 Conformable fractional derivatives (CFPD) of some functions:

We have the following

$$D_x^{\nu}(c)=0,\ D_{\tau}^{\beta}(c)=0, c \text{ is constant.}$$

$$D_x^{\nu}\left(a\left(\frac{x^{\nu}}{\nu}\right)^{n}\left(\frac{\tau^{\beta}}{\beta}\right)^{m}\right)=na\left(\frac{x^{\nu}}{\nu}\right)^{n-\nu}\left(\frac{\tau^{\beta}}{\beta}\right)^{m},$$

$$D_{\tau}^{\beta}\left(a\left(\frac{x^{\nu}}{\nu}\right)^{n}\left(\frac{\tau^{\beta}}{\beta}\right)^{m}\right)=ma\left(\frac{x^{\nu}}{\nu}\right)^{r}\left(\frac{\tau^{\beta}}{\beta}\right)^{m-\beta}, \forall\ a,m,\mathrm{n}\in\mathbb{R}$$

$$D_x^{\nu}\left(e^{c\left(\frac{x^{\nu}}{\nu}\right)+d\left(\frac{\tau^{\beta}}{\beta}\right)}\right)=ce^{c\left(\frac{x^{\nu}}{\nu}\right)+d\left(\frac{\tau^{\beta}}{\beta}\right)},\ D_{\tau}^{\beta}\left(e^{c\left(\frac{x^{\nu}}{\nu}\right)+d\left(\frac{\tau^{\beta}}{\beta}\right)}\right)=de^{c\left(\frac{x^{\nu}}{\nu}\right)+d\left(\frac{\tau^{\beta}}{\beta}\right)}, \forall\ \mathrm{c},\ \mathrm{d}\in\mathbb{R}$$

$$D_x^{\nu}\left(e^{c\left(\frac{x^{\nu}}{\nu}\right)+d\left(\frac{\tau^{\beta}}{\beta}\right)}\right)=ce^{c\left(\frac{x^{\nu}}{\nu}\right)+d\left(\frac{\tau^{\beta}}{\beta}\right)}, D_{\tau}^{\beta}\left(e^{c\left(\frac{x^{\nu}}{\nu}\right)+d\left(\frac{\tau^{\beta}}{\beta}\right)}\right)=de^{c\left(\frac{x^{\nu}}{\nu}\right)+d\left(\frac{\tau^{\beta}}{\beta}\right)}, \forall\ \mathrm{c},\ \mathrm{d}\in\mathbb{R}$$

$$D_x^{\nu}\left(\sin\left(c\left(\frac{x^{\nu}}{\nu}\right)\right)\sin\left(d\left(\frac{\tau^{\beta}}{\beta}\right)\right)\right)=c.\cos\left(c\left(\frac{x^{\nu}}{\nu}\right)\right)\sin\left(d\left(\frac{\tau^{\beta}}{\beta}\right)\right),$$

$$D_{\tau}^{\beta}\left(\sin\left(c\left(\frac{x^{\nu}}{\nu}\right)\right)\sin\left(d\left(\frac{\tau^{\beta}}{\beta}\right)\right)\right)=d\sin\left(c\left(\frac{x^{\nu}}{\nu}\right)\right)\cos\left(\lambda\left(\frac{\tau^{\beta}}{\beta}\right)\right), \forall\ \mathrm{c},\ \mathrm{d}\in\mathbb{R}$$

$$D_x^{\nu}\left(\cos\left(c\left(\frac{x^{\nu}}{\nu}\right)\right)\cos\left(d\left(\frac{\tau^{\beta}}{\beta}\right)\right)\right)=-c.\sin\left(c\left(\frac{x^{\nu}}{\nu}\right)\right)\sin\left(d\left(\frac{\tau^{\beta}}{\beta}\right)\right),$$

$$D_{\tau}^{\beta}\left(\cos\left(c\left(\frac{x^{\nu}}{\nu}\right)\right)\cos\left(d\left(\frac{\tau^{\beta}}{\beta}\right)\right)\right)=-d\cos\left(c\left(\frac{x^{\nu}}{\nu}\right)\right)\sin\left(d\left(\frac{\tau^{\beta}}{\beta}\right)\right), \forall\ \mathrm{c},\ \mathrm{d}\in\mathbb{R}$$

3 Conformable Double Sumudu Transform (CDST)

In this section, we introduce the definition of (CDST).

The (CDST) of a piecewise continuous function $\varphi:[0,\infty)\times[0,\infty)\to\mathbb{R}$ of exponential order is defined on the set;

$$\Omega=\left\{\varphi(x,\tau):\exists\ \lambda_1,\lambda_2>0,\ |\varphi(x,\tau)|<K\exp\left(\frac{\left(\frac{x^{\nu}}{\nu}\right)+\left(\frac{\tau^{\beta}}{\beta}\right)}{\lambda_j^2}\right),\ j=1,\ 2 \text{ and } (x,\tau)\in\mathbb{R}_+^2\right\},$$

by the following integral

$$\bar{\varphi}(p,q)=S_x^{\nu}S_{\tau}^{\beta}\left(\varphi(x,\tau):(p,q)\right)=\int\limits_0^{\infty}\int\limits_0^{\infty}e^{-\left(\frac{x^{\nu}}{\nu}+\frac{\tau^{\beta}}{\beta}\right)}\varphi(px,q\tau)\,x^{\nu-1}\tau^{\beta-1}dxd\tau,$$

where $x>0,\ \tau>0$, and $p,\ q$ are the transform variables of x and τ respectively.

The single Conformable Sumudu transforms (CST) of a real valued function $\varphi:[0,\infty)\times[0,\infty)\to\mathbb{R}$ with respect to the variables x and τ respectively, are defined by

$$\bar{\varphi}(p) = S_x^{\nu}\left(\varphi\left(x,\tau\right):p\right) = \int\limits_0^{\infty} e^{-\frac{x^{\nu}}{\nu}}\varphi\left(px,\tau\right)x^{\nu-1}dx,$$

$$\bar{\varphi}(q) = S_{\tau}^{\beta}\left(\varphi\left(x,\tau\right):q\right) = \int\limits_0^{\infty} e^{-\frac{\tau^{\beta}}{\beta}}\varphi\left(x,q\tau\right)\tau^{\beta-1}d\tau.$$

Let $\varphi:[0,\infty)\times[0,\infty)\to\mathbb{R}$ be a function such that $\bar{\varphi}(p,q) = S_x^{\nu}S_{\tau}^{\beta}\left(\varphi\left(\frac{x^{\nu}}{\nu},\frac{\tau^{\beta}}{\beta}\right)\right)$ exist, then

$$S_x^{\nu}S_{\tau}^{\beta}\left(\varphi\left(\frac{x^{\nu}}{\nu},\frac{\tau^{\beta}}{\beta}\right)\right) = S_xS_{\tau}\left(\varphi(x,\tau)\right),$$

where $S_xS_{\tau}\left(\varphi(x,\tau)\right) = \int\limits_0^{\infty}\int\limits_0^{\infty} e^{-x-\tau}\varphi(px,q\tau)dxd\tau.$
By using the definition of (CDST), one can easily see the proof by letting $w = \left(\frac{x^{\nu}}{\nu}\right)$, $y = \left(\frac{\tau^{\beta}}{\beta}\right)$.

The (CDST) for certain functions is given by:

1. $S_x^{\nu}S_{\tau}^{\beta}\left(c\right) = S_xS_{\tau}\left(c\right) = c,$
2. $S_x^{\nu}S_{\tau}^{\beta}\left(\left(\frac{x^{\nu}}{\nu}\right)^r\left(\frac{\tau^{\beta}}{\beta}\right)^s\right) = S_xS_{\tau}\left[\left(x^r\tau^s\right)\right] = \Gamma(r+1)\Gamma(s+1)p^rq^s,$
3. $S_x^{\nu}S_{\tau}^{\beta}\left(e^{a\left(\frac{x^{\nu}}{\nu}\right)+b\left(\frac{\tau^{\beta}}{\beta}\right)}\right) = S_xS_{\tau}\left(e^{ax+b\tau}\right) = \frac{1}{(1-ap)(1-bq)},$
4. $S_x^{\nu}S_{\tau}^{\beta}\left(sin\left(a\frac{x^{\nu}}{\nu}\right)\sin\left(b\frac{\tau^{\beta}}{\beta}\right)\right) = \frac{abpq}{(1+(a^2p^2))(1+(b^2q^2))},$
5. $S_x^{\nu}S_{\tau}^{\beta}\left(\cos\left(a\frac{x^{\nu}}{\nu}\right)\cos\left(b\frac{\tau^{\beta}}{\beta}\right)\right) = \frac{1}{(1+(a^2p^2))(1+(b^2q^2))}.$

If $\bar{\varphi}(p,q) = S_x^{\nu}S_{\tau}^{\beta}\left(\varphi\left(\frac{x^{\nu}}{\nu},\frac{\tau^{\beta}}{\beta}\right)\right)$ exist for $p,q>0$, then for $a,b,r,s\in\mathbb{R}$, we have

$$S_x^{\nu}S_{\tau}^{\beta}\left(\varphi\left(cx,d\tau\right)\right) = \bar{\varphi}\left(cp,dq\right),$$
$$S_x^{\nu}S_{\tau}^{\beta}\left(c\varphi\left(x,\tau\right)+d\phi\left(x,\tau\right)\right) = c\bar{\varphi}\left(p,q\right)+d\bar{\phi}\left(p,q\right),$$
$$S_x^{\nu}S_{\tau}^{\beta}\left(f\left(x\right)g\left(\tau\right)\right) = \bar{f}(p)\bar{g}(q).$$

By using the definition of (CDST), one can easily show the proofs.
For $0<\nu,\ \beta\leq 1$, the (CDST) of $\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}, \frac{\partial^{\beta}\varphi}{\partial\tau^{\beta}}$ are given respectively as follows:

i. $S_x^{\nu}S_{\tau}^{\beta}\left(\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}\right) = p^{-1}\left[\bar{\varphi}\left(p,q\right)-\bar{\varphi}\left(0,q\right)\right],$

ii. $S_x^{\nu}S_{\tau}^{\beta}\left(\frac{\partial^{\beta}\varphi}{\partial\tau^{\beta}}\right) = q^{-1}\left[\bar{\varphi}\left(p,q\right)-\bar{\varphi}\left(p,0\right)\right].$

(i) Using definition of (CDST) and definition of (CFPD), we get

$$S_x^\nu S_\tau^\beta\left(\frac{\partial^\nu\varphi}{\partial x^\nu}\right)=\frac{1}{pq}\int_0^\infty\int_0^\infty e^{-\left(\frac{x^\nu}{p\nu}+\frac{\tau^\beta}{q\beta}\right)}\frac{\partial^\nu\varphi\left(x,\tau\right)}{\partial x^\nu}x^{\nu-1}\tau^{\beta-1}dxd\tau$$

using definition 4,

$$=\frac{1}{pq}\int_0^\infty\int_0^\infty e^{-\left(\frac{x^\nu}{p\nu}+\frac{\tau^\beta}{q\beta}\right)}\left(x\right)^{1-\nu}\frac{\partial\varphi\left(x,\tau\right)}{\partial x}x^{\nu-1}\tau^{\beta-1}dxd\tau$$

simplyfing,

$$=\frac{1}{pq}\int_0^\infty\int_0^\infty e^{-\left(\frac{x^\nu}{p\nu}+\frac{\tau^\beta}{q\beta}\right)}\frac{\partial\varphi\left(x,\tau\right)}{\partial x}\tau^{\beta-1}dxd\tau$$

by integration by parts,

$$\begin{aligned}&=-\frac{1}{pq}\int_0^\infty e^{-\left(\frac{\tau^\beta}{q\beta}\right)}\tau^{\beta-1}\varphi\left(0,\tau\right)+\frac{1}{pq}\frac{1}{p}\int_0^\infty\int_0^\infty e^{-\left(\frac{x^\nu}{p\nu}+\frac{\tau^\beta}{q\beta}\right)}\varphi\left(x,\tau\right)x^{\nu-1}\tau^{\beta-1}dxd\tau\\&=p^{-1}\left[\,\bar\varphi\left(p,q\right)-\bar\varphi\left(0,q\right)\right].\end{aligned}$$

Which completes the proof.

(ii) The proof is analogous to the proof of (i).

The (CDST) of the, (CFPDs) $\frac{\partial^{n\nu}\varphi}{\partial x^{n\nu}}$, $\frac{\partial^{m\beta}\varphi}{\partial\tau^{m\beta}}$, of the function φ is given by,

$$\begin{aligned}S_x^\nu S_\tau^\beta\left(\frac{\partial^{n\nu}\varphi}{\partial x^{n\nu}}\right)&=p^{-n}\left[\bar\varphi(p,q)-\sum_{i=0}^{n-1}p^iS_\tau^\beta\left(\frac{\partial^{i\nu}}{\partial x^{i\nu}}\varphi\left(0,\tau\right)\right)\right],\\S_x^\nu S_\tau^\beta\left(\frac{\partial^{m\beta}\varphi}{\partial\tau^{m\beta}}\right)&=q^{-m}\left[\bar\varphi(p,q)-\sum_{k=0}^{m-1}q^kS_x^\nu\left(\frac{\partial^{k\beta}}{\partial\tau^{k\beta}}\varphi\left(x,0\right)\right)\right].\end{aligned}$$

The proof can be done by mathematical induction.

(Conformable Double Laplace- Conformable Double Sumudu duality). If the (CDST) of a function $\varphi\left(x,\ \tau\right)$ exists, then

$$S_x^\nu S_\tau^\beta\left(\varphi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right),(p,q)\right)=\frac{1}{pq}L_x^\nu L_\tau^\beta\left(\varphi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right),\left(\frac{1}{p},\frac{1}{q}\right)\right),$$

where $L_x^\nu L_\tau^\beta\left(\varphi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right)\right)=\int_0^\infty\int_0^\infty e^{-\left(p\frac{x^\nu}{\nu}+q\frac{t^\beta}{\beta}\right)}\varphi\left(\frac{x^\nu}{\nu},\frac{\tau^\beta}{\beta}\right)x^{\nu-1}\tau^{\beta-1}dxd\tau$ denotes the (CDLT) of the function φ.

By using the definition of (CDST), one can easily show the proof. If $\bar{\varphi}(p,q) = S_x^{\nu} S_{\tau}^{\beta}\left(\varphi\left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta}\right)\right)$ and $\bar{\phi}(p,q) = S_x^{\nu} S_{\tau}^{\beta}\left(\phi\left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta}\right)\right)$ both exist for
$p > 0$ *and* $q > 0$. Then the Conformable double Sumudu transform of the convolution of φ and ϕ

$$[\varphi * * \phi](x,\ \tau) = \int_0^{\frac{x^{\nu}}{\nu}} \int_0^{\frac{\tau^{\beta}}{\beta}} \varphi\left(\frac{x^{\nu}}{\nu} - \kappa, \frac{\tau^{\beta}}{\beta} - \lambda\right) \phi(\kappa, \lambda)\, d\kappa d\lambda,$$

is given by

$$S_x^{\nu} S_{\tau}^{\beta}\left([\varphi * * \phi]\left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta}\right)\right) = pq\bar{\varphi}(p,q)\bar{\phi}(p,q).$$

Using Lemma () and Theorem 2.2 in Ref.[14] we have

$$\begin{aligned} S_x^{\nu} S_{\tau}^{\beta}\left(\left([\varphi * * \phi]\left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta}\right)\right)\right) &= \frac{1}{pq} L_x^{\nu} L_{\tau}^{\beta}\left(\left([\varphi * * \phi]\left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta}\right)\right), \left(\frac{1}{p}, \frac{1}{q}\right)\right) \\ &= \frac{1}{pq} L_x^{\nu} L_{\tau}^{\beta}\left(\varphi\left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta}\right), \left(\frac{1}{p}, \frac{1}{q}\right)\right) L_x^{\nu} L_{\tau}^{\beta}\left(\phi\left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta}\right), \left(\frac{1}{p}, \frac{1}{q}\right)\right) \\ &= pq S_x^{\nu} S_{\tau}^{\beta}\left(\varphi\left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta}\right)\right) S_x^{\nu} S_{\tau}^{\beta}\left(\phi\left(\frac{x^{\nu}}{\nu}, \frac{\tau^{\beta}}{\beta}\right)\right) \end{aligned}$$

4 Applications

In this section, the (CDSTM) is applied to obtain the solutions of a wide class of linear fractional partial differential equations involving (CFD).

Consider the conformable fractional partial Telegraph equation

$$\frac{\partial^{2\nu}\varphi}{\partial x^{2\nu}} - \frac{\partial^{2\beta}\varphi}{\partial \tau^{2\beta}} - \frac{\partial^{\beta}\varphi}{\partial \tau^{\beta}} - \varphi + \left(\frac{x^{\nu}}{\nu}\right)^2 + \left(\frac{\tau^{\beta}}{\beta}\right) - 1 = 0, \tag{1}$$

with respect to the initial and boundary conditions

$$\begin{aligned} \varphi\left(\frac{x^{\nu}}{\nu},\ 0\right) &= \left(\frac{x^{\nu}}{\nu}\right)^2, \frac{\partial^{\tau}}{\partial \tau^{\nu}}\varphi\left(\frac{x^{\nu}}{\nu},\ 0\right) = 1, \\ \varphi\left(0, \frac{\tau^{\beta}}{\beta}\right) &= \left(\frac{\tau^{\beta}}{\beta}\right), \frac{\partial^{\nu}}{\partial x^{\nu}}\varphi\left(0, \frac{\tau^{\beta}}{\beta}\right) = 0. \end{aligned} \tag{2}$$

Applying the (CDST) to both side of equation (1) and single (CLT) to initial and boundary conditions (2), we get

$$\begin{aligned} & p^{-2}\bar{\varphi}(p,q) - p^{-2}\bar{\varphi}(0,q) - p^{-1}\frac{\partial^{\nu}\bar{\varphi}(0,q)}{\partial x^{\nu}} - q^{-2}\bar{\varphi}(p,q) + q^{-2}\bar{\varphi}(p,0) \\ & + q^{-1}\frac{\partial^{\beta}\bar{\varphi}(p,0)}{\partial \tau^{\beta}} - q^{-1}\bar{\varphi}(p,q) + q^{-1}\bar{\varphi}(p,0) - \bar{\varphi}(p,q) + 2p^2 + q - 1 = 0, \end{aligned} \tag{3}$$

substituting the single (CST) of initial and boundary conditions

$$\bar{\varphi}(0,q)=2q^2,\frac{\partial^\beta\bar{\varphi}(p,0)}{\partial\tau^\beta}=1,\ \bar{\varphi}(0,q)=q\frac{\partial^\nu}{\partial x^\nu}\bar{\varphi}(0,q\ )=0,\ \bar{\varphi}(p,0)=2p^2,$$

in equation (3), we get

$$\left(p^{-2}-q^{-2}-q^{-1}-1\right)\bar{\varphi}(p,q)=\left(qp^{-2}-2q^{-2}p^2-q^{-1}-2q^{-1}p^2-2p^2-q+1\right),$$

simplifying, we obtain

$$\bar{\varphi}(p,q)=\left(2p^2+q\right),$$

taking the inverse of (CDST), we get

$$\varphi(x,\tau)=S_x^{-1}S_\tau^{-1}\left[\left(2p^2+q\right)\right]=\left(\frac{x^\nu}{\nu}\right)^2+\left(\frac{\tau^\beta}{\beta}\right),$$

for $\nu=1$ and $\beta=1$, the exact solution of (1) becomes

$$\varphi(x,\tau)=x^2+\tau.$$

Consider the conformable fractional Advection- Diffusion equation

(4)
$$\frac{\partial^\beta\varphi}{\partial\tau^\beta}=\frac{\partial^{2\nu}\varphi}{\partial x^{2\nu}}+\frac{\partial^\nu\varphi}{\partial x^\nu},\ 0<\nu,\ \beta\leq 1,$$

subject to the initial and boundary conditions:

(5)
$$\begin{aligned}&\varphi\left(\frac{x^\rho}{\rho},\ 0\right)=e^{\left(\frac{x^\nu}{\nu}\right)}-\left(\frac{x^\nu}{\nu}\right),\\&\varphi\left(0,\frac{\tau^\beta}{\beta}\right)=1+\left(\frac{\tau^\beta}{\beta}\right),\ \frac{\partial^\nu}{\partial x^\nu}\varphi\left(0,\frac{\tau^\beta}{\beta}\right)=0.\end{aligned}$$

Applying the (CDST) to both side of equation (4) and single (CLT) to initial and boundary conditions (5), we get

(6)
$$q^{-1}\left[\bar{\varphi}(p,q)-\bar{\varphi}(p,0)\right]=p^{-2}\bar{\varphi}(p,q)-p^{-2}\bar{\varphi}(0,q)-p^{-1}\frac{\partial\bar{\varphi}(0,q)}{\partial x^\nu}-p^{-1}\left[\bar{\varphi}(p,q)-\bar{\varphi}(0,q)\right],$$

substituting,

$$\bar{\varphi}(p,0)=\frac{1}{1-p}-p,\bar{\varphi}(0,q)=1+q,\ \frac{\partial\bar{\varphi}(0,q)}{\partial x^\nu}=0.$$

In equation (6), we get

$$\left(q^{-1}-p^{-2}+p^{-1}\right)\bar{\varphi}(p,q)=q^{-1}\left(\frac{1}{1-p}-p\right)-p^{-2}(1+q)+p^{-1}(1+q).$$

Consequently,

$$\varphi(x,\tau) = S_x^{-1} S_\tau^{-1}\left[\frac{q^{-1}\left(\frac{1}{1-p}-p\right)-p^{-2}(1+q)+p^{-1}(1+q)}{(q^{-1}-p^{-2}+p^{-1})}\right],$$

simplifying,

$$\varphi(x,\tau) = S_x^{-1} S_\tau^{-1}\left[\frac{1}{1-p}-p+q\right] = e^{\left(\frac{x^\nu}{\nu}\right)} - \left(\frac{x^\nu}{\nu}\right) + \left(\frac{\tau^\beta}{\beta}\right).$$

when $\nu = 1$ and $\beta = 1$ the exact solution of (4) becomes

$$\varphi(x,\tau) = e^x - x + \tau.$$

Consider the conformable fractional Klein-Gordon equation

$$\frac{\partial^{2\beta}\varphi}{\partial\tau^\beta} - \frac{\partial^{2\nu}\varphi}{\partial x^{2\nu}} - 2\varphi = -2\sin\left(\frac{x^\nu}{\nu}\right)\sin\left(\frac{\tau^\beta}{\beta}\right),\ 0 <\nu,\ \beta \leq 1 \tag{7}$$

subject to the initial and boundary conditions:

$$\begin{aligned} &\varphi\left(\frac{x^\nu}{\nu},\ 0\right) = 0, \quad \frac{\partial^\tau}{\partial\tau^\nu}\varphi\left(\frac{x^\nu}{\nu}, 0\right) = \sin\left(\frac{x^\nu}{\nu}\right), \\ &\varphi\left(0, \frac{\tau^\beta}{\beta}\right) = 0, \quad \frac{\partial^\nu}{\partial x^\nu}\varphi\left(0, \frac{\tau^\beta}{\beta}\right) = \sin\left(\frac{\tau^\beta}{\beta}\right). \end{aligned} \tag{8}$$

Applying the (CDST) to both side of equation (7) and single (CLT) to initial and boundary conditions (8), we get

$$\begin{aligned} &q^{-2}\bar{\varphi}(p,q) - q^{-2}\bar{\varphi}(p,0) - q^{-1}\frac{\partial^\beta\bar{\varphi}(p,0)}{\partial\tau^\beta} - p^{-2}\bar{\varphi}(p,q) \\ &+ p^{-2}\bar{\varphi}(0,q) + p^{-1}\frac{\partial^\nu\bar{\varphi}(0,q)}{\partial x^\nu} - 2\bar{\varphi}(p,q) = -2\frac{p}{1+p^2}\frac{q}{1+q^2}, \end{aligned} \tag{9}$$

substituting,

$$\bar{\varphi}(p,0) = 0,\ \bar{\varphi}(0,q) = 0,\ \frac{\partial^\beta\bar{\varphi}(p,0)}{\partial\tau^\beta} = \frac{p}{1+p^2}, \frac{\partial^\nu\bar{\varphi}(0,q)}{\partial x^\nu} = \frac{q}{1+q^2},$$

in equation (9), we obtain

$$\left(q^{-2}-p^{-2}-2\right)\bar{\varphi}(p,q) = q^{-1}\left(\frac{p}{1+p^2}\right) - p^{-1}\left(\frac{q}{1+q^2}\right) - 2\left(\frac{p}{1+p^2}\frac{q}{1+q^2}\right).$$

Consequently,

$$\varphi(x,\tau) = S_x^{-1} S_\tau^{-1}\left[\frac{1}{(q^{-2}-p^{-2}-2)}\left(q^{-1}\left(\frac{p}{1+p^2}\right) - p^{-1}\left(\frac{q}{1+q^2}\right) - 2\left(\frac{p}{1+p^2}\frac{q}{1+q^2}\right)\right)\right],$$

simplifying, we get

$$\varphi(x,\tau)=S_x^{-1}S_\tau^{-1}\left[\frac{p}{1+p^2}\frac{q}{1+q^2}\right]=\sin\left(\frac{x^\nu}{\nu}\right)\sin\left(\frac{\tau^\beta}{\beta}\right).$$

When $\nu=1$ and $\beta=1$ the exact solution of (9) becomes

$$\varphi(x,\tau)=\sin(x)\sin(\tau).$$

Consider the CFPD Korteweg-de Vries equation

$$\frac{\partial^\beta\varphi}{\partial\tau^\beta}+\frac{\partial^{3\nu}\varphi}{\partial x^{3\nu}}+\frac{\partial^\nu\varphi}{\partial x^\nu}=0,\ \ 0<\nu,\ \beta\leq 1 \tag{10}$$

subject to the initial and boundary conditions:

(11)
$$\varphi\left(\frac{x^\nu}{\nu},\,0\right)=e^{-\frac{x^\nu}{\nu}},\ \varphi\left(0,\frac{\tau^\beta}{\beta}\right)=e^{2\frac{\tau^\beta}{\beta}},\ \frac{\partial^\nu}{\partial x^\nu}\varphi\left(0,\frac{\tau^\beta}{\beta}\right)=-e^{2\frac{\tau^\beta}{\beta}},\ \frac{\partial^{2\nu}}{\partial x^{2\nu}}\varphi\left(0,\frac{\tau^\beta}{\beta}\right)=e^{2\frac{\tau^\beta}{\beta}}.$$

Applying the (CDST) to both side of equation (10) and single (CLT) to initial and boundary conditions (11), we get

$$\begin{aligned}&q^{-1}\bar{\varphi}(p,q)-q^{-1}\bar{\varphi}(p,0)+p^{-3}\bar{\varphi}(p,q)-p^{-3}\bar{\varphi}(0,q)-\\&p^{-2}\frac{\partial^\nu\bar{\varphi}(0,q)}{\partial x^\nu}-p^{-1}\frac{\partial^{2\nu}\bar{\varphi}(0,q)}{\partial x^{2\nu}}+p^{-\nu}\bar{\varphi}(p,q)-p^{-1}\bar{\varphi}(0,q)=0,\end{aligned} \tag{12}$$

substituting the single (CLT) of the initial and boundary conditions

$$\bar{\varphi}(p,0)=\frac{1}{1+p},\ \bar{\varphi}(0,q)=\frac{1}{1+2q},\ \frac{\partial^\nu\bar{\varphi}(0,q)}{\partial x^\nu}=\frac{-1}{1-2q},\ \frac{\partial^{2\nu}\bar{\varphi}(0,q)}{\partial x^{2\nu}}=\frac{1}{1-2q},$$

in equation(12), we obtain

$$\left(q^{-1}+p^{-3}+p^{-1}\right)\bar{\varphi}(p,q)=\left(\frac{q^{-1}}{1+p^2}\right)+\left(\frac{p^{-3}}{1+2q}\right)-\left(\frac{p^{-2}}{1-2q}\right)+\left(\frac{p^{-1}}{1-2q}\right)+\left(\frac{p^{-1}}{1-2q}\right).$$

Consequently,

$$\varphi(x,\tau)=S_x^{-1}S_\tau^{-1}\left[\frac{1}{(q^{-1}+p^{-3}+p^{-1})}\left(\begin{array}{l}\left(\frac{q^{-1}}{1+p^2}\right)+\left(\frac{p^{-3}}{1+2q}\right)-\left(\frac{p^{-2}}{1-2q}\right)+\\\left(\frac{p^{-1}}{1-2q}\right)+\left(\frac{p^{-1}}{1-2q}\right),\end{array}\right)\right],$$

simplifying,

$$\varphi(x,\tau)=L_x^{-1}L_\tau^{-1}\left[\frac{1}{(1-2q)(1+p)}\right]=e^{2\left(\frac{\tau^\beta}{\beta}\right)-\left(\frac{x^\nu}{\nu}\right)}.$$

In the case of $\nu=1$ and $\beta=1$, the exact solution of (10) becomes

$$\varphi(x,\tau)=e^{2\tau-x}.$$

Consider the (CFPD) Euler-Bernoulli equation

$$\frac{\partial^{4\nu}\varphi}{\partial x^{4\nu}}+\frac{\partial^{2\beta}\varphi}{\partial \tau^{2\beta}}=\left(\frac{x^{\nu}}{\nu}\right)\left(\frac{\tau^{\beta}}{\beta}\right)+\left(\frac{\tau^{\beta}}{\beta}\right)^{2}, \tag{13}$$

subject to the initial and boundary conditions:

$$\begin{aligned}
&\varphi\left(\frac{x^{\nu}}{\nu},\,0\right)=0,\;\frac{\partial^{\tau}}{\partial \tau^{\nu}}\varphi\left(\frac{x^{\nu}}{\nu},\,0\right)=\frac{1}{120}\left(\frac{x^{\nu}}{\nu}\right)^{5},\\
&\varphi\left(0,\frac{\tau^{\beta}}{\beta}\right)=\frac{1}{12}\left(\frac{\tau^{\beta}}{\beta}\right)^{4},\;\frac{\partial^{i\nu}}{\partial x^{i\nu}}\varphi\left(0,\frac{\tau^{\beta}}{\beta}\right)=0,\;\text{for } i=1,2,3.
\end{aligned} \tag{14}$$

Applying the (CDST) to both side of equation (13) and single (CLT) to initial and boundary conditions (14), we get

$$\begin{aligned}
&p^{-4}\bar{\varphi}\left(p,q\right)-p^{-4}\bar{\varphi}\left(0,q\right)-p^{-3}\frac{\partial^{\nu}\bar{\varphi}\left(0,q\right)}{\partial x^{\nu}}-p^{-2}\frac{\partial^{2\nu}\bar{\varphi}\left(0,q\right)}{\partial x^{2\nu}}\\
&-p^{-1}\frac{\partial^{3\nu}\bar{\varphi}\left(0,q\right)}{\partial x^{3\nu}}+q^{-2}\bar{\varphi}\left(p,q\right)-q^{-2}\bar{\varphi}\left(p,0\right)-q^{-1}\frac{\partial^{\beta}\bar{\varphi}\left(p,0\right)}{\partial \tau^{\beta}}=pq+2q^{2},
\end{aligned} \tag{15}$$

substituting

$$\bar{\varphi}\left(p,0\right)=0,\frac{\partial^{\beta}\bar{\varphi}\left(p,0\right)}{\partial \tau^{\beta}}=p^{5},\;\bar{\varphi}\left(0,q\right)=2q^{4},\;\frac{\partial^{i\nu}}{\partial x^{ri\nu}}\bar{\varphi}\left(0,q\;\right)=0,\;\text{for, i}=1,2,3.$$

In equation (15), we obtain

$$\left(p^{-4}+q^{-2}\right)\bar{\varphi}\left(p,q\right)=\left(2p^{-4}q^{4}+q^{-1}p^{5}+pq+2q^{2}\right).$$

Consequently,

$$\varphi(x,\tau)=S_{x}^{-1}S_{x}^{-1}\left[\frac{\left(2p^{-4}q^{4}+q^{-1}p^{5\nu}+pq+2q^{2}\right)}{\left(p^{-4}+q^{-2}\right)}\right],$$

simplifying,

$$\varphi(x,\tau)=S_{x}^{-1}S_{x}^{-1}\left[2q^{4}+qp^{5}\right]=\frac{1}{12}\left(\frac{\tau^{\beta}}{\beta}\right)^{4}+\frac{1}{5!}\left(\frac{\tau^{\beta}}{\beta}\right)\left(\frac{x^{\nu}}{\nu}\right)^{5}.$$

When $\nu=1$ and $\beta=1$, the exact solution of (10) becomes

$$\varphi\left(x,\tau\right)=\frac{2\tau^{4}}{4!}+\frac{\tau x^{5}}{5!}.$$

5 Conclusions

In this article, we have introduced the definition of the comformable double Sumudu transform (CDST). First, we have applied the (CDST) to some certain functions, next, some theorems and properties related to (CDST) are presented and proved. To illustrate the applicability and efficiency of the proposed transform, we have implemented (CDST) to obtain the exact solution of a wide class of (FPDE) involving (CFD). Based on the results obtained, we conclude that the (CDSTM) is efficient, appropriate, reliable and sufficient to acquire the exact solutions of (CFPDEs) subject to the considered initial and boundary conditions. Moreover, the calculations involved in (CDST) have small computational size as compared to other methods. Finally, it's worthwhile to mention that the (CDSTM) can be coupled with some other methods to solve non-linear (CFPDEs) arising in many field of sciences and engineering, which will be discussed in a subsequent article.

References

[1] Klcman, A., Gadain, H.E. On the applications of Laplace and Sumudu transforms. Journal of the Franklin Institute 2010; 347(5): 848-862.

[2] Abdeljawad, T. On conformable fractional calculus. Journal of computational and Applied Mathematics 2015; 279: 57-66.

[3] Chen, C., Jiang, Y.L. Simplest equation method for some time-fractional partial differential equations with conformable derivative. Computers and Mathematics with Applications 2018; 75(8): 2978-2988.

[4] Belgacem, F.B.M. Introducing and analyzing deeper Sumudu properties. Nonlinear Studies 2006; 13(1): 23.

[5] Watugala, G.K. Sumudu transform: a new integral transform to solve differential equations and control engineering problems. International Journal of Mathematical Education in Science and Technology 1993; 24(1): 35-43.

[6] Hashemi, M.S. Invariant subspaces admitted by fractional differential equations with conformable derivatives. Chaos Solitons Fractals 2018; 107; 161-169.

[7] Hosseini, K., Mayeli, P., Ansari, R. Bright and singular soliton solutions of the conformable time-fractional KleinGordon equations with different nonlinearities. Waves Random Complex Media 2017; 26: 1-9 .

[8] Khalil R., Horani M. A., Yousef A., Sababheh M. A new definition of fractional derivative. Journal of Computational and Applied Mathematics 2014; 264: 65-70.

[9] Kilbas A.A, Srivastava H.M, Trujillo J.J. Theory and applications of fractional differential equations. Elsevier, Amsterdam 2006; 204: 540.

[10] Korkmaz, A., Hosseini, K. Exact solutions of a nonlinear conformable time-fractional parabolic equation with exponential nonlinearity using reliable methods. Optical and Quantum Electronics 2017; 49(8): 278.

[11] Korkmaz, A. On the wave solutions of conformable fractional evolution equations. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 2018; 67(1); 68-79.

[12] Lakshmikantham, V., Vatsala, A.S. Basic theory of fractional differential equations. Nonlinear Analysis: Theory, Methods and Applications 2008; 69(8): 2677-2682.

[13] Asiru, M.A. Further properties of the Sumudu transform and its applications. International Journal of Mathematical Education in Science and Technology 2002; 33(3): 441-449.

[14] Ozkan, O., Kurt, A. Conformable Fractional Double Laplace Transform and Its Applications to Fractional Partial Integro-Differential Equations. Journal of Fractional Calculus and Applications 2020; 11(1): 70-81.

[15] Ozkan, O., Kurt, A. On conformable double Laplace transform. Optical and Quantum Electronics 2018; 50(2); 103.

[16] Podlubny, I. Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Elsevier 1998.

[17] Tchuenche, J.M., Mbare, N.S. An application of the double Sumudu transform. Applied Mathematical Sciences 2007; 1(1-4): 31-39.

[18] Thabet, H., Kendre, S. Analytical solutions for conformable space time fractional partial differential equations via fractional differential transform. Chaos, Solitons and Fractals 2018; 109: 238245.

[19] Yaslan, H. C. New analytic solutions of the conformable spacetime fractional Kawahara equation. Optik 2017; 140: 123-126.

[20] Al-Zhour, Z., Alrawajeh, F., Al-Mutairi, N., Alkhasawneh, R. New results on the conformable fractional Sumudu transform: theories and applications. International Journal of Analysis and Applications 2019; 17(6); 1019-1033.